Ten Mathematical Essays on Approximation in Analysis and Topology
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The stability of the equilibrium: a search for the right approximation

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Abstract

To be done....

Key words: Include them,

1. Introduction

In some textbooks in Mechanics, the phenomenon of parametric resonance is illustrated with the pendulum of variable length. After a change of the time variable, this class of pendula is modeled by the equation

$$\ddot{\theta} + \alpha(t) \sin \theta = 0 \tag{1}$$

where

$$\alpha(t) = g\ell(t)^3$$

and $\ell(t) > 0$ is the length at an instant t. It is traditional to assume that α is periodic, say of period $T > 0$. This model leads to suggestive examples of resonance because the equilibrium $\theta = 0$ becomes unstable if $\alpha(t)$ oscillates in an appropriate way. Although the system has only one degree of freedom[1], the study of the stability of

$$\theta = 0$$

is not elementary. This probably explains why it is customary to substitute the original equation by its linear approximation

$$\ddot{\theta} + \alpha(t)\theta = 0. \tag{2}$$

[1] or one and a half if the dependence on time is counted

The main theme of these notes will be the validity of this procedure. It will turn out that the linearization principle leads to the right conclusions in most cases, but there are exceptions. Sometimes the equation (2) can be unstable while the equilibrium $\theta = 0$ is stable for (1). In contrast, we shall find that the third order approximation (of Duffing type)

$$\ddot{\theta} + \alpha(t)\theta - \frac{1}{3!}\alpha(t)\theta^3 = 0 \tag{3}$$

is faithful. This means that the equilibrium $\theta = 0$ is stable for (1) if and only if the same holds for (3). It must be noticed that the positivity of α is crucial for this result. Indeed, if $\alpha(t)$ can change sign, probably none of the approximations obtained by truncating the expansion of the sine function is faithful.

The idea of replacing a complicated equation by an approximation is central in Stability Theory. The first Lyapunov's method is the simplest instance. It can be applied to our equation to prove instability in the easiest cases but it does not help in the proofs of stability. This is so because the notion of asymptotic stability (considered in Lyapunov's first method) is strange to Hamiltonian mechanics. The study of the stability of the equilibrium requires sophisticated techniques (KAM theory) which use the information on nonlinear approximations. We refer to [2,6] for the perturbative case, where

$$\alpha(t) = \omega^2 + \epsilon\beta(t),$$

and to [26,23] for the general case. On the other hand, the results on instability also use nonlinear approximations but are of a more elementary nature. Already in [15], Levi-Civita obtained instability criteria using the quadratic approximation. His results were presented for abstract mappings and applied to the study of a three body problem. The basic technique in [15] is a detailed analysis of the dynamics around the equilibrium and it could be adapted to the pendulum of variable length. Also, it would be possible to employ Lyapunov functions as in [32]. In these notes we shall show how to obtain instability criteria using a less standard approach. Topological degree will be employed to reduce instability proofs to the computation of certain indexes (localized versions of the degree). The rest of these notes is organized in six sections. The notion of stability and its connection with the dynamics of planar mappings is discussed in §2. The next section, §3, analyzes the linearized equation and the symplectic group $Sp(\mathbb{R}^2)$. In particular, the conjugacy classes in this group are found. The basic facts about degree theory are collected in §4. The degree is useful to define the index of the equilibrium of our differential equation, as shown in §5. Some links between stability and index can be found in §6. Finally, in §7, several characterizations of the stability of the equilibrium of the pendulum are presented. They are obtained in terms of the index, the third approximation or the conjugacy classes of $Sp(\mathbb{R}^2)$. The notes are concluded with some discussions about equations with more degrees of freedom.

2. Perpetual stability and discrete dynamical systems

We shall work with the class of differential equations

$$\ddot{\theta} = f(t, \theta) \tag{4}$$

where f is defined around $\theta = 0$, say $f : \mathbb{R} \times (-\epsilon, \epsilon) \to \mathbb{R}$ with $\epsilon > 0$. The function f satisfies

$$f(t, 0) = 0 \quad \forall t \in \mathbb{R}$$

and so $\theta = 0$ is an equilibrium of the equation. In addition, f is continuous, T-periodic in t and there is uniqueness for the initial value problem associated to (4).

Given a point

$$(\theta_0, \omega_0) \in (-\epsilon, \epsilon) \times \mathbb{R},$$

the solution satisfying

$$\theta(0) = \theta_0 \quad \text{and} \quad \dot{\theta}(0) = \omega_0$$

will be denoted by $\theta(t; \theta_0, \omega_0)$. In general one cannot say that this solution is defined in the whole real line but it is at least defined in a large interval for small values of $|\theta_0|$ and $|\omega_0|$.

The equilibrium $\theta = 0$ is said to be stable if given any neighborhood of the origin in $\mathbb{R}^2$, say $\mathcal{U}$, there exists another neighborhood $\mathcal{V}$ such that if (θ_0, ω_0) belongs to $\mathcal{V}$, then the solution $\theta(t; \theta_0, \omega_0)$ is defined in $(-\infty, \infty)$ and

$$(\theta(t; \theta_0, \omega_0), \dot{\theta}(t; \theta_0, \omega_0)) \in \mathcal{U} \quad \forall t \in \mathbb{R}.$$

This is the notion of perpetual stability, often employed in Hamiltonian dynamics (see Chapter 3 of [32]). The reader who is familiar with stability theory will notice that it means Lyapunov stability for the future and the past. Two simple examples are the equations

$$\ddot{\theta} + \theta = 0 \quad \text{and} \quad \ddot{\theta} - \theta = 0.$$

The equilibrium is stable only for the first.

