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Dynamics of a forced oscillator with obstacle

Rafael Ortega
Departamento de Matemática Aplicada
Facultad de Ciencias, Universidad de Granada
18071 Granada, Spain
E-mail address: rortega@ugr.es

1 Introduction

Consider the scalar differential equation

$$\ddot{x} + g(x) = f(t) \tag{1}$$

where f is 2π-periodic, say $f \in C(\mathbb{T})$ with $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, and g satisfies

$$\lim_{x\to\pm\infty} g(x) = \pm\infty, \quad \limsup_{|x|\to\infty} \frac{g(x)}{x} < \infty. \tag{2}$$

The existence of 2π-periodic solutions has been analyzed by many authors using different variational and topological methods. For the linear case ($g(x) = \omega^2 x$) it is well known that the existence of a periodic solution is equivalent to the boundedness of all solutions and one can ask whether such an equivalence still holds in nonlinear cases. In this paper I shall report on several results which give partial answers to this question. First we shall assume that g satisfies the assumptions of Lazer and Leach in [12] and we shall show that the condition for existence of a periodic solution obtained in that paper guarantees, in many cases, the boundedness of all solutions. For this class of nonlinearities the situation resembles the linear theory. Later we shall consider the asymmetric nonlinearities that were first discussed by Fucik [10] and Dancer [5, 4]. The situation now is more delicate because unbounded and periodic solutions can coexist. After this brief review of published results we shall analyze in detail the problem of boundedness for a forced linear oscillator which bounces elastically against a wall. This problem has not been considered previously and it will be employed to illustrate the techniques developed in [20] and [21]. We notice that the periodic problem for this bouncing oscillator was already studied by Lazer and McKenna in [14]. They interpreted the model as a limiting case of the asymmetric oscillator.

Moser's Theorem on existence of invariant curves will be crucial in the proofs. The use of this Theorem in the study of boundedness for (1) is classical and one can refer to [19, 6, 15]. In all those papers the function g was superlinear at infinity and did not satisfy (2). In many cases we shall be able to obtain additional information on the dynamics around infinity. For instance, when there is boundedness, the existence of large subharmonic and quasi-periodic solutions follows as a consequence of the method of proof and the theory of twist mappings.

2 Remarks on the Lazer-Leach's condition

Let us now assume that the function g in (1) is of the type

$$g(x) = n^2 x + h(x)$$

where $n = 1, 2, \dots$ and h is a continuous and bounded function having limits at infinity, $h(+\infty)$ and $h(-\infty)$. The main result in [12] implies that (1) has a 2π-periodic solution if the condition below hold

$$|\hat{f}_n| < \frac{1}{\pi}|h(+\infty) - h(-\infty)|, \tag{3}$$

where $\hat{f}_n = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{int}dt$.

We shall refer to (3) as to the Lazer-Leach condition. When h is not constant and satisfies

$$h(-\infty) \le h(x) \le h(+\infty) \quad \forall x \in \mathbb{R} \tag{4}$$

this condition becomes necessary and sufficient for the solvability of the periodic problem. This is a remarkable consequence of [12].

Next we shall show that (3) also plays a role in the problem of boundedness. Let us first assume that h is the piecewise linear function

$$h_L(x) = \begin{cases} -L & \text{if } x \le -1 \\ Lx & \text{if } |x| \le 1 \\ L & \text{if } x \ge 1 \end{cases}$$

for some $L > 0$. This function satisfies (4) and the Lazer-Leach condition becomes

$$|\hat{f}_n| < \frac{2L}{\pi}. \tag{5}$$

It was proved in [21] that if $f \in C^5(\mathbb{R})$ and (5) holds then all solutions of (1) with $g(x) = n^2 x + h_L(x)$ are bounded. In this case (5) is sharp for the boundedness problem because all the solutions are unbounded when it does not hold. This follows from [23] and [1]. More recently Liu has obtained similar results for a class of functions g including the model nonlinearity $g(x) = \arctan x$ (see [17] and also [11]).

