A criterion for asymptotic stability based on topological degree

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Abstract. There exist some connections between the stability properties of fixed points of maps and the corresponding topological degree. A sufficient condition for asymptotic stability based on this idea is obtained. This condition is used to find new results on the stability of periodic solutions of a class of differential equations of dissipative type.

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1. Introduction

The aim of this paper is to study some relationships between the Leray-Schauder degree and the theory of stability. The degree is sometimes used to prove the existence of stationary or periodic solutions of certain differential equations of evolutionary type. Often the only information on the solution given by this method is a rough estimate and the value of the degree of a related operator. In such case, the standard techniques of stability theory do not seem to apply and it is natural to ask what can be said about the stability of the solution using degree theory. In general the degree is not sufficient to characterize stability and one has to impose some additional conditions.

This kind of question has been considered in the context of the theory of differential equations of periodic type [1], [12], [13], [14], in the case of positive mappings in [2] and in bifurcation theory [16]. A different point of view on the use of the degree in stability theory can be seen in [9]. In these works, excepting [2], [9] and [14], the principle of linearized stability is employed and the solution under study has to be of hyperbolic type.

In this paper we obtain a general result, relating the fixed point index (local version of the degree) and the stability of fixed points, that unifies and extends to the non-hyperbolic case the previous results. It can be summarized as follows: "Let Ψ be a map acting on the phase space Z and let $p \in Z$ be a fixed point of Ψ which is isolated (in an appropriate sense). Assume that all the eigenvalues of $\Psi'(p)$, except perhaps one, lie in the open unit disk. Then the stability properties of p with respect to Ψ only depend on certain fixed point indexes of p."

The precise statements are given in Section 2. These abstract results are then applied to the study of the stability of periodic solutions of nonautonomous ordinary differential equations in Section 3. There the map Ψ is the Poincaré-Andronov operator associated to the equation and $\Psi'(p)$ is identified with a monodromy matrix of the variational equation at the periodic solution. In practice, the Floquet multipliers (eigenvalues of $\Psi'(p)$) are difficult to compute and it is of interest to find explicit conditions on the equation that guarantee the applicability of the main result. The condition on the truncated trace of the field introduced in [17] is useful in this context. Originally this condition was used in [17] to extend a result in [11] on the number of periodic solutions of plane dissipative systems with negative divergence. Some new information on this class of systems will also be obtained from our results. Section 4 is devoted to the proof of the abstract result of Section 2. This result has an elementary proof if Z has dimension one. In the general case the key idea is to restrict Ψ to a locally invariant curve on which the properties of stability and degree of the fixed point are preserved. The main tools in the proof are some of the techniques developed in [14] and a variant of the center manifold theorem. The center manifold was already used in [2], but there the order-preserving character of the operator played a role in the proof. We shall give a detailed proof valid for general operators.

2. The abstract result

In the following Z denotes a real Banach space with norm $|\,.\,|$ and Ω an open and bounded neighborhood of the origin $z = 0$. Consider the map $\Psi : \overline{\Omega} \subset Z \longrightarrow Z$ and assume that $\Psi(0) = 0$. We are interested in the properties of stability of $z = 0$ as a fixed point of Ψ; in particular, Lyapunov stability (L.s.), asymptotic stability (a.s.) and uniform asymptotic stability (u.a.s.). Under the general assumptions :

$$\Psi \text{ is completely continuous and of class } C^1 \tag{2.1}$$

$$z = 0 \quad \text{is an isolated fixed point of } \Psi \tag{2.2}$$

the fixed point is defined as $i[\Psi, 0] = \deg[I - \Psi, \Omega_0, 0]$ where deg refers to the Leray-Schauder degree, I is the identity in Z and $\Omega_0 \subset \Omega$ is a neighborhood of the origin containig no other fixed points of Ψ. When Ψ is completely continuous, the concepts of asymptotic stability and uniform asymptotic stability of fixed points are equivalent (see [3]). Next result shows that the asymptotic stability of $z = 0$ imposes a severe restriction on the index.

Proposition 2.1. *In the previous setting assume that $z = 0$ is asymptotically stable. Then $i[\Psi, 0] = 1$.*

(The proof follows from Theorem 39.1 in [7]).