Let us now consider the difference equation

$$\xi_{n+1} = M(\xi_n) \tag{5}$$

where

$$M : \mathcal{D} \subset \mathbb{R}^2 \to \mathbb{R}^2$$

is a one-to-one and continuous mapping defined in an open set $\mathcal{D}$. It is also assumed that the origin lies in $\mathcal{D}$ and it is a fixed point of M. Given an initial condition $\xi_0 \in \mathcal{D}$, the solution

$$\{\xi_n\}_{n \in I}, \quad \xi_n = M^n(\xi_0),$$

is defined on some subset I of $\mathbb{Z}$. The fixed point $\xi = 0$ is said to be stable if for each neighborhood $\mathcal{U}(0)$, there exists another neighborhood $\mathcal{V}(0)$ such that if $\xi_0 \in \mathcal{V}$ then $\{\xi_n\}$ is defined in $\mathbb{Z}$ and

$$\xi_n \in \mathcal{U} \quad \forall n \in \mathbb{Z}.$$

To practice with this definition the reader can consider the linear mappings M defined by the matrices

$$R[\Theta] = \begin{pmatrix} \cos\Theta & \sin\Theta \\ -\sin\Theta & \cos\Theta \end{pmatrix}, \qquad H_+[\Theta] = \begin{pmatrix} \cosh\Theta & \sinh\Theta \\ \sinh\Theta & \cosh\Theta \end{pmatrix}, \qquad \Theta \neq 0.$$

In the first case $\xi = 0$ is stable while in the second it is unstable.

There is of course a complete analogy between the definitions of stability for the continuous and discrete situations. Now we are going to immerse the study of stability for differential equations in the theory of difference equations. This is a central idea in dynamical systems that goes back to Poincaré.

The mapping

$$P(\theta_0, \omega_0) = (\theta(T; \theta_0, \omega_0), \dot{\theta}(T; \theta_0, \omega_0))$$

is well defined in a neighborhood of

$$\theta_0 = \omega_0 = 0$$

and, due to the uniqueness for the initial value problem, it is one-to-one and continuous. Moreover, the iterates P^n are obtained by evaluating the solutions at time $t = nT$. This property is crucial to prove that the equilibrium $\theta = 0$ is stable for (4) if and only if the fixed point $\theta_0 = \omega_0 = 0$ is stable for the mapping $M = P$. The mapping P is usually called the Poincaré map associated to the equation (4) and it has an important property: it preserves area and orientation. For smooth equations this is equivalent to the identity

$$\det P'(\theta_0, \omega_0) = 1$$

and it is a consequence of Liouville's theorem in Hamiltonian mechanics. The general case can be treated with the techniques in [31], Chapter IX.

To finish this section we notice that the notion of stability is invariant under changes of variables. For example, if φ is a local homeomorphism fixing the origin, the change

$$\xi = \varphi(\eta)$$

transforms

$$\xi_{n+1} = M(\xi_n)$$

into

$$\eta_{n+1} = M^\star(\eta_n)$$

with

$$M^\star = \varphi^{-1} \circ M \circ \varphi,$$

and the stability of $\xi = 0$ and $\eta = 0$ are equivalent.

3. The linear equation and the symplectic group

The linear equation

$$\ddot{\theta} + \alpha(t)\theta = 0, \tag{6}$$

where $\alpha(t)$ is continuous and T-periodic, is called Hill's equation and there are many studies about it. The book by Magnus and Winkler [20] is a classical reference. After passing to a first order system

$$\dot{\xi} = A(t)\xi, \qquad \xi = \begin{pmatrix} \theta \\ \omega \end{pmatrix}, \qquad A(t) = \begin{pmatrix} 0 & 1 \\ -\alpha(t) & 0 \end{pmatrix},$$

we find the matrix solution $X(t)$ satisfying

$$X(0) = I$$

(I is the 2×2 identity matrix). The Poincaré map associated to (6) is linear, namely

$$P\begin{pmatrix}\theta_0\\ \omega_0\end{pmatrix} = L\begin{pmatrix}\theta_0\\ \omega_0\end{pmatrix}, \qquad L = X(T).$$

We present two examples. For the harmonic oscillator ($\alpha \equiv 1$) and a fixed period T, L is the rotation $R[T]$ defined in the previous section. For the repulsive case ($\alpha \equiv -1$), L is the matrix $H_+[T]$.

Liouville's theorem implies that the matrix solution $X(t)$ always satisfies

$$\det X(t) = 1.$$

This property motivates our interest in the symplectic group. The group of 2×2 matrices with nonzero determinant will be denoted by $Gl(\mathbb{R}^2)$. The subgroup of $Gl(\mathbb{R}^2)$ composed by the matrices satisfying

$$\det L = 1$$

is the symplectic group, denoted by $Sp(\mathbb{R}^2)$. Given a matrix L in $Sp(\mathbb{R}^2)$, the eigenvalues μ_1, μ_2 satisfy

$$\mu_1 \mu_2 = 1$$

and one can distinguish three cases:

- *elliptic*: $\mu_1 = \overline{\mu_2}$, $|\mu_1| = 1$, $\mu_1 \neq \pm 1$
- *hyperbolic*: $\mu_1, \mu_2 \in \mathbb{R}$, $0 < |\mu_1| < 1 < |\mu_2|$
- *parabolic*: $\mu_1 = \mu_2 = \pm 1$

The conjugacy classes in the group $Sp(\mathbb{R}^2)$ can be described according to this classification. For an elliptic matrix L there exists $Q \in Sp(\mathbb{R}^2)$ such that $Q^{-1}LQ$ is a rotation

$$Q^{-1}LQ = R[\Theta], \quad \Theta \in (0, \pi) \cup (\pi, 2\pi).$$

A hyperbolic matrix is conjugate to a matrix in one of the two families

$$H_\pm[\Theta] = \begin{pmatrix}\pm\cosh\Theta & \sinh\Theta\\ \sinh\Theta & \pm\cosh\Theta\end{pmatrix}, \qquad \Theta \in (0, \infty).$$

Finally, a parabolic matrix will be conjugate to one of the six matrices

$$I, -I, P_+, P_-, -P_+, -P_- \quad \textit{where } P_\pm = \begin{pmatrix}1 & \pm 1\\ 0 & 1\end{pmatrix}.$$