3 The asymmetric oscillator

Let us now consider another piecewise linear function. Namely, $g(x) = ax^+ - bx^-$ where $a, b > 0$ with $a \neq b$. The corresponding equation

$$\ddot{x} + ax^+ - bx^- = f(t) \tag{6}$$

can be thought as a model of the motion of a particle subjected to an asymmetric restoring force. The periodic problem for (6) was analyzed by Fucik [10] and Dancer [5, 4] in the seventies. In this context they realized the importance of the following set, lying in the plane of parameters (a, b),

$$\Sigma = \bigcup_{p=1}^{\infty} C_p, \quad C_p = \{(a,b) \in \mathbb{R}^2_+ : \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} = \frac{2}{p}\}.$$

It can be proved that if $(a, b) \notin \Sigma$ then (6) has a 2π-periodic solution for every $f \in C(\mathbb{T})$. On the contrary, when $(a, b) \in \Sigma$ the solvability of the periodic problem depends upon f (see [4, 13, 7]). The set Σ can be thought as a sort of periodic spectrum and sometimes it is called the Fucik spectrum. In contrast to the Lazer-Leach's situation, now there is no direct connection between the periodic problem and the boundedness of all solutions. In a joint paper with Alonso [2] we noticed that, given any (a, b) with

$$\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} \in \mathbb{Q},$$

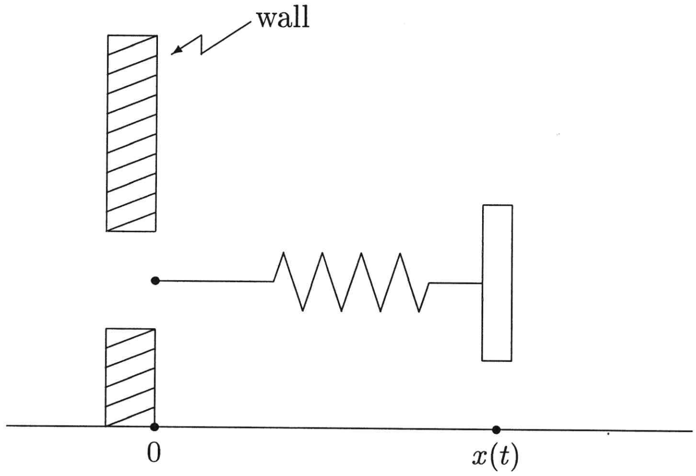
it is possible to construct many functions f for which (6) has unbounded solutions. Selecting the couple (a, b) so that it is not in Σ one finds examples of coexistence of unbounded and periodic solutions. Sufficient conditions on f for the boundedness of all solutions have been obtained in [18] and also in [22].

The function $g(x) = ax^+ - bx^-$ is possibly the simplest example of a function satisfying $g'(+\infty) \neq g'(-\infty)$. Results for more general jumping nonlinearities can be seen in [8, 3, 25].

In the next sections we shall consider the following limit case of (6),

$$\ddot{x} + ax^+ - \infty x^- = f(t). \tag{7}$$

Of course this is not a well defined differential equation but it will be interpreted in the sense proposed by Lazer and McKenna in [14]. The equation (7) can be thought as the model of the motion of a particle which is attached to a spring ($-ax$) that pushes the particle against a barrier situated at $x = 0$. At this barrier the particle bounces elastically. See the figure below.



Some discussions on the periodic problem for (7) can be found in [14]. We also mention [9] for some connections between jumping nonlinearities and bouncing problems.