The converse of this proposition is not valid excepting in certain special cases. In the simplest situation, $Z = \mathbf{R}$, the converse is true if Ψ preserves the orientation (i.e. $\Psi'(0) \geq 0$). If Ψ reverses the orientation it is no longer true, as shown by the map $\Psi_\lambda(z) = \lambda z$ $(\lambda < -1)$. However $\Psi^2 = \Psi \circ \Psi$ is always orientation preserving and the asymptotic stability can be characterized in terms of $i[\Psi^2, 0]$. When $\dim Z \geq 2$ a characterization of this type is not possible, unless one imposes additional conditions on Ψ.

It follows from (2.1) that the linear operator $\Psi'(0)$ is compact. The nonzero eigenvalues can be arranged in an ordered (and sometimes finite) sequence,

$$\{\lambda_n\}_{n\geq 1}, \quad |\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n| \geq \ldots$$

where each eigenvalue λ_n has algebraic multiplicity μ_n. Our last condition on Ψ is

$$\lambda_1 \geq 0, \quad \mu_1 = 1, \quad |\lambda_n| < 1 \quad \forall n \geq 2. \tag{2.3}$$

Roughly speaking, this condition says that the unstable manifold of $z = 0$ has at most dimension one and Ψ preserves the orientation on it. The main result of this section is

Theorem 2.2. *Assume that* (2.1), (2.2) *and* (2.3) *hold. The following statements are equivalent:*

(i) $z = 0$ *is asymptotically stable*
(ii) $z = 0$ *is Lyapunov stable*
(iii) $i[\Psi, 0] = 1$.

(A proof is given in Section 4).

Remarks.

1. The implication $(ii) \Rightarrow (i)$ proves that, in our setting, a stable fixed point which is isolated is automatically asymptotically stable. This kind of result is classical in the theory of scalar first order periodic equations (see [15] and also [18] for an extension to certain systems).
2. It will follow from the proof of the theorem that if (2.1), (2.2) and (2.3) hold then $|i[\Psi, 0]| \leq 1$. Therefore the instability corresponds to the indexes $i = -1, 0$.
3. The assumption (2.1) can be probably weakened using more general degree theories (see [8]).
4. The result in [2] assumed that Z was an ordered Banach space and replaced (2.3) by an assumption of monotonicity of Ψ that in particular implied (2.3). In the context of [2] the asymptotic stability of $z = 0$ is also equivalent to the existence of ordered upper and lower solutions arbitrarily close to the origin from above and below respectively. This conclusion is not valid for operators that do not preserve the order (see Lemma 3.2 in [14]).

The map $\Psi^2 = \Psi \circ \Psi$ is well defined in some neighborhood of the origin (remark that $\Psi(0) = 0$). The application of the previous result to Ψ^2, instead of Ψ, will lead to a corollary of interest. Assumptions (2.2) and (2.3) need to be modified to

$$z = 0 \text{ is an isolated fixed point of } \Psi^2 \tag{2.4}$$

$$\mu_1 = 1, \quad |\lambda_n| < 1 \quad \forall n \geq 2 \tag{2.5}$$

Note that (2.4) is stronger than (2.2) while (2.5) is weaker than (2.3). In particular, (2.5) implies that periodic points of Ψ of period 2 do not exist in some neighborhood of $z = 0$.

Corollary 2.3. *Assume that* (2.1), (2.4) *and* (2.5) *hold. The following statements are equivalent:*

(i) $z = 0$ *is asymptotically stable*
(ii) $z = 0$ *is Lyapunov stable*
(iii) $i[\Psi^2, 0] = 1$.

Remark. In this setting, a combination of Proposition 2.1 with this corollary leads to the implication $i[\Psi^2, 0] = 1 \Rightarrow i[\Psi, 0] = 1$.