All these facts can be proven from the theory of Jordan canonical forms. In fact that theory can be seen as the classification of the conjugacy classes in $Gl(\mathbb{R}^2)$. There is a more subtle point which does not follow from Jordan canonical form. From the point of view of the group $Gl(\mathbb{R}^2)$, the rotations $R[\Theta]$ and $R[2\pi - \Theta]$ are conjugate. This is not true in the symplectic group, for if $Q \in Gl(\mathbb{R}^2)$ satisfies

$$Q^{-1}R[\Theta]Q = R[2\pi - \Theta]$$

then

$$\det Q < 0.$$

In view of this property we can say that the angle Θ is a symplectic invariant. Similar situations appear in the parabolic case for the matrices P_+ and P_- (or $-P_+$ and $-P_-$). More details and geometric insights about this group can be found in the paper by Broer and Levi [5]. The reader can deduce from the previous discussions that the origin $\theta = 0$ is stable for (6) if and only if the monodromy matrix $X(T)$ is elliptic or parabolic with

$$X(T) = \pm I.$$

Hill's equation is invariant under translation and rescaling of time. This means that the change

$$t = \lambda(s + \tau), \quad x = x(s),$$

with $\lambda > 0$ and $\tau \in \mathbb{R}$, transforms the Hill's equation in another equation of the same type, namely

$$\frac{d^2\theta}{ds^2} + \alpha^\star(s)\theta = 0 \tag{7}$$

with

$$\alpha^\star(s) = \lambda^2\alpha(\lambda(s + \tau)).$$

The new period is

$$T^\star = \frac{T}{\lambda}.$$

We have made reference to this class of changes because they have a remarkable property, they are sufficient to arrive at the canonical form of monodromy matrices. More precisely,

Proposition 1. *Given $\alpha(t)$, continuous and T-periodic, there exists $\tau \in \mathbb{R}$ and $\lambda > 0$ such that the monodromy matrix associated to (7) is one of the matrices:*

- $R[\Theta]$, $\Theta \in (0, \pi) \cup (\pi, 2\pi)$ *(elliptic case)*
- $H_\pm[\Theta]$, $\Theta \neq 0$ *(hyperbolic case)*
- $I, -I, P_+, P_-, -P_+, -P_-$ *(parabolic case).*

The proof of this result can be seen in [25], Proposition 8, for the elliptic case and in [23], Lemma 2.1, for the parabolic case. Recently Yan and Zhang have found in [33] new applications of this result in the elliptic case. For the reader interested in details a proof in the hyperbolic case is presented.

Proof. Assume that the eigenvalues are μ_1 and μ_2. We find Floquet solutions associated to these eigenvalues. These are non-trivial solutions satisfying

$$\varphi(t + T) = \mu_1\varphi(t), \quad \psi(t + T) = \mu_2\psi(t).$$

The product

$$\Pi = \varphi\psi$$

is T-periodic and so there exists $\tau \in \mathbb{R}$ with

$$\dot{\Pi}(\tau) = 0.$$

The linear independence of φ and ψ implies that $\varphi(\tau)$ and $\psi(\tau)$ do not vanish. We select

$$\varphi(\tau) = \psi(\tau) = 1$$

and define

$$u = \frac{1}{2}(\varphi + \psi), \quad v = \frac{1}{2}(\varphi - \psi).$$

Then

$$u(\tau) = 1, \quad v(\tau) = 0, \quad \dot{u}(\tau) = 0, \quad \text{and } \dot{v}(\tau) \neq 0.$$

Here one uses the definition of τ. From now on we shall assume

$$\dot{v}(\tau) > 0.$$

If this derivative is negative we exchange the roles of φ and ψ. The function

$$u^2 - v^2 = \Pi$$

is T-periodic and so

$$u(\tau + T)^2 - v(\tau + T)^2 = 1.$$

From

$$\dot{\Pi}(\tau + T) = 0,$$

we find that

$$\dot{u}(\tau + T)u(\tau + T) - \dot{v}(\tau + T)v(\tau + T) = 0.$$

The Wronskian formula implies that

$$\dot{v}(\tau + T)u(\tau + T) - \dot{u}(\tau + T)v(\tau + T) = \dot{v}(\tau).$$

From these equations one obtains

$$\dot{u}(\tau + T) = \dot{v}(\tau)v(\tau + T), \quad \dot{v}(\tau + T) = \dot{v}(\tau)u(\tau + T).$$

After the change

$$s = t + \tau,$$

the monodromy matrix takes the form $Q^{-1}MQ$ with

$$M = \begin{pmatrix} u(\tau + T) & v(\tau + T) \\ v(\tau + T) & u(\tau + T) \end{pmatrix}, \qquad Q = \begin{pmatrix} \dot{v}(\tau)^{1/2} & 0 \\ 0 & \dot{v}(\tau)^{-1/2} \end{pmatrix}.$$

We notice that M is of the type $H_{\pm}[\Theta]$ with

$$u(\tau + T) = \pm\cosh\Theta, \qquad v(\tau + T) = \sinh\Theta.$$

The matrix Q is eliminated with a change of scale. □

4. Degree theory and index of zeros

Let us fix Ω, bounded and open subset of $\mathbb{R}^d$, $d \geq 1$. The degree is defined for continuous mappings from $\overline{\Omega}$ into $\mathbb{R}^d$ which do not vanish on the boundary. More precisely, given

$$F \in C(\overline{\Omega}, \mathbb{R}^d), \quad F(\xi) \neq 0 \ \forall \xi \in \partial\Omega, \tag{8}$$

we can assign to it an integer which will be denoted by $\deg(F, \Omega)$. Among many other properties of degree we mention:

- *Existence.* If $\deg(F, \Omega) \neq 0$ then $F(\xi) = 0$ has at least one solution in Ω.
- *Invariance by homotopy.* If

$$\mathcal{F} : \overline{\Omega} \times [0,1] \to \mathbb{R}^d, \quad \mathcal{F} = \mathcal{F}(\xi, \lambda),$$

  is continuous and $\mathcal{F}(\cdot, \lambda)$ satisfies (8) for each $\lambda \in [0,1]$, then

$$\deg(\mathcal{F}(\cdot, \lambda), \Omega) \quad \text{is independent of } \lambda.$$

- *Excision.* If K is a compact subset of $\overline{\Omega}$ and $F(\xi) \neq 0$ for $\xi \in K$, then

$$\deg(F, \Omega) = \deg(F, \Omega \setminus K).$$

In the properties of existence and excision it was assumed that F satisfied (8). There are many books about degree theory and we refer to [18] or [30] for more details.