4 A linear equation with obstacle

Given $a > 0$ and $f \in C(\mathbb{T})$ we consider the equation with obstacle

$$\begin{cases} \ddot{x} + ax = f(t), \\ x(t) \geq 0, \\ x(t_0) = 0 \;\Rightarrow\; \dot{x}(t_0+) = -\dot{x}(t_0-). \end{cases} \tag{8}$$

By a solution of (8) we understand a continuous function $x : I \to [0, \infty)$, defined on some closed interval $I \subset \mathbb{R}$, such that the conditions below hold,

(i) the set of zeros $Z = \{t \in I : x(t) = 0\}$ is discrete (*ii*) for any interval $J = [t_1, t_2]$ with $Z \cap (t_1, t_2) = \emptyset$, the function $x(t)$ belongs to $C^2(J)$ and satisfies

$$\ddot{x}(t) + ax(t) = f(t), \quad t \in J$$

(*iii*) given $t_0 \in Z \cap \text{int}\,(I)$, $\dot{x}(t_0+) = -\dot{x}(t_0-)$.
(Here $\dot{x}(t_0\pm)$ denote the right and left derivatives of x at t_0. The condition (*ii*) guarantees that they are well defined).

In the previous definition the set Z can be empty. Then $x(t)$ is just a positive solution of the linear equation. When Z is non-empty we shall say that $x(t)$ is a bouncing solution. As an example consider the function $x(t) = 1 + c \sin \sqrt{a} t$. It is a solution of (8) with $f \equiv a$ for any c with $|c| \leq 1$ but it is a bouncing solution if and only if $|c| = 1$.

Given $\tau \in \mathbb{R}$ and $(x_0, v_0) \in \mathbb{R}^2$ with $x_0 > 0$ or $x_0 = 0$ and $v_0 > 0$, we can always find a unique solution of (8) satisfying $x(\tau) = x_0$, $\dot{x}(\tau) = v_0$. Sometimes this solution cannot be defined in the whole line. For instance, assume that we can find a solution $x(t)$ of the linear equation $\ddot{x} + ax = f(t)$ satisfying (for some $t^* \in \mathbb{R}$): a) $x(t^*) = \dot{x}(t^*) = 0$, b) $x(t) > 0$ if $0 < t^* - t < \epsilon$, c) there exists a sequence $t_n \downarrow t^*$ such that $x(t_n) \leq 0$. Then we can construct a solution of (8) which coincides with $x(t)$ on the interval $(t^* - \epsilon, t^*]$. It is clear that this solution cannot be continued to the right of t^*. We also notice that all the solutions of (8) are well defined over $(-\infty, +\infty)$ if $f(t) \neq 0$ for every $t \in \mathbb{R}$.

The homogeneous equation ($f \equiv 0$) can be easily analyzed. Actually the solutions are

$$x(t) = A|\sin(\sqrt{a}t + \phi)|, \quad A > 0,\ \phi \in \mathbb{T}.$$

All of them are periodic with period

$$T = \frac{\pi}{\sqrt{a}}.$$

We shall distinguish the solution with initial conditions $x(0) = 0$, $\dot{x}(0) = 1$ and denote it by

$$\varphi_a(t) = \frac{1}{\sqrt{a}}|\sin(\sqrt{a}t)|.$$

The analysis of the non-homogeneous case ($f \not\equiv 0$) is more delicate and we shall distinguish two cases depending on whether the period T is commensurable with 2π or not. In the first case we can find positive integers p, q such that

$$\sqrt{a} = \frac{q}{2p}, \quad p \text{ and } q \text{ are relatively prime.} \tag{9}$$

Assuming that this condition holds we define the function

$$\Phi(\tau) = \frac{1}{q}\int_0^{2\pi p} f(s+\tau)\varphi_a(s)ds.$$

This function is 2π-periodic and will play an important role in what follows. Sometimes it is more convenient to employ another expression of Φ, namely

$$\Phi(\tau) = \frac{1}{q}\sum_{h=0}^{q-1}\mu(\tau + hT)$$

where

$$\mu(\tau) = \int_0^T f(s+\tau)\varphi_a(s)ds = \frac{1}{\sqrt{a}}\int_\tau^{\tau+T} f(s)\sin(\sqrt{a}(s-\tau))ds. \qquad (10)$$

These last expressions reveal that Φ is of class C^2.