3. The index of a periodic solution

Consider the differential equation

$$x' = f(t, x) \tag{3.1}$$

where $f : \mathbf{R} \times S \longrightarrow \mathbf{R}^N$, $N \geq 1$, and $S \subset \mathbf{R}^N$ is open. It is assumed that f is continuous, T-periodic with respect to t and such that $f_x(t, x)$ is defined and continuous on $\mathbf{R} \times S$. Given $p \in S$ let $\phi(t, p)$ denote the solution of (3.1) satisfying $\phi(0, p) = p$. The Poincaré-Andronov map is defined on some open subset of S by $P(p) = \phi(T, p)$. Let θ be a T-periodic solution of (3.1). It is well known that the stability properties of θ as a solution of the differential equation are those of $\theta(0)$ considered as a fixed point of P. The solution θ is called isolated (period kT) with $k \geq 1$ if $\theta(0)$ is isolated as a fixed point of P^k. In such case the index of period kT is defined as

$$\gamma_{kT}(\theta) = i[P^k, \theta(0)].$$

The map P is C^1 and the jacobian matrix $P'(\theta(0))$ is a monodromy matrix of the variational equation

$$y' = f_x(t, \theta(t))y. \tag{3.2}$$

In consequence, the eigenvalues of $P'(\theta(0))$ are precisely the Floquet multipliers of (3.2). (See for instance [6]). To verify condition (1.3) or (1.5) when the multipliers are not explicitly known one needs some conditions on f. For example, conditions (1.2) in [12] and (H-1) in [14] imply (1.3). (See Lemma 1.4 of [12] and Lemma 4.4 of [14]). In this paper we shall use a condition employed in [17] that implies (1.5). Let $\tau_1(t, x) \geq \tau_2(t, x) \geq \ldots \geq \tau_N(t, x)$ be the eigenvalues of the symmetric matrix $(1/2)[f_x(t, x) + f_x(t, x)^*]$, where * denotes the transpose matrix. Then $\tau_1(t, x) + \tau_2(t, x) + \ldots + \tau_N(t, x)$ coincides with the trace of $f_x(t, x)$ (the divergence of f) and a partial sum $\tau_1(t, x) + \tau_2(t, x) + \ldots + \tau_k(t, x)$, $1 \leq k < N$ is called a truncated trace.

Theorem 3.1. *In the previous notations assume that*

$$\tau_1(t,x)+\tau_2(t,x)<0, \forall (t,x)\in \mathbf{R}\times S, \tag{3.3}$$

and let θ *be a* T*-periodic solution of* (3.1) *which is isolated (period* $2T$*). Then*

$$\theta \text{ is asymptotically stable [resp. unstable]} \iff \gamma_{2T}(\theta)=1 \text{ [resp. } \neq 1].$$

Remark. When $N=2$ (3.3) reduces to $\mathrm{div}_x f<0$ in $\mathbf{R}\times S$. In this case the conclusion of Theorem 3.1 was obtained in [13] when θ is hyperbolic.

Proof. To apply corollary 2.3 with $\Psi=P$ it is enough to prove that (2.5) holds when $\{\lambda_n\}$ is the sequence of multipliers of (3.2). Let $y_1(t)$, $y_2(t)$ be linearly independent solutions of (3.2). Define $\Delta(t)=|y_1(t)|^2|y_2(t)|^2-\langle y_1(t)^2,y_2(t)\rangle^2$ where $\langle .,. \rangle$ is the usual inner product in $\mathbf{R}^N$ and $|\,.\,|$ is the Euclidean norm. It follows from Theorem 1 in [17] that $\Delta(T)\leq\Delta(0)\exp\{2\int_0^T[\tau_1(t,\theta(t))+\tau_2(t,\theta(t))]dt\}$ and from (3.3), $\Delta(T)<\Delta(0)$. Since y_1, y_2 can be chosen arbitrarily, it is now easy to show that only one multiplier can be greater or equal to 1 in absolute value. This proves that (2.5) holds.

Remark. It is interesting to notice the geometrical meaning of this proof. The quantity $\Delta(t)^{1/2}$ is the area of the parallelogram determined by $0, y_1(t), y_2(t)$. The inequality $\Delta(T)<\Delta(0)$ says that the flow determined by (3.2) contracts this area after one period. A classical result when $N=2$.

The condition (3.3) was introduced in [17] together with

$$S=\mathbf{R}^N \text{ and the system (3.1) is analytic and dissipative.} \tag{3.4}$$

The system (3.1) is said to be analytic when f is a real analytic function of the coordinates (t,x). Also, it is said that the system is dissipative if there exists a constant $b>0$ such that every solution $x(t)$ of (3.1) is defined up to $+\infty$ and satisfies $limsup_{t\to+\infty}|x(t)|\leq b$. It was proved in [17], and previously in [11] for $N=2$, that the number of T-periodic solutions is finite when (3.3) and (3.4) hold.

