

On the Lambert problem with drag

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Dedicated to Prof. Alain Chenciner

Abstract

The Lambert problem consists in connecting two given points in a given lapse of time under the gravitational influence of a fixed center. While this problem is very classical, we are concerned here with situations where friction forces act alongside the Newtonian attraction. Under some boundedness assumptions on the friction, there exists exactly one rectilinear solution if the two points lie on the same ray, and at least two solutions travelling in opposite directions otherwise.

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1 Introduction

Finding a solution of the Kepler problem from two specified times and the corresponding respective positions is usually referred to as the *Lambert problem*. In other terms, it is the combination of the Kepler problem with Dirichlet boundary conditions.

The history of this problem goes back in time to the dawn of Celestial Mechanics, having been briefly mentioned by Lambert in a letter to Euler [5, p. 24], and subsequently by Lagrange in his *Mécanique Analytique* [13, §34, p. 39]. In Gauss' *Theoria Motus* [10, §84, p. 108] we read

‘Hence, inversely, it is apparent that two radii vectors given in magnitude and position, together with the time in which the heavenly body describes the intermediate space, determine the whole orbit. But this problem, to be considered among the most important in the theory of the motions of the heavenly bodies, is not so easily solved, since the expression of the time in terms of the elements is transcendental, and moreover, very complicated.’

Fast forward to the second half of the twentieth century. In the sixties the development of computers and the needs of the aerospace industry signalled the beginning of an important literature on numerical iterative algorithms designed to approximate solutions of the Lambert problem [8, 14, 11]. From a more theoretical point of view, the first results on existence and uniqueness are due to Simó [22], whose approach was based on the Levi-Civita transformation. More recently, Albouy [1, §38], [2], has resorted to a related result, the so-called *Lambert theorem* to throw some new light on Simó's result. See also [3, 4].

If the particle moves in the vacuum and is not affected by any forces other than the gravity of the fixed center, the problem is integrable, a fact already known to Newton. If on the other hand our particle crosses a cloud of gas or dust (or is so close to the Earth that it interacts with its atmosphere), then one should take into account the influence of the drag. Friction forces in Celestial Mechanics have also a tradition spanning for centuries. Their effect was already studied by Euler [9] or Poincaré [21, Chapter VI], but research in this direction continues to this day [24, 6, 17, 18, 12]. See also the recent work [20], which studies numerically the Lambert problem in a frictional environment.

Many forms of friction make sense from a physical point of view. See, e.g., the discussion in [18, p. 266-267]. In this paper we shall always assume that the friction force is linear in the velocity and acts in the opposite direction of motion. On the other hand, its intensity may depend in a complicated way on the position of the particle, especially if the environment is heterogeneous. Mathematically we are led to a system of differential equations of the form

$$\ddot{x} + D(x)\dot{x} = -\frac{x}{|x|^3}, \quad x \in \mathbb{R}^2 \setminus \{0\}, \quad (K)$$

where $D : \mathbb{R}^2 \setminus \{0\} \rightarrow \mathbb{R}$ is nonnegative and continuously differentiable. Unless explicitly stated otherwise, solutions of (K) are understood in a classical sense, i.e., they are assumed to be twice continuously differentiable and avoid the collision singularity.

Given a solution $x = x(t)$ of (K), its angular momentum $c(t) := \det(x(t), \dot{x}(t))$ satisfies the first-order linear equation $\dot{c} = -D(x(t))c$ and therefore, it cannot change sign. Passage to polar coordinates $x = r(\cos \theta, \sin \theta)$ leads to the classical equality $c = r^2\dot{\theta}$, and consequently the sign of c divides the set of solutions of (K) into three nonoverlapping classes: *solutions living in a ray* (or rectilinear), *solutions rotating counterclockwise*, and *solutions rotating clockwise*.

Assuming that our solution is defined on the time interval $[-T, 0]$, it will be called an *arc from* $A := x(-T)$ *to* $B := x(0)$ provided that $|\theta(0) - \theta(-T)| < 2\pi$. (For reasons that will be clear below we name the time interval $[-T, 0]$ instead of the more conventional option $[0, T]$). On the other hand, the number T is usually referred to as the *flight (or transfer) time* of the arc.