Given an open set $\mathcal{U} \subset \mathbb{R}^d$ and $F \in C(\mathcal{U}, \mathbb{R}^d)$, let us assume that $\xi_\star \in \mathcal{U}$ is an isolated root of $F(\xi) = 0$. This means that

$$F(\xi_\star) = 0$$

and, for some $\delta > 0$,

$$F(\xi) \neq 0 \ \text{ if } 0 < |\xi - \xi_\star| \leq \delta.$$

We define the index of F at $\xi_\star$ by

$$\text{ind}[F, \xi_\star] = \deg(F, B_\delta(\xi_\star)),$$

where $B_\delta(\xi_\star)$ is the ball of radius δ centered at $\xi_\star$. The property of excision shows that this ball could be replaced by any small neighborhood of $\xi_\star$.

In dimension one ($d = 1$), the index can only take the values ± 1 and 0. Namely,

$$\text{ind}[F, \xi_\star] = 1 \quad \text{if } F(\xi_\star - \delta) < 0 < F(\xi_\star + \delta),$$

,

$$\text{ind}[F, \xi_\star] = -1 \quad \text{if } F(\xi_\star - \delta) > 0 > F(\xi_\star + \delta)$$

and

$$\text{ind}[F, \xi_\star] = 0 \quad \text{otherwise.}$$

In dimension two ($d = 2$) the index can take any integer value. The prototypes are (in complex notation) $z \mapsto z^n$ for positive index n and $z \mapsto \overline{z}^n$ for index $-n$. The simplest procedure to compute the index is by linearization. Given $F \in C^1(\mathcal{U}, \mathbb{R}^d)$ and $\xi_\star \in \mathcal{U}$ with $F(\xi_\star) = 0$, if the Jacobian matrix $F'(\xi_\star)$ is non-singular then

$$\text{ind}[F, \xi_\star] = \text{sign}\{\det F'(\xi_\star)\}.$$

The linearization technique is also useful for degenerate zeros ($\det F'(\xi_\star) = 0$) as long as the Jacobian matrix is not identically zero. In such a case the computation of the index is not direct, but at least one can reduce the dimension. This idea can be found in the book [11]. Next we describe the simplest situation.

Assume that $d = 2$ and $F = (F_1, F_2)$ is a C^1 mapping with

$$F_1(0,0) = F_2(0,0) = 0 \quad \text{and} \quad \frac{\partial F_1}{\partial \xi_2}(0,0) \neq 0.$$

We can apply the Implicit Function Theorem to

$$F_1(\xi_1, \xi_2) = 0$$

and solve in ξ_2, say

$$\xi_2 = \varphi(\xi_1).$$

Define the function

$$\Phi(\xi_1) = F_2(\xi_1, \varphi(\xi_1))$$

and assume that $\xi_1 = 0$ is an isolated zero of this function. Then

$$\xi_1 = \xi_2 = 0$$

is an isolated zero of F and

$$\text{ind}_{\mathbb{R}^2}[F, 0] = -\sigma \, \text{ind}_{\mathbb{R}}[\Phi, 0]$$

where

$$\sigma = \text{sign}\{\frac{\partial F_1}{\partial \xi_2}(0,0)\}.$$

To exercise with the properties of degree we sketch a proof. Assume that we are in the case

$$\frac{\partial F_1}{\partial \xi_2}(0,0) > 0$$

and, for $\lambda \in [0,1]$, consider the system of equations

$$\begin{cases} \lambda F_1(\xi_1, \xi_2) + (1-\lambda)(\xi_2 - \varphi(\xi_1)) = 0, \\ \lambda F_2(\xi_1, \xi_2) + (1-\lambda)\Phi(\xi_1) = 0. \end{cases}$$

This is well defined in a neighborhood of the origin and the only solutions of the first equation are $\xi_2 = \varphi(\xi_1)$. This is a consequence of the uniqueness of the implicit function since

$$\frac{\partial \mathcal{F}_1(0,0,\lambda)}{\partial \xi_2} = \lambda \frac{\partial F_1}{\partial \xi_2}(0,0) + (1-\lambda) > 0.$$

Once the first equation is solved, we substitute into the second and deduce that this last equation becomes equivalent to

$$\Phi(\xi_1) = 0.$$

In consequence the only solution of the system is $\xi_1 = \xi_2 = 0$ because $\xi_1 = 0$ was isolated for $\Phi = 0$. The invariance by homotopies leads us to the computation of the index for

$$(\xi_1, \xi_2) \mapsto (\xi_2 - \varphi(\xi_1), \Phi(\xi_1)).$$

When $\Phi'(0) \neq 0$ the computation of this index can be done by linearization. In the case $\Phi'(0) = 0$ the reader could try a proof or read more about the computation of indexes in [12].

We finish this section with an example on how to compute the index using the third approximation. Consider a planar and smooth mapping F with Taylor expansion

$$(\xi_1, \xi_2) \mapsto (k\xi_2 + \alpha\xi_1^3 + \beta\xi_1^2\xi_2 + \gamma\xi_1\xi_2^2 + \delta\xi_2^3 + \cdots, a\xi_1^3 + b\xi_1^2\xi_2 + c\xi_1\xi_2^2 + d\xi_2^3 + \cdots).$$

Then, if $k \neq 0$ and $a \neq 0$,

$$\mathrm{ind}[F, (0,0)] = -\mathrm{sign}(ka).$$

To prove this we first solve $F_1 = 0$ and find

$$\xi_2 = \varphi(\xi_1) = O(\xi_1^3).$$

Thus,

$$\Phi(\xi_1) = a\xi_1^3 + O(\xi_1^4).$$

5. The index of an equilibrium

Again we consider the differential equation

$$\ddot{\theta} = f(t, \theta), \tag{9}$$

in the same conditions as in Section 2. The equilibrium $\theta = 0$ is isolated (period T) if there exists $\delta > 0$ such that the equation (9) has no T-periodic solutions satisfying

$$0 < |\theta(t)| + |\dot{\theta}(t)| \leq \delta, \quad \forall t \in \mathbb{R}.$$

As an example consider the equation

$$\ddot{\theta} + \theta = 0,$$

then $\theta = 0$ is isolated (period T) if T is not a multiple of 2π. In general, if $\theta = 0$ is isolated (period T), the origin of $\mathbb{R}^2$ is an isolated root of the equation

$$(I - P)(\xi) = 0,$$

where I is the identity and P is the Poincaré map. We define the index of $\theta = 0$ as

$$\gamma_T(0) = \mathrm{ind}[I - P, 0].$$

The differential equation is periodic in time and we have fixed the period as T. The multiples nT, $n \geq 2$, are also admissible as periods of (9) and so one can consider iterated indexes

$$\gamma_{nT}(0) = \mathrm{ind}[I - P^n, 0]$$

when $\theta = 0$ is isolated (period nT).