A periodic solution with minimal period $kT, k\geq 1$, is called a subharmonic solution of k^{th}order. The set of all subharmonic solutions of (3.1) is nonempty and countable if (3.3) and (3.4) are assumed. Next results give sufficient conditions for the existence of stable harmonic and subharmonic solutions.

Corollary 3.2. *Assume that* (3.3) *and* (3.4) *hold. If the system* (3.1) *has not subharmonic solutions of the* 2^{nd}*order, then it has at least one asymptotically stable* T*-periodic solution.*

Proof. It follows from [17] that the number of fixed points of P is finite. Also, from the assumption on nonexistence of subharmonics, the fixed points of P and P^2 are the same. Since the system is dissipative it is possible to find a ball $B\subset\mathbf{R}^N$ such that every periodic point of P lies in B and $P^k(\overline{B})\subset B, k\geq k_0$ (see [15]). From Theorem 39.1 in [7], $\deg[I-P^2,B,0]=1$. The excision property of degree and remark 2 after Theorem 2.2 imply the existence of $p\in B$, isolated fixed point of P, such that $i[P^2, p] = 1$. Then $\theta(t) = \phi(t, p)$ is a T-periodic solution with $\gamma_{2T}(\theta) = 1$ and it only remains to apply Theorem 3.1.

Corollary 3.3. *Assume that* (3.3), (3.4) *hold and that the system* (3.1) *has a finite number of subharmonic solutions. Then at least one of them is asymptotically stable.*

Proof. Let $m > 0$ be an integer such that the order of each subharmonic solution of (3.1) divides m. Then every subharmonic solution admits the period mT and the previous corollary can be applied with initial period mT.

4. Proofs

We divide this section in three parts. In the first part, we collect some facts on the theory of center manifolds that will be used later. Next we present a perturbation of Ψ that is of interest when $z = 0$ is not hyperbolic. Finally all the pieces are put together and Theorem 2.2 is proved.

4.1 Remarks on center manifolds

Let X and Y be Banach spaces with $\dim Y < \infty$ and assume that Z is the product space $Z = X \times Y$ with norm $| z | = \max\{| x |, | y |\}, z = (x, y)$. Let $F : \mathrm{dom}(F) \subset Z \longrightarrow Z$ be a map such that $\mathrm{dom}(F)$ contains a neighborhood of the origin and $F(0) = 0$. It is assumed that F is Lipschitz-continuous. We shall also consider maps of the kind

$$u : \{y \in Y / \mid y \mid \leq \delta\} \longrightarrow X \quad (\delta > 0) \tag{4.1}$$

satisfying

$$u \text{ is Lipschitz} - \text{continuous and } u(0) = 0. \tag{4.2}$$

Denote by Σ the corresponding graph $\Sigma = \{(u(y), y) / \mid y \mid \leq \delta\}$. The following definitions concern this manifold.

Definition 4.1. Σ is locally invariant if for each $z \in Z$ such that $| F(z) | \leq \delta$ then $F(z) \in \Sigma$.

Definition 4.2. Σ is locally attracting if for each $z \in Z$ such that $| z | \leq \delta$ and $| \hat{z} | \leq \delta$ then $| \hat{x} - u(\hat{y}) | \leq \alpha \mid x - u(y) \mid$, where $z = (x, y), \hat{z} = F(z) = (\hat{x}, \hat{y})$ and $\alpha \in (0, 1)$ is a fixed constant independent of z.

Both concepts are of course relative to the map F. The definition of local attractivity is more restrictive than those in [10] and [4, page 262]. Σ is said to be a Center Manifold if it is locally invariant and attracting. The local invariance of Σ allows the consideration of the restricted map $F_\Sigma : \Sigma' \subset \Sigma \longrightarrow \Sigma, F_\Sigma(z) = F(z)$, where Σ' is a neighborhood (relative to Σ) of the origin. Next result shows that the information on asymptotic stability is preserved by this reduction.