Example 1. Computation of Φ.
Assume $a = 1$. Then (9) holds with $p = 1$, $q = 2$ and

$$\Phi(\tau) = \frac{1}{2}\int_0^{2\pi} f(s+\tau)|\sin s|ds.$$

Assuming that $f(t) = \alpha + \beta\sin t + \gamma\sin 2t$, with $\alpha, \beta, \gamma \in \mathbb{R}$, a computation shows that

$$\Phi(\tau) = 2\alpha - \frac{2}{3}\gamma\sin 2\tau.$$

We notice that the function Φ changes sign if and only if $3|\alpha| < |\gamma|$. Next result will imply that in such a case there are unbounded solutions of (8).

Theorem 1 *Assume that $\sqrt{a} \in \mathbb{Q}$ and it satisfies (9). In addition Φ changes sign and all zeros of Φ are nondegenerate; that is,*

$$\Phi(\tau)^2 + \Phi'(\tau)^2 > 0 \quad \forall\tau \in \mathbb{R}.$$

Then there exists $R > 0$ such that any solution of (8) with

$$x(\tau) + |\dot{x}(\tau)| > R \quad (\text{for some } \tau \in \mathbb{R})$$

is unbounded.

Next we present a complementary result about the boundedness of solutions.

Theorem 2 *Assume that*

$$f \in C^4(\mathbb{T})$$

and one the conditions below holds,

(i) $\sqrt{a} \in \mathbb{Q}$ *and it satisfies (9) with* $\Phi(\tau) \neq 0 \; \forall \tau \in \mathbb{R}$

(ii) $\sqrt{a} \notin \mathbb{Q}$ *and* $\int_0^{2\pi} f(t)dt \neq 0$.

Then there exists $R > 0$ *such that every solution of (8) satisfying*

$$x(\tau) + |\dot{x}(\tau)| > R \quad \text{(for some } \tau \in \mathbb{R})$$

is well defined in $(-\infty, +\infty)$ *and bounded.*

Remarks. 1. The proof of these theorems will give more insight on the dynamics of the equation as well as precise information about the oscillatory properties of solutions.
2. Going back to the example before Theorem 1 one notices that Theorem 2 applies when $3|\alpha| > |\gamma|$. The case $3|\alpha| = |\gamma|$ and β arbitrary, is left open by the previous theorems.
3. In the second theorem we need some extra regularity for the forcing, $f \in C^4$. There are known examples where the regularity plays a role in the problem of boundedness (see [16, 26]) and so this condition seems reasonable. I will present an example in this direction, but first it is convenient to state explicitly an intuitive consequence of Theorem 2.

Corollary 3 *Assume that*

$$f \in C^4(\mathbb{T})$$

and

$$f(t) \geq 0 \;\; \forall t \in \mathbb{R}. \tag{11}$$

Then, for arbitrary $a > 0$, *all solutions of (8) are bounded.*

Example 2. A spring with impulses.

Let $\delta(t)$ denote the 2π-periodic extension of the Dirac mass concentrated at $t = 0$. More precisely, δ is the measure on $\mathbb{T}$ defined by

$$\delta \in C(\mathbb{T})^*, \;\; \langle \delta, \phi \rangle = \phi(0), \;\; \phi \in C(\mathbb{T}).$$

We shall consider the equation

$$\begin{cases} \ddot{x} + x = \delta(t), \\ x(t) \geq 0, \\ x(t_0) = 0 \;\; \Rightarrow \;\; \dot{x}(t_0+) = -\dot{x}(t_0-) \end{cases} \tag{12}$$

and we shall see that all solutions are unbounded. Since δ satisfies the condition (11) when it is interpreted in a liberal way, this example shows that the previous corollary is not valid when f is a measure.