To understand this definition one must recall that the Poincaré map for period nT is just the iteration

$$P^n = P \circ^{(n} \cdots \circ P.$$

Also, we notice that if $\gamma_{nT}(0)$ is well defined then the same happens for $\gamma_{kT}(0)$ if k is a divisor of n. In this section we shall concentrate on the first index $\gamma_T(0)$ and describe some methods to compute it. To this end we shall assume that the force $f(t,\theta)$ is smooth in θ. This will be understood in the following sense: the partial derivatives $\frac{\partial^n f}{\partial \theta^n}(t,\theta)$ exist everywhere in $\mathbb{R} \times (-\epsilon, \epsilon)$ and the functions

$$(t,\theta) \mapsto \frac{\partial^n f}{\partial \theta^n}(t,\theta)$$

are continuous for each $n \geq 1$. The first approximation to (9) is

$$\ddot{\theta} + a(t)\theta = 0, \qquad a(t) = \frac{\partial f}{\partial \theta}(t,0). \tag{10}$$

The general theory of differential equations says that P is smooth and the matrix $P'(0)$ is precisely the monodromy matrix $X(T)$ which was defined in Section 3. If $X(T)$ is elliptic, hyperbolic or parabolic with

$$\mu_1 = \mu_2 = -1,$$

the index can be computed by linearization. Namely,

$$\gamma_T(0) = \operatorname{sign}\{\det(I - X(T))\} = \operatorname{sign}\{(1-\mu_1)(1-\mu_2)\},$$

where μ_1, μ_2 are the eigenvalues of $X(T)$, often called the Floquet multipliers. The computation of the index in the degenerate case

$$\mu_1 = \mu_2 = 1$$

is more delicate and requires information on nonlinear approximations. Assume now that (9) can be expanded as

$$\ddot{\theta} + a(t)\theta + c(t)\theta^p + \cdots = 0,$$

where $c(t)$ is not identically zero. From now on, the dots in an expansion will refer to terms with an order higher than the order of those explicitly mentioned. The nonlinear expansion of P up to the order p is

$$\begin{pmatrix} \theta_0 \\ \omega_0 \end{pmatrix} \mapsto X(T) \begin{pmatrix} \theta_0 + \partial_{\omega_0} H(\theta_0,\omega_0) + \cdots \\ \omega_0 - \partial_{\theta_0} H(\theta_0,\omega_0) + \cdots \end{pmatrix}$$

where

$$H(\theta_0,\omega_0) = \frac{1}{p+1}\int_0^T c(t)(\phi_1(t)\theta_0 + \phi_2(t)\omega_0)^{p+1} dt$$

and $\phi_1(t)$, $\phi_2(t)$ are the solutions of the linear equation (10) satisfying

$$\phi_1(0) = \dot{\phi}_2(0) = 1, \qquad \dot{\phi}_1(0) = \phi_2(0) = 0.$$

The way to obtain this formula is rather standard but we outline a proof. First we observe that the solution of

$$\ddot{\delta} + a(t)\delta = b(t), \qquad \delta(0) = \dot{\delta}(0) = 0,$$

is

$$\delta(t) = \int_0^t [\phi_1(s)\phi_2(t) - \phi_1(t)\phi_2(s)]b(s)ds.$$

Since the solution of (9) satisfies

$$\theta(t;\theta_0,\omega_0) = \phi_1(t)\theta_0 + \phi_2(t)\omega_0 + \cdots$$

we can apply the previous formula with

$$\delta(t) = \theta(t;\theta_0,\omega_0) - \phi_1(t)\theta_0 - \phi_2(t)\omega_0$$

and

$$b(t) = -c(t)\theta(t;\theta_0,\omega_0)^p + \cdots$$

to obtain

$$\theta(t;\theta_0,\omega_0) = \phi_1(t)\theta_0 + \phi_2(t)\omega_0 - \int_0^t [\phi_1(s)\phi_2(t) - \phi_1(t)\phi_2(s)]c(s)(\phi_1(s)\theta_0 + \phi_2(s)\omega_0)^p ds + \cdots$$

A similar formula is valid for the derivative and the conclusion follows for $t = T$.

Once we have a nonlinear expansion of P, we can employ the methods of the previous section to compute the index. Let us assume first that $X(T)$ is one of the matrices P_+, P_-. The function

$$F = I - P, \quad F = F(\theta_0,\omega_0),$$

satisfies

$$\frac{\partial F_1}{\partial \omega_0}(0,0) = \mp 1.$$

By solving $F_1 = 0$ one obtains

$$\omega_0 = \varphi(\theta_0) = O(\theta_0^p).$$

Hence,

$$\Phi(\theta_0) = F_2(\theta_0,\varphi(\theta_0)) = \partial_{\theta_0}H(\theta_0,\varphi(\theta_0)) + \cdots = (\int_0^T c(t)\phi_1(t)^{p+1}dt)\theta_0^p + \cdots$$

Assume now that the integral appearing in the previous formula is not zero, then $\theta_0 = 0$ is isolated for Φ and so $\theta = 0$ is isolated (period T). Moreover,

$$\gamma_T(0) = \begin{cases} 0 & \text{if } p \text{ is even} \\ \nu \,\mathrm{sign}\{\int_0^T c(t)\phi_1(t)^{p+1}dt\} & \text{if } p \text{ is odd.} \end{cases}$$

Here

$$\nu = \begin{cases} 1 & \text{if } X(T) = P_+, \\ -1 & \text{if } X(T) = P_-. \end{cases}$$