Proposition 4.3. *In the previous setting assume that Σ is a center manifold. Then, if the origin is a.s. with respect to F_Σ, it is u.a.s. with respect to F.*

The proof of this result is proposed as an exercise in [4, page 262] and we now give a sketch of it. Since Σ' is locally compact, 0 is u.a.s. with respect to F_Σ and one can find a Lyapunov function V as in [4, page 88]. Define

$$W(z) = V(u(y), y) + \lambda \mid x - u(y) \mid, z = (x, y) \in Z.$$

Using the local attractivity of Σ it can be seen that W is a Lyapunov function with respect to F when λ is large.

Later we shall need a consequence of the proof of the center manifold theorem for C^1-maps. Before stating it we introduce some notation. Df denotes the Fréchet derivative of f, $L(X)$ denotes the Banach algebra of bounded linear endomorphisms of X and $\| \, . \, \|$ is the associated norm, $[f]_{Lip}$ is the best Lipschitz constant of f.

Proposition 4.4. *Let R and S be completely continuous C^1-mappings of a neighborhood of the origin of Z into X and Y respectively, such that $R = 0, S = 0, DR = 0, DS = 0$ at $z = 0$. Let $\{A_n\}_{n\geq 1}$ and $\{B_n\}_{n\geq 1}$ be convergent sequences in $L(X)$ and $L(Y)$ respectively, with $A_n \longrightarrow A_\infty, B_n \longrightarrow B_\infty$ in the uniform topology. Assume that A_∞ is compact and*

$$\| A_n \| < 1, \| A_n \| \| B_n^{-1} \| < 1, 1 \leq n \leq \infty. \tag{4.3}$$

Define the sequence of maps $F_n(x, y) = (A_n x + R(x, y), B_n y + S(x, y)), 1 \leq n \leq \infty$. Then there exist $\delta > 0$, (independent of n), a subsequence of F_n denoted by $\{F_k\}$ and functions u_k satisfying (4.1) *and* (4.2) *and such that the corresponding graphs Σ_k are center manifolds with respect to $F_k, 1 \leq k \leq \infty$. In addition,*

$$[u_k]_{Lip} \leq 1, 1 \leq k \leq \infty \tag{4.4}$$

$$u_k \to u_\infty \text{ uniformly in } \mid y \mid \leq \delta. \tag{4.5}$$

Proof. It follows from the proof of the center manifold theorem given in [5] that there exists $\delta_1 > 0$ (independent of n) and Lipschitz-continuous functions

$$u_n : \{y \in Y / \mid y \mid \leq \delta_1\} \longrightarrow X, u_n(0) = 0, 1 \leq n < \infty$$

such that $[u_n]_{Lip} \leq 1$, the graphs $\Sigma_n = \{(u_n(y), y) / \mid y \mid \leq \delta_1\}$ are center manifolds with respect to F_n and the constant $\alpha \in (0, 1)$ appearing in Definition 4.2 is also independent of n. The independence with respect to n of δ_1 and α comes from (4.3), which keeps the quantities $\| A_n \|$ and $\| A_n \| \| B_n^{-1} \|$ uniformly less than 1. It also follows from the mentioned proof that there exist $\delta_2, \delta_3 \in (0, \delta_1), \delta_3 < \delta_2$, independent of n, such that

a) $z \in Z, \mid z \mid \leq \delta_2 \Rightarrow \mid F_n(z) \mid \leq \delta_1, 1 \leq n < \infty$

b) For each $y \in Y, \mid y \mid \leq \delta_3$, and n, $1 \leq n < \infty$, there exists $y_n \in Y, \mid y_n \mid \leq \delta_2$, satisfying $y = B_n y_n + S(u_n(y_n), y_n)$.

We now claim that for each $y \in Y, |y| \leq \delta_3$ the set $H(y) = \{u_n(y)/1 \leq n < \infty\}$ is relatively compact in X. In fact, let (y_n) be the sequence obtained in b). The local invariance of Σ_n implies that $u_n(y) = A_n u_n(y_n) + R(u_n(y_n), y_n) = A_\infty u_n(y_n) + (A_n - A_\infty)u_n(y_n) + R(u_n(y_n), y_n)$. Since $|y_n| \leq \delta_2$ and A_∞, R are completely continuous it is clear that $H(y)$ has a compact closure. From $[u_n]_{Lip} \leq 1, u_n(0) = 0$, we conclude that (u_n) is equicontinuous and uniformly bounded in $|y| \leq \delta_3$. The Ascoli theorem implies the existence of a subsequence (u_k) converging uniformly in $|y| \leq \delta_3$ to a certain function u_∞. An argument of passage to the limit allows us to prove that $\Sigma_\infty = \{(u_\infty(y), y)/ |y| \leq \delta\}, \delta = \delta_3/2$, is locally invariant and locally attracting (the constant α is fixed).