A solution of (12) can be defined as a continuous function satisfying (8) with $f \equiv 0$ in each interval $[2\pi n, 2\pi(n+1)]$ and such that

$$\dot{x}(2\pi n+) = \dot{x}(2\pi n-) + 1, \qquad \text{if } x(2n\pi) > 0$$

and

$$\dot{x}(2\pi n+) = -\dot{x}(2\pi n-) + 1, \qquad \text{if } x(2n\pi) = 0.$$

Intuitively we can describe the situation as follows: in the absence of external force our particle would bounce periodically with period π. Now we are adding an external force which is localized at times $t = 0, \pm 2\pi, \pm 4\pi, \ldots$ and has the effect of increasing the velocity of the particle in one unit.

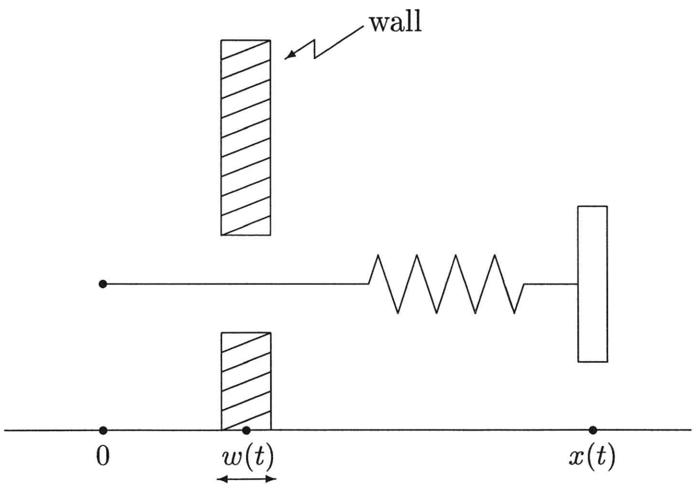
We consider the Poincaré mapping

$$P : (x(0), \dot{x}(0+)) \mapsto (x(2\pi), \dot{x}(2\pi+))$$

where $x(t)$ is a solution of (12). Since the period of δ is twice the period of the free oscillation, it is clear that P is just the translation of vector $(0, 1)$, that is $x(2\pi) = x(0)$, $\dot{x}(2\pi+) = \dot{x}(0+) + 1$. In consequence all solutions are defined up to $+\infty$ and the corresponding energy goes to infinity .

Example 3. A spring with oscillating wall.

In principle one could consider more general oscillators by letting the barrier to oscillate. More concretely, let us now assume that the wall is not fixed at $x = 0$ but it moves according to the known law $w = w(t)$. See the figure below.



We assume that w is smooth, positive and 2π-periodic. The particle follows the model

$$\begin{cases} \ddot{x} + ax = f(t) & \text{if } x(t) > w(t) \\ x(t) \geq w(t) \\ x(t_0) = w(t_0) & \Rightarrow \quad \dot{x}(t_0+) = -\dot{x}(t_0-) + 2\dot{w}(t_0). \end{cases}$$

The last condition reflects that the bouncing against the wall is elastic.

The change of reference system

$$x = y + w(t)$$

transforms the model into (8) where the new external force depends upon f, w and $\ddot{w}$. More discussions on this kind of oscillators as well as some connections with billiards can be seen in [3]. I thank R. Ramírez-Ros for informing me of this reference.

5 The successor map

Given $\tau \in \mathbb{R}$ and $v > 0$ let $x(t; \tau, v)$ be the solution of

$$\ddot{x} + ax = f(t), \quad x(\tau) = 0, \ \dot{x}(\tau) = v.$$

We denote by $\hat{\tau} > \tau$ the first zero of $x(t; \tau, v)$ to the right of τ. The corresponding velocity after bouncing will be denoted by

$$\hat{v} = -\dot{x}(\hat{\tau}; \tau, v).$$

The properties of the map $S : (\tau, v) \mapsto (\hat{\tau}, \hat{v})$ have been studied in [21] and [22] and we shall use the results in these papers. First of all we notice that S is well defined and one-to-one in the domain