The computation of the index when

$$X(T) = I$$

is based on different ideas. Assume that the origin is the only critical point of H, we shall prove that $\theta = 0$ is isolated (period T) and

$$\gamma_T(0) = \mathrm{ind}[\nabla H, 0].$$

To prove this statement we notice that, since $X(T)$ is the identity, the map

$$F = I - P$$

has the form

$$F = J\nabla H + R,$$

where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and $R = R(\theta_0, \omega_0)$ is a remainder of order higher than p. Consider the system of equations

$$J\nabla H(\theta_0, \omega_0) + \lambda R(\theta_0, \omega_0) = 0, \qquad \lambda \in [0,1].$$

We are going to prove that, in a small neighborhood of the origin, there are no solutions different from

$$\theta_0 = \omega_0 = 0.$$

This is done by a contradiction argument. Assume that there is a sequence of solutions

$$\xi_n = (\theta_{0n}, \omega_{0n}), \qquad \lambda_n \in [0,1],$$

with

$$\xi_n \neq 0 \quad \text{and} \quad |\xi_n| \to 0.$$

Define

$$\eta_n = \frac{\xi_n}{|\xi_n|}$$

and extract a convergent subsequence, say

$$\eta_n \to \eta_\star, \qquad \lambda_n \to \lambda_\star.$$

Dividing the equations by $|\xi_n|^p$ and passing to the limit we conclude that $\eta_\star$ is a critical point of H. This is impossible because $|\eta_\star| = 1$ and 0 was the only critical point. The invariance by homotopies implies that

$$\gamma_T(0) = \mathrm{ind}[J\nabla H, 0].$$

It remains to prove that

$$\mathrm{ind}[J\nabla H, 0] = \mathrm{ind}[\nabla H, 0].$$

To do this we notice that J has positive determinant and so J and I are in the same component of $Gl(\mathbb{R}^2)$. Let J_λ be a continuous arc in $Gl(\mathbb{R}^2)$ joining I and J. The equation

$$J_\lambda \nabla H = 0$$

is equivalent to $\nabla H = 0$ and this homotopy proves the identity between the two indexes.

We sum up the discussions of this section. If the Floquet multipliers are different from 1, the index $\gamma_T(0)$ can be computed from the variational equation (10). In the critical case $\mu_1 = \mu_2 = 1$ one can use the first nonlinear approximation to compute the index in many cases.

6. Stability and index

The connections between stability and index are delicate and some interesting questions remain unanswered. For an autonomous system

$$\dot{x} = X(x), \qquad x \in \mathbb{R}^d$$

having $x = 0$ as an isolated and stable equilibrium, we would like to know the index of the vector field X. For dimension $d = 2$ it is known that

$$\text{ind}[X, 0] = 1.$$

For dimension $d \geq 3$ the index can be any integer (at least for C^∞ vector fields). We refer to the interesting book by Krasnosel'skii and Zabreiko [12] for more information. See also the papers by Erle [9] and Bonati and Villadelprat [4]. In the recent paper [4] there is a construction which contradicts some of the assertions in [12].

Given a T-periodic equation

$$\dot{x} = X(t, x), \qquad x \in \mathbb{R}^d$$

having $x = 0$ as an isolated (period T) and stable equilibrium, the interest is in the index $\gamma_T(0)$. For dimensions $d \geq 3$, the construction for the autonomous case can be adapted and so the index can take any value (see [27] for more details). In [10] Krasnosel'skii stated that the index is always 1 in dimension $d = 2$. He also said that this fact could be proved easily, but he did not present the proof. Dancer and I found a proof which probably cannot be called elementary. It is based on the arc translation lemma due to Brouwer. In particular we adapted the proof of this lemma given by M. Brown in [7]. Our equation (9) can be transformed, in the usual way, into a periodic system in $\mathbb{R}^2$. Hence, as a consequence of the results in [8] we have,

Theorem 2. *Assume that $\theta = 0$ is an equilibrium of* (9) *which is isolated (period T) and stable. Then $\gamma_T(0) = 1$.*

Equations of period T are also of period nT, $n \geq 2$. We are lead to the following consequence,

Corollary 3. *Assume that $\theta = 0$ is an equilibrium of* (9) *which is isolated (period nT, $n \geq 2$) and stable. Then $\gamma_{nT}(0) = 1$.*

This result is of practical value. It allows to obtain instability criteria via degree theory. In fact, if one of the indexes is different from 1, we can say that $\theta = 0$ is unstable. As an example consider the equation

$$\ddot{\theta} + \theta + c(t)\theta^2 = 0 \tag{11}$$

and assume that $c(t)$ has period

$$T = \frac{2\pi}{3}.$$

The linearization ($\ddot{\theta} + \theta = 0$) has monodromy matrices (for periods T, $2T$ and $3T$),

$$X(T) = R[\frac{2\pi}{3}], \qquad X(2T) = R[\frac{4\pi}{3}], \qquad X(3T) = R[2\pi] = I.$$

This implies that, for periods T and $2T$, it is possible to compute the index by linearization. Namely, $\theta = 0$ is isolated (period $2T$) and

$$\gamma_T(0) = \text{sign}\{\det(I - X(T))\} = 1,$$
$$\gamma_{2T}(0) = \text{sign}\{\det(I - X(2T))\} = 1.$$

To compute the third index we must employ the discussions about the degenerate case in Section 5. Since

$$X(3T) = I$$

and

$$\phi_1(t) = \cos t, \qquad \phi_2(t) = \sin t,$$

for period $3T$ the function H becomes

$$H(\theta_0, \omega_0) = \frac{1}{3}\int_0^{2\pi} c(t)(\theta_0 \cos t + \omega_0 \sin t)^3 dt.$$

In complex notation,

$$\xi = \theta_0 + i\omega_0, \qquad \overline{\xi} = \theta_0 - i\omega_0,$$

equals

$$H(\xi, \overline{\xi}) = \frac{1}{24}\int_0^{2\pi} c(t)(\xi e^{-it} + \overline{\xi} e^{it})^3 dt.$$

The function $c(t)$ has period $\frac{2\pi}{3}$ and this implies that

$$\int_0^{2\pi} c(t)e^{it} dt = 0.$$

Some computations lead to

$$H(\xi, \overline{\xi}) = \frac{1}{8}\left(\gamma\overline{\xi}^3 + \overline{\gamma}\xi^3\right)$$

where

$$\gamma = \int_0^{\frac{2\pi}{3}} c(t)e^{3it} dt. \tag{12}$$

The derivative

$$H_{\overline{\xi}} = \frac{1}{2}(H_{\theta_0} + iH_{\omega_0})$$

is $\frac{3}{8}\gamma\overline{\xi}^2$ and so, if $\gamma \neq 0$, the only critical point of H is the origin. It follows that $\theta = 0$ is isolated (period $3T$) and

$$\gamma_{3T}(0) = \text{ind}[\nabla H, 0] = -2.$$

The conclusion is that $\theta = 0$ is unstable as soon as the quantity defined by (12) does not vanish.