We are now ready to formulate our problem in a more precise way. *Given points* $A, B \in \mathbb{R}^2 \setminus \{0\}$, *and given some flight time* $T > 0$, *are there arcs from* A *to* B *having flight time* T ? With other words, we are concerned with the Dirichlet problem arising from the combination of (K) with the boundary conditions

$$x(-T) = A, \quad x(0) = B, \quad (BC)$$

focusing our attention on solutions rotating for less than one tour on the given time interval. Our precise assumptions on the (nonnegative, continuously differentiable) friction coefficient $D = D(x)$ will be as follows:

$$[\mathbf{D}_1] \quad D : \mathbb{R}^2 \setminus \{0\} \rightarrow \mathbb{R} \text{ is bounded.}$$

$$[\mathbf{D}_2] \quad \lim_{x \rightarrow 0} \sqrt{|x|} \nabla D(x) = 0.$$

The main result of this paper is given below:

Theorem 1.1. *Assuming $[\mathbf{D}_{1-2}]$, fix some flight time $T > 0$ and points $A, B \in \mathbb{R}^2 \setminus \{0\}$. Then:*

- (a) *If A and B lie on the same ray starting at the origin then there exists a unique rectilinear arc going from A to B in the flight time T .*

- (b) If A, B do not lie on the same ray starting at the origin then there exists at least one arc from A to B with flight time T and rotating clockwise, and at least one arc from A to B with flight time T and rotating counterclockwise.

Some remarks are in order:

- (i) It seems reasonable to ask whether assumptions $[\mathbf{D}_{1-2}]$ are actually necessary. While we cannot entirely answer to this question, it will be clear from our discussion (see Corollary 3.3) that $[\mathbf{D}_2]$ may indeed be fully dropped in the case of the rectilinear statement (a). Assumption $[\mathbf{D}_2]$ will be required only in connection with the Levi-Civita regularization (Subsection 8.2 and Section 9) to complete the proof of (b), and we do not know whether one could construct another proof without this hypothesis. It implies that the function $z \in \mathbb{C} \setminus \{0\} \mapsto D(z^2)$ can be extended to a continuously differentiable function of two real variables on $\mathbb{C} \equiv \mathbb{R}^2$, and in particular, D has a limit at $x = 0$.
- (ii) Concerning the nonrectilinear situation (b), the two arcs rotating in opposite directions are well-known to be unique in the frictionless case $D \equiv 0$ (see [22]). We do not know whether uniqueness still holds under the presence of friction.
- (iii) Throughout this paper we focus our attention on motions making less than one full tour on the given time interval, which we call *arcs* (they are sometimes called *simple arcs* in the literature). It would be interesting to study the existence and multiplicity of solutions turning more than one tour; see, e.g. the recent paper [4] on the frictionless situation $D \equiv 0$. We point out that solutions making exactly one tour -or a given integer number of tours- in a given flight time might exist only rarely; for instance, in absence of friction such a solution exists if and only if $A = B$ and the flight time T is not too small (the solution must be elliptic and Kepler's Third Law applies).
- (iv) In the rectilinear case (a), the unique solution is actually nondegenerate (see Lemma 3.4). Thus, when the two endpoints are slightly pertubated so as not lie on the same line, our solution can be continued in such a way that it sweeps a small angle (the so-called direct arc). These problems admit also solutions rotating in the opposite direction and therefore sweeping an angle close to 2π (indirect arcs); in the limit such solutions converge to a *generalized solution* which bounces at the origin. Bouncing solutions will not appear explicitly in this paper, but they will be somehow behind the arguments of Subsection 8.2 and Section 9.