$$\mathcal{R}_+ = \{(\tau, v) \in \mathbb{R}^2 \, : \, v > 0\}.$$

Moreover, it satisfies

$$S(\tau + 2\pi) = S(\tau, v) + (2\pi, 0).$$

In view of this property it is natural to identify τ with $\tau + 2\pi$ and we shall interpret τ and v as polar coordinates (τ =angle, v = radius). In this way the mapping S is defined on the cylinder,

$$S : \mathbb{T} \times (0, \infty) \to \mathbb{T} \times [0, \infty).$$

The iteration

$$(\tau_{n+1}, v_{n+1}) = S(\tau_n, v_n)$$

will reflect the dynamical properties of (8) as well as the oscillatory properties of solutions. Given an orbit of S, $\{(\tau_n, v_n)\}_{n\in\Lambda}$, $\Lambda \subset \mathbb{Z}$, such that $\{\tau_n : n \in \Lambda\}$ is a closed and discrete subset of $\mathbb{R}$, we can construct a bouncing solution of (8) defined as

$$x(t) = x(t; \tau_n, v_n) \quad \text{if } t \in [\tau_n, \tau_{n+1}].$$

Conversely, given a solution $x(t)$ of (8) we can label the set Z as a sequence $\{\tau_n\}$. If we define $v_n = \dot{x}(\tau_n+)$ and assume $v_n \neq 0$ then the sequence $\{(\tau_n, v_n)\}$ is an orbit of S. We notice that positive solutions of (8) do not correspond to any orbit of S. These solutions are always bounded. On the other hand a bouncing solution will be bounded if and only if $\sup_n v_n < \infty$ (see Lemma 4.3 in [21]).

We are now interested in the regularity of S. To this end we consider the singularity set

$$\Sigma = \{(\tau, v) : \hat{v} = 0\}.$$

Then S is of class C^1 on $\mathcal{R}_+ - \Sigma$. Another useful fact is that Σ is bounded in the cylinder, this means that there exists $\nu > 0$ such that

$$\Sigma \subset \mathbb{R} \times (0, \nu].$$

All these facts are proven in [21]. Now we present the expansion of S at infinity as obtained at Section 6 of [22]:

$$\begin{cases} \hat{\tau} = \tau + \frac{\pi}{\sqrt{a}} + \frac{1}{\sqrt{a}v}\sigma_0(\tau) + F(\tau, v) \\ \hat{v} = v + k_0(\tau) + G(\tau, v) \end{cases}$$

where

$$\sigma_0(\tau) = \int_\tau^{\tau+\frac{\pi}{\sqrt{a}}} f(t) \sin\sqrt{a}(t-\tau)dt, \quad k_0(\tau) = \int_\tau^{\tau+\frac{\pi}{\sqrt{a}}} f(t) \cos\sqrt{a}(t-\tau)dt.$$

The remainders F and G satisfy

$$F(\tau, v) = O(\frac{1}{v^2}), \quad G(\tau, v) = O(\frac{1}{v}) \quad \text{as } v \to +\infty,$$

uniformly in $\tau \in \mathbb{R}$. Moreover, if $f \in C^p(\mathbb{T})$, $p \geq 1$, then one can estimate the derivatives of the order $\alpha = (\alpha_1, \alpha_2)$, $\alpha_1 + \alpha_2 \leq p$, in the form

$$\partial^\alpha F(\tau, v) = O(\frac{1}{v^{2+\alpha_2}}), \quad \partial^\alpha G(\tau, v) = O(\frac{1}{v^{1+\alpha_2}}) \quad \text{as } v \to +\infty.$$

We are now in a position to prove the results of the previous Section. To prove Theorem 1 we shall apply the results in Section 3 of [2]. When a satisfies (9) one can rewrite the expansion of S at infinity as

$$\begin{cases} \hat{\tau} = \tau + \frac{2\pi p}{q} + \frac{\mu(\tau)}{v} + F(\tau, v) \\ \hat{v} = v - \mu'(\tau) + G(\tau, v), \end{cases}$$

where $\mu(\tau)$ was defined by (10). At this point it is convenient to notice that $\sigma_0' = -\sqrt{a}k_0$.