This example shows that the linearization procedure is not valid for a general equation of the type (9). In this example $\theta = 0$ was stable for the linearization and unstable for the original equation. The reader who is familiar with hamiltonian dynamics will have recognized the phenomenon of resonance at the roots of the unity. In this case it was the third root

$$\omega = e^{\frac{2\pi i}{3}}$$

and we refer to [32,19] for more details.

7. The pendulum of variable length

Consider again the equation

$$\ddot{\theta} + a(t)\sin\theta = 0 \tag{13}$$

where $a(t)$ is continuous, T-periodic and positive. We shall compute the second index $\gamma_{2T}(0)$. Let us start with the linearization principle. If μ_1 and μ_2 are the Floquet multipliers of the linearized equation (period T), the eigenvalues of

$$X(2T) = X(T)^2$$

are μ_1^2 and μ_2^2. In the elliptic case,

$$\mu_1 = \overline{\mu_2}, \qquad \mu_1 \neq \pm 1,$$

and

$$\gamma_{2T}(0) = \text{sign}\{(1-\mu_1^2)(1-\mu_2^2)\} = \text{sign}|1-\mu_1^2|^2 = 1.$$

In the hyperbolic case,

$$|\mu_1| < 1 < |\mu_2|$$

and

$$\gamma_{2T}(0) = \text{sign}\{(1-\mu_1^2)(1-\mu_2^2)\} = -1.$$

In the parabolic case

$$\mu_1 = \mu_2 = \pm 1,$$

we notice that $X(2T)$ must be conjugate in $Sp(\mathbb{R}^2)$ to one of the matrices I, P_+, P_-. Going back to the methods of computation in the degenerate case and considering the third order approximation

$$\ddot{\theta} + a(t)\theta - \frac{1}{3!}a(t)\theta^3 = 0 \tag{14}$$

we obtain an expansion of the Poincaré map like in Section 5, with

$$H(\theta_0, \omega_0) = -\frac{1}{24}\int_0^{2T} a(t)(\phi_1(t)\theta_0 + \phi_2(t)\omega_0)^4 dt.$$

Since H has a strict maximum at the origin $\xi = 0$, we can use Euler's theorem for homogeneous functions to deduce that

$$\xi \cdot \nabla H(\xi) = 4H(\xi) < 0 \qquad \forall \xi \neq 0.$$

This inequality implies that $\xi = 0$ is the only critical point of H and so we can discuss the case

$$X(2T) = I.$$

More precisely, $\theta = 0$ is isolated (period $2T$) with

$$\gamma_{2T}(0) = \text{ind}[\nabla H, 0] = 1.$$

Here we have used a typical property of the index for gradient operators (see [12] or [1]).

Let us now assume that $X(2T)$ is conjugate to P_+ or P_-. We apply the proposition in Section 3 and find a change of independent variable

$$t = \lambda(s + \tau)$$

such that the equation (13) becomes

$$\frac{d^2\theta}{ds^2} + a^\star(s)\theta - \frac{a^\star(s)}{3!}\theta^3 + \cdots = 0, \qquad a^\star(s) = \lambda^2 a(\lambda(s+\tau))$$

and the monodromy matrix $X^\star(2T^\star)$ of the linearization is precisely P_+ or P_-. This transformation in time does not alter the index of $\theta = 0$. For if $P^\star$ is the Poincaré map of the new equation, then

$$L^{-1}P^\star L = P$$

with

$$L(\theta_0, \omega_0) = (\theta_0, \lambda\omega_0).$$

The commutativity theorem for degree shows that the indexes of $I - P$ and $I - P^\star$ coincide. Incidentally we notice that the stability of $\theta = 0$ is also preserved. We apply once again the discussions of Section 5 and conclude that

$$\gamma_{2T}(0) = \nu \text{sign}\{-\frac{1}{3!}\int_0^{T^\star} a^\star(s)\phi_1^\star(s)^4 ds\} = -\nu$$

with $\nu = 1$ if $X(2T) \sim P_+$ and $\nu = -1$ if $X(2T) \sim P_-$.

At this point the reader may think that the computation of other indexes $\gamma_{kT}(0)$ could lead to more instability criteria. However this is not the case, as can be seen after computing all indexes. The next step is to discuss the stability of $\theta = 0$. This can be done but the techniques which are required go beyond the scope of these notes. The details can be seen in [26,23], the second paper in collaboration with Núñez. Summing up the previous discussions and the results in these papers one obtains

Theorem 4. *The following statements are equivalent:*

(i) $\theta = 0$ *is stable for* (13)

(ii) $\theta = 0$ *is isolated (period* $2T$*) and* $\gamma_{2T}(0) = 1$

(iii) *$\theta = 0$ is stable for the Duffing equation* (14)

(iv) *the monodromy matrix $X(2T)$ is conjugate in $Sp(\mathbb{R}^2)$ to $R[\Theta]$, for some $\Theta \in \mathbb{R}$, or to P_-.*

We notice that the assertion (iv) is the answer to the question posed in the introduction of these notes. The linearization procedure is valid for the pendulum of variable length excepting when

$$X(2T) \sim P_-.$$

In this situation $\theta = 0$ is stable for the original equation (13) but unstable for the linearization. The analysis leading to Theorem 4 is not exclusive of the pendulum and can be applied to other equations. The crucial property is that the coefficient of the cubic term has a sign. Other results about stability using the third approximation can be found in [28,16,17,22,24,29,34,13,14].