4.2 A perturbation of Ψ.

In the sequel we follow the notation of Section 2. In this paragraph, in addition to (2.1), (2.2) and (2.3), it is assumed that $z = 0$ is not hyperbolic as a fixed point so that $\lambda_1 = 1$. Let P be the projection onto the eigenspace of λ_1 given by $P = (1/2\pi i)\oint_\Gamma(\lambda I - L)^{-1}d\lambda$, where $L = \Psi'(0)$ and Γ is a small circle around λ_1 (with counterclockwise orientation). We also use the notations

$$X = \text{Ker}P, Y = \text{Im}P, Q = I - P. \tag{4.6}$$

Since λ_1 is simple, Y is one-dimensional and one can find a bounded linear functional $\Pi : X \longrightarrow \mathbf{R}$ and $v \in Y, |v| = 1$, such that $Pz = \Pi(z)v, z \in Z$. Also the hyperplane X is invariant under L and the restriction $L_X : X \longrightarrow X$ satisfies

$$r(L_X) = |\lambda_2| < 1 \tag{4.7}$$

where r denotes the spectral radius. In the sequel we can use the decomposition $z = x + \xi v, (x, \xi) \in X \times \mathbf{R}, \forall z \in Z$ and assume that Z has been renormed by $|z| = \max\{|x|_*, |\xi|\}$, where $|\,.\,|_*$ is a norm in X (equivalent to the original one) such that $\| L_X \|_* < 1$. (This is useful to remain in the framework of 4.1 whenever needed.)

Given $\epsilon \in \mathbf{R}$ define the perturbed map

$$\Psi_\epsilon(z) = \Psi(z) - \epsilon Pz, z \in \overline{\Omega}. \tag{4.8}$$

Then Ψ_ϵ also satisfies (2.1) and $\Psi_\epsilon(0) = 0$. Since $\Psi'_\epsilon(0) = L - \epsilon P$ and P and L commute, the eigenvalues of $\Psi'_\epsilon(0)$ are given by

$$\sigma(\Psi'_\epsilon(0)) - \{0\} = \{1 - \epsilon\} \cup \{\lambda_n/n \geq 2\}. \tag{4.9}$$

If $\epsilon \neq 0$, 1 is not an eigenvalue of $\Psi'_\epsilon(0)$ so that $z = 0$ is an isolated fixed point of Ψ_ϵ and, from the general properties of degree (see [7]),

$$i[\Psi_\epsilon, 0] = \text{sign}\epsilon. \tag{4.10}$$

Next result studies the index when $\epsilon = 0$.

Lemma 4.5. *Assume that* (2.1), (2.2) *and* (2.3) *hold and in addition* $\lambda_1 = 1$. *Then* $|i[\Psi, 0]| \leq 1$. *Moreover there exists* $\omega \subset \Omega$, *an open neighborhood of* $z = 0$, *such that*

$$i[\Psi,0]=1[resp.-1] \Longleftrightarrow \Psi_\epsilon(z)\neq z$$

$$\forall z\in\overline{\omega}-\{0\}, \forall\epsilon\geq 0 \ \ [resp. \forall\epsilon\leq 0].$$

Proof. We start with an application of the Lyapunov-Schmidt method to the equation

$$\Psi(z)=z \tag{4.11}$$

The equation is split into

$$Q\Psi(x+\xi v)=x \tag{4.12}$$

$$\Pi\Psi(x+\xi v)=\xi. \tag{4.13}$$

Since $I-L_X$ is an isomorphism in X by (4.7) and $\partial_x Q\Psi(0)=L_X$, the implicit function theorem implies the existence of ω, open subset of Z with $0\in\omega\subset\Omega$, and $w:[-\tau,\tau]\longrightarrow X$ continuous ($\tau>0$) such that $z\in\overline{\omega}$ solves (4.12) if and only if $x=w(\xi)$. Since (2.2) holds it is not restrictive to assume also that

$$\Psi(z)\neq z \quad \forall z\in\overline{\omega}-\{0\}. \tag{4.14}$$

Define $\Gamma:[-\tau,\tau]\longrightarrow\mathbf{R}, \Gamma(\xi)=\xi-\Pi\Psi(w(\xi)+\xi v)$. Then (4.11) is equivalent in $\overline{\omega}$ to $x=w(\xi),\Gamma(\xi)=0$. The function Γ satisfies $\Gamma(0)=0$ and $\Gamma(\xi)\neq 0$ if $0<|\xi|\leq\tau$, because $z=0$ is a fixed point and (4.14) holds. We distinguish four possible cases:

1) $\Gamma(\xi)>0, \forall\xi\in[-\tau,\tau]-\{0\}$,
2) $\Gamma(\xi)<0, \forall\xi\in[-\tau,\tau]-\{0\}$,
3) $\xi\Gamma(\xi)>0, \forall\xi\in[-\tau,\tau]-\{0\}$,
4) $\xi\Gamma(\xi)<0, \forall\xi\in[-\tau,\tau]-\{0\}$

and prove the result in several claims.

Claim 1. If 1) or 2) hold then $i[\Psi,0]=0$.

Assume for instance that 1) holds and consider the equation $\Psi(z)=z+\lambda v$, ($\lambda>0$). Applying the Lyapunov-Schmidt method to this equation one obtains (in $\overline{\omega}$) $x=w(\xi),\Gamma(\xi)+\lambda=0$. The second equation cannot have a solution if Γ is positive so that $\deg[I-\Psi,\omega,-\lambda v]=0$ if $\lambda>0$. The continuity properties of degree imply that $0=\deg[I-\Psi,\omega,0]=i[\Psi,0]$.

Claim 2. If 3) [resp. 4)] holds then $\Psi_\epsilon(z)\neq z \ \ \forall z\in\overline{\omega}-\{0\}, \forall\epsilon\geq 0$ [resp. $\forall\epsilon\leq 0$].

The equation $\Psi_\epsilon(z)=z$ is equivalent in $\overline{\omega}$ to $x=w(\xi),\Gamma(\xi)+\epsilon\xi=0$. If ϵ is positive and 3) holds the second equation only admits the solution $\xi=0$.

Claim 3. If 3) [resp. 4)] holds then $i[\Psi,0]=1$ [resp. -1].

If 3) holds, claim 2 proves that $I-\Psi$ is homotopic in ω to $I-\Psi_\epsilon$ when $\epsilon>0$. Then $\deg[I-\Psi_\epsilon,\omega,0]=\deg[I-\Psi,\omega,0]=i[\Psi,0]$. Applying (4.10) and claim 2 again one obtains $\deg[I-\Psi_\epsilon,\omega,0]=i[\Psi_\epsilon,0]=1$ if $\epsilon>0$.

4.3 Conclusion.

Let us start with a preliminary result on the dynamics of one-dimensional maps.

Lemma 4.6. *Let* $f : [-\delta,\delta] \longrightarrow \mathbf{R}$ $(\delta > 0)$ *be a continuous function with* $f(0) = 0$ *and satisfying*

$$\xi f(\xi) > 0 \ \text{and} \ \xi \neq f(\xi) \ \text{for each} \ \xi \in [-\delta,\delta] - \{0\}. \tag{4.15}$$

The following statements are equivalent:

a) $\xi = 0$ is a.s. with respect to f,
b) $\xi = 0$ is L.s. with respect to f,
c) $-\delta < f(-\delta) < 0 < f(\delta) < \delta$.

This result is essentially a reformulation of Theorem 2.2 when $\dim Z = 1$ and the proof is elementary. We are now able to prove Theorem 2.2. The implication i) $\Rightarrow$ ii) is trivial, while i) $\Rightarrow$ iii) follows from Proposition 2.1. We prove iii) $\Rightarrow$ i) and ii) $\Rightarrow$ i).

iii) $\Longrightarrow$ i).