The expansion for the q-iterate of S is

$$\begin{cases} \tau_q = \tau + 2\pi p + \frac{q\Phi(\tau)}{v} + \hat{F}(\tau, v) \\ v_q = v - q\Phi'(\tau) + \hat{G}(\tau, v) \end{cases}$$

where $\hat{F}$ and $\hat{G}$ are remainders satisfying the same conditions as F and G. We can now apply Proposition 3.1 in [2] to deduce that, when the conditions of Theorem 1 hold, there exists $R_1 > 0$ such that if $v_0 \geq R_1$ then $\{(\tau_n, v_n)\}$ is well defined in the future or in the past and satisfies $v_n \to +\infty$ as $n \to +\infty$ or $n \to -\infty$. Thus, any solution of (8) satisfying $x(\tau) = 0$, $\dot{x}(\tau) \geq R_1$ (for some τ) is unbounded. The proof of the theorem can be finished by an application of Lemma 4.3 in [21].

To prove Theorem 2 it is sufficient to find $R_1 > 0$ such that any orbit $\{(\tau_n, v_n)\}$ with $v_0 \geq R_1$ is well defined for $n \in \mathbb{Z}$ and $\sup_n v_n < \infty$. This will be achieved by means of the theory of invariant curves. First we assume that the condition (i) holds. We shall find a sequence of Jordan curves $\{\Gamma_n\}$ in $\mathbb{T} \times (0, \infty)$ which are homotopic to the circle $v = \textit{constant}$ and such that $S^q(\Gamma_n) = \Gamma_n$. These curves are ordered and go to infinity as $n \to +\infty$. This means that Γ_{n+1} lies in the unbounded component of $[\mathbb{T} \times (0, \infty)] - \Gamma_n$ and $\min\{v : (\tau, v) \in \Gamma_n\} \to +\infty$ as $n \to \infty$. Since S is a topological mapping (for large v) and $\hat{v} \to +\infty$ as $v \to +\infty$, one deduces that Γ_n acts as a barrier for the orbits of S^q. Thus, if (τ_0, v_0) lies between Γ_N and Γ_{N+1} then (τ_{kq}, v_{kq}) will also lie on this region for any $k \in \mathbb{Z}$. From here it is easy to prove that v_n is bounded.

To prove the existence of Γ_n one can proceed exactly in the same way as in the proof of Theorem 1.1 in [22]. First one notices that S^q has the intersection property and then, after the change of variables

$$\theta = \tau, \quad \delta r = \frac{1}{v}, \quad (\delta > 0 \text{ parameter})$$

one can apply Theorem 3.1 of [22] to S^q. Notice that in that Theorem one can replace C^5 by C^4. To realize this it is sufficient to employ a C^4 version of the Small Twist Theorem (see for instance the appendix in [23]).

To prove Theorem 2 when *(ii)* holds one proves the existence of invariant curves of S. This is achieved by employing the main result in [23]. See also Example 2 in the same paper.

Final remarks.
1. In the assumptions of Theorem 1 it is possible to give an almost complete description of the dynamics of S around infinity. This can be achieved by combining the expansion of S^q previously obtained with the proof of Proposition 3.1 in [2].
2. In the assumptions of Theorem 2 we find the typical situation where KAM theory can be applied. In the annulus between two invariant curves we can apply the Poincaré-Birkhoff Theorem to deduce the existence of periodic points of S. This lead to subharmonic solutions of large amplitude. The solutions with initial conditions on an invariant curve will be quasi-periodic. All this is explained in the book [25].

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