A natural question about Theorem 4 is its possible extension to more degrees of freedom. To fix the ideas consider the system

$$\ddot{\theta} + A\theta + \alpha(t)S(\theta) = 0, \qquad \theta = \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_N \end{pmatrix}$$

where A is the $N \times N$ tridiagonal matrix, coming from the discretization of the Laplacian,

$$A = \epsilon \begin{pmatrix} -2 & 1 & 0 & \dots & 0 & 0 \\ 1 & -2 & 1 & \dots & 0 & 0 \\ 0 & 1 & -2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -2 & 1 \\ 0 & 0 & 0 & \dots & 1 & -2 \end{pmatrix}, \qquad \epsilon > 0,$$

and

$$S(\theta) = \begin{pmatrix} \sin\theta_1 \\ \vdots \\ \sin\theta_N \end{pmatrix}.$$

It is not clear if the approach in these notes can be extended. In principle one cannot expect a result like Theorem 2 because we are in more dimensions and the index of a stable equilibrium can be any number. However, our system is analytic and Hamiltonian and we are in a rather special situation. Is there a version of Theorem 2 applicable to this example? In any case it is probable that one can obtain instability criteria for the third approximation using Lyapunov functions. The stability is more delicate. Probably the notion of perpetual stability is too demanding as to obtain reasonable results. There is the notion of formal stability, associated to the Birkhoff normal form [3], which seems easier to study. This formal stability implies (via KAM theory) the notion of stability introduced by Moser in his conference in ICM Berlin 99 [21]. We finish these notes by recalling Moser's definition of *stability in measure*: instead of requiring that all orbits of a certain neighborhood are bounded for all times, one asks that most orbits (in the sense of measure) are bounded.

References

1. Amann, H., A note on degree theory for gradient mappings, *Proc. Amer. Math. Soc.* **85** (1982), 591–595.
2. Arnold, V.I., and Avez, A., *Ergodic Problems of Classical Mechanics*, Addison Wesley, 1989.
3. Birkhoff, G.D., *Dynamical Systems*, Amer. Math. Soc., 1927.
4. Bonati, C., and Villadelprat, J., The index of stable critical points, *Topology Appl.* **126** (2002), 263–271.
5. Broer, H.W., and Levi, M., Geometrical aspects of stability theory for Hill's equation, *Arch. Rat. Mech. Anal.* **131** (1995), 225–240.
6. Broer, H.W., and Vegter, G., Bifurcational aspects of parametric resonance, *Dynamics Reported (N.S.)* **1** (1992), 1–53.
7. Brown, M., A new proof of Brouwer's lemma on translation arcs, *Houston Math. J.* **10** (1984), 35–41.
8. Dancer, E.N., and Ortega, R., The index of Lyapunov stable fixed points in two dimensions, *J. Dynam. Diff. Eqns.* **6** (1994), 631–637.
9. Erle, D., Stable equilibria and vector field index, *Topol. Appl.* **49** (1993), 231–235.
10. Krasnosel'skii, M.A., *Translations Along Trajectories of Differential Equations*, Trans. Math. Monographs 19, Amer. Math. Soc., 1968.
11. Krasnosel'skii, M.A., Perov, A.I., Povolotskiy, A.I., and Zabreiko, P.P., *Plane Vector Fields*, Academic Press, 1966.
12. Krasnosel'skii, M.A., and Zabreiko, P.P., *Geometrical Methods of Nonlinear Analysis*, Springer-Verlag, 1984.
13. Lei, J., and Zhang, M., Twist property of periodic motion of an atom near a charged wire, *Letters in Math. Phys.* **60** (2002), 9–17.
14. Lei, J., Li, X., Yan, P., and Zhang, M., Twist character of the least amplitude periodic solution of the forced pendulum, to appear.
15. Levi-Civita, T., Sopra alcuni criteri di instabilita, *Annali di Matematica* **V** (1901), 221–307.
16. Liu, B., The stability of the equilibrium of a conservative system, *J. Math. Anal. App.* **202** (1996), 133–149.
17. Liu, B., The stability of the equilibrium of reversible systems, *Trans. Am. Math. Soc.* **351** (1999), 515–531.
18. Lloyd, N.G., *Degree Theory*, Cambridge Univ. Press, 1978.
19. Meyer, K., Counter-examples in dynamical systems via normal form theory, *SIAM Rev.* **28** (1986), 41–51.

20. Magnus, W., and Winkler, S., *Hill's Equation*, Dover, 1979.
21. Moser, J., Dynamical systems-past and present, *Doc. Math. J., DMV Extra Volume* **ICM I** (1998), 381–402.
22. Núñez, D., The method of lower and upper solutions and the stability of periodic oscillations, *Nonl. Anal.* **51** (2002), 1207–1222.
23. Núñez, D., and Ortega, R., Parabolic fixed points and stability criteria for nonlinear Hill's equation, *Z. Angew. Math. Phys.* **51** (2000), 890–911.
24. Núñez, D., and Torres, P., Periodic solutions of twist type of an earth satelite equation, *Discr. Con. Dyn. Syst.* **7** (2001), 303–306.
25. Ortega, R., The twist coefficient of periodic solutions of a time-dependent Newton's equation, *J. Dynam. Diff. Eqns.* **4** (1992), 651–665.
26. Ortega, R., The stability of the equilibrium of a nonlinear Hill's equation, *SIAM J. Math. Anal.* **25** (1994), 1393–1401.
27. Ortega, R., Some applications of the topological degree to stability theory, in *Topological Methods in Differential Equations and Inclusions* (Granas, A., and Frigon, M., Eds.), pages 377–409, Kluwer Academic, 1995.
28. Ortega, R., Periodic solutions of a newtonian equation: stability by the third approximation, *J. Diff. Eqns.* **128** (1996), 491–518.
29. Torres, P., Twist solutions of a Hill's equation with singular term, *Adv. Nonl. Studies* **2** (2002), 279–287.
30. Rothe, E.H., *Introduction to Various Aspects of Degree Theory in Banach Spaces*, Math. Surveys 23, Amer. Math. Soc., 1986.
31. Sell, G., *Topological Dynamics and Ordinary Differential Equations*, Van Nostrand-Reinhold, 1971.
32. Siegel, C., and Moser, J., *Celestial Mechanics*, Springer-Verlag, 1971.
33. Yan, P., and Zhang, M., Higher order nonresonance for differential equations with singularities, *Math. Meth. in App. Sci.*, to appear.
34. Zhang, M., The best bound on the rotations in the stability of periodic solutions of a newtonian equation, to appear.