After this introduction the paper is organized as follows. We begin with Section 2, where we discuss some general properties of the damped Kepler problem (K). Subsequently, Section 3 is devoted to study its rectilinear solutions; in particular, we prove the rectilinear statement (a) of Theorem 1.1. In Section 4 we state, without proof, three important results labelled as Propositions 4.1, 4.2 and 4.3, which, in combination with a continuation argument from degree theory, will lead to the proof of Theorem 1.1(b) in Section 5. The second part of the paper is devoted to prove the three propositions advanced in Section 4; more precisely Proposition 4.1 is proved in Section 6, Proposition 4.2 is established in Section 7, and the purpose of the remaining Sections 8-9 is to validate Proposition 4.3.

The proof of the nonrectilinear statement (b) of Theorem 1.1 is based in embedding (K)-(BC) into a continuous family of Lambert problems depending on a parameter $\lambda \in [0, 1]$. This family will be obtained by letting the initial point A_λ rotate on a circumference of radius $|A|$ centered at the origin. One starts at the rectilinear situation $A_0 := (|A|/|B|)B$ and ends at the

desired point $A_1 := A$ (see Figure 1(b) below). At $\lambda = 0$, the results of Section 3 show that the corresponding rectilinear problem has a unique solution which is nondegenerate. Thus, the implicit function theorem may be used to obtain a *local* branch of solutions for small $\lambda > 0$. In fact, well-known arguments from Brouwer degree theory show the existence of a *global* branch of solutions spanning for all $\lambda \in [0, 1]$ -and thus guaranteeing the existence of solution for our original Lambert problem- provided that two conditions hold. On the one hand, we need to ensure that the branch of solutions does not blow up to infinity, a fact provided by Proposition 4.2. On the other, solutions should not escape through the boundary of the domain before arriving to $\lambda = 1$. Since this boundary is made of collinear motions (by Proposition 4.1), the result follows from Corollary 4.4. The two possible directions of rotation will then give rise to the two solutions described in Theorem 1.1(b).

2 Catastrophes are due to collisions

Equation (K) admits several alternative rewritings which can be used to reveal a number of its features. To start with, let the nonvanishing function $x :]\alpha, 0] \rightarrow \mathbb{R}^2 \setminus \{0\}$, $x = x(t)$, be continuously differentiable (our solutions will always be defined on time intervals ending at $t = 0$ unless explicitly stated otherwise). We set

$$p(t) := \exp\left(-\int_t^0 D(x(s))ds\right). \quad (1)$$

Then (K) becomes

$$\frac{d}{dt}(p(t)\dot{x}) = -\frac{p(t)}{|x|^3}x, \quad x \neq 0,$$

as one can readily check. While this equality is reminiscent of the usual presentation of Sturm-Liouville systems, the function p appearing here depends on x in a nonlinear, nonlocal fashion. Notice that

$$e^{D_*t} \leq p(t) \leq 1, \quad t \in]\alpha, 0], \quad (2)$$

where $D_* \geq 0$ stands for some upper bound of D on $\mathbb{R}^2 \setminus \{0\}$ (which exists by assumption $[\mathbf{D}_1]$). If $x = x(t)$ is a solution of (K), then it will satisfy

$$|p(t)\dot{x}(t) - \dot{x}(0)| = \left| \int_0^t \frac{p(s)}{|x(s)|^3}x(s)ds \right| \leq \int_t^0 \frac{1}{|x(s)|^2}ds \leq \frac{|t|}{\min_{t \leq s \leq 0} |x(s)|^2}, \quad t \in]\alpha, 0], \quad (3)$$

and the combination of (2)-(3) gives

$$|\dot{x}(t)| \leq e^{D_*|t|}|\dot{x}(0)| + \frac{|t| e^{D_*|t|}}{\min_{t \leq s \leq 0} |x(s)|^2}, \quad \alpha < t < 0. \quad (4)$$

We shall often work with solutions $x :]\alpha, 0] \rightarrow \mathbb{R}^2 \setminus \{0\}$ of (K) which are *maximal in the past*. It means that if the extended solution $\hat{x} :]\hat{\alpha}, 0] \rightarrow \mathbb{R}^2 \setminus \{0\}$ satisfies that $\hat{\alpha} \leq \alpha$ and $\hat{x}(t) = x(t)$ for every $t \in]\alpha, 0]$, then $\hat{\alpha} = \alpha$. In this situation, if $\alpha > -\infty$ we may say that x is not globally defined in the past, and solutions of this kind are the target of the following