If $\lambda_1 \neq 1, i[\Psi, 0] = \text{sign}(1-\lambda_1)$ and iii) imply that $\lambda_1 < 1$. Therefore $r(\Psi'(0)) < 1$ and the linearization principle proves i). From now on we assume $\lambda_1 = 1$. Let $\epsilon_n \to 0$ be a sequence with $0 < \epsilon_n < 1- \parallel L_X \parallel_*$ for each $n \geq 1$ and let Ψ_{ϵ_n} denote the perturbation of Ψ given by (4.8). We apply Proposition 4.4 where Z is split as in (4.6), $A_n = L_X, B_n = (1-\epsilon_n)I_Y, F_n = \Psi_{\epsilon_n}, F_\infty = \Psi$. Then (4.3) is verified and there exist $\delta > 0$, a subsequence of F_n that we call again F_n and a sequence of functions $u_n : [-\delta,\delta] \longrightarrow X, u_n(0) = 0$, such that (4.4), (4.5) hold and the curve $\Sigma_n = \{u_n(\xi) + \xi v / \mid \xi \mid \leq \delta\}$ is a center manifold with respect to F_n. Define $f_n : [-\delta,\delta] \longrightarrow \mathbf{R}, f_n(\xi) = \Pi F_n(u_n(\xi) + \xi v), 1 \leq n \leq \infty$. Then f_n is continuous and, from (4.5), $f_n \to f_\infty$ uniformly in $[-\delta,\delta]$. Also, $f_n(\xi) = (1-\epsilon_n)\xi + \Pi T(u_n(\xi) + \xi v)$ where $T(z) = \Psi(z) - \Psi'(0)z$. It follows from (4.4) that the asymptotic formula stated below is valid uniformly with respect to n, $f_n(\xi) = (1-\epsilon)\xi + o(\xi)$ as $\xi \to 0$. It is assumed that δ is chosen small enough in order to get $\xi f_n(\xi) > 0$ if $0 < \mid \xi \mid \leq \delta, 1 \leq n \leq \infty$. Using the local invariance of Σ_n and restricting the size of δ if needed, it can be assumed that if $\xi \in [-\delta,\delta]$ is a fixed point of f_n then $u_n(\xi) + \xi v$ is a fixed point of $F_n, 1 \leq n \leq \infty$. Let ω be the open set in Z given by Lemma 4.5 and such that the only fixed point of F_n in $\overline{\omega}$ is the origin (again iii) has been used). It is not restrictive to assume $\Sigma_n \subset \omega \quad \forall \ n, 1 \leq n \leq \infty$. In consequence $\xi = 0$ is the only fixed point of f_n and (4.15) is valid for $f = f_n$. The projection Π becomes a homeomorphism when it is restricted to Σ_n. Denote it by $\Pi_n : \Sigma_n \longrightarrow [-\delta,\delta]$. From the definition of f_n one has $f_n = \Pi_n \circ F_n \circ \Pi_n^{-1}$, so that f_n and $F_{n|\Sigma_n}$ are conjugate maps. Therefore the stability properties of $\xi = 0$ with respect to f_n and of $z = 0$ with respect to $F_{n|\Sigma_n}$ are the same. Now if $n < \infty$ it follows from (4.9) that $r(F_n'(0)) < 1$ so that $z = 0$ is asymptotically stable with respect to F_n. Therefore $\xi = 0$ is also a.s. with respect to f_n. From Lemma 4.6, $-\delta < f_n(-\delta) < 0 < f_n(\delta) < \delta, \forall n < \infty$.

Letting $n \to \infty$, $-\delta \leq f_\infty(-\delta) \leq 0 \leq f_\infty(\delta) \leq \delta$. The previous inequalities must be strict since (4.15) is valid for $f_n = f_\infty$. Again Lemma 4.6 implies that $\xi = 0$ is a.s. with respect to f_∞. Finally we apply Proposition 4.3 because $F_{\infty|\Sigma_\infty}$ and f_∞ are conjugate.

ii) $\Longrightarrow$ i)

The classical linearization results say in our case that $\lambda_1 \leq 1$ if ii) holds while i) holds if $\lambda_1 < 1$. Thus, we can restrict to the case $\lambda_1 = 1$. As in the previous implication construct a center manifold Σ_∞ with respect to Ψ and define the function $f_\infty : [-\delta, \delta] \longrightarrow \mathbf{R}$ which is conjugate to $\Psi_{|\Sigma_\infty}$ and satisfies (4.15). From Lemma 4.6 and ii) we conclude that $\xi = 0$ is a.s. with respect to f_∞ and again use Proposition 4.3.

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