Lemma 2.1. *Let $x :]\alpha, 0] \rightarrow \mathbb{R}^2 \setminus \{0\}$ be a solution of (K), maximal in the past. If $\alpha > -\infty$ then $\liminf_{t \downarrow \alpha} |x(t)| = 0$.*

Proof. Using a contradiction argument, we assume that $\liminf_{t \downarrow \alpha} |x(t)| > 0$. Then, (4) implies that $\limsup_{t \downarrow \alpha} |\dot{x}(t)| < +\infty$, and (K) gives $\limsup_{t \downarrow \alpha} |\ddot{x}(t)| < +\infty$. Therefore both $x(t)$ and $\dot{x}(t)$ have limits when $t \downarrow \alpha$, the limit of x being nonzero. The standard continuation theory for solutions of ordinary differential equations states that our solution can be extended to some time interval containing α . This is a contradiction and concludes the proof. \square

Assume now that $T > 0$ is fixed and $x_n : [-T, 0] \rightarrow \mathbb{R}^2 \setminus \{0\}$ is a sequence of solutions of (K) satisfying

$$x_n(0) \rightarrow x_0 \neq 0, \quad \dot{x}_n(0) \rightarrow \dot{x}_0.$$

Lemma 2.2. *If the solution $x = x(t)$ of (K) with $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$ cannot be extended to $[-T, 0]$, then $\min_{[-T, 0]} |x_n| \rightarrow 0$ as $n \rightarrow +\infty$.*

Proof. We argue by contradiction and assume that, after possibly passing to a subsequence, $\{\min_{[-T, 0]} |x_n|\}_n$ is bounded from below by a positive constant. Combining (4) with the fact that $\{\dot{x}_n(0)\}$ is bounded we conclude that $\{|\dot{x}_n(t)|\}_n$ is uniformly bounded on $[-T, 0]$. In addition $\{|x_n(0)|\}$ is bounded, and it follows that also $\{|x_n(t)|\}_n$ is uniformly bounded on $[-T, 0]$. Moreover, both sequences $\{x_n(t)\}_n$, $\{\dot{x}_n(t)\}_n$ are equicontinuous -as a consequence of equation (K) in the latter case-, and the Ascoli-Arzelà lemma states that they are both uniformly convergent, at least along some subsequence. Then, $x(t) := \lim_{n \rightarrow +\infty} x_n(t)$ must be a solution of (K) satisfying $x(0) = x_0$, $\dot{x}(0) = \dot{x}_0$ and defined on $[-T, 0]$. It contradicts our assumptions and concludes the proof. \square

3 Existence, uniqueness and nondegeneracy in the rectilinear problem

In this section we are concerned with the collinear motions of the frictional Kepler problem (K). While our main goal will be to prove the rectilinear part (a) of Theorem 1.1, we shall conclude by showing the nondegeneracy of the unique rectilinear solution.

Thus, let $w_0 \in \mathbb{R}^2$ with $|w_0| = 1$ be fixed and consider motions $x :]\alpha, 0] \rightarrow \mathbb{R}^2 \setminus \{0\}$ of the form $x(t) := r(t)w_0$ with $r = r(t) > 0$. Setting $\delta(r) := D(rw_0)$, system (K) becomes

$$\ddot{r} + \delta(r)\dot{r} = -\frac{1}{r^2}, \quad r > 0. \quad (5)$$

Throughout this section we shall analyze this scalar equation under the assumptions (modelled on \mathbf{D}_1) that $\delta :]0, +\infty[\rightarrow \mathbb{R}$ is *nonnegative, bounded and continuously differentiable*. We shall start with the following observation:

Lemma 3.1. *Two solutions $r_1 \neq r_2$ of (5) intersect at most once. With other words, if $r_1(t_*) = r_2(t_*)$ for some t_* , then $r_1(t) \neq r_2(t)$ for any $t \neq t_*$ in the common definition interval of r_1 and r_2 .*

Proof. We use a contradiction argument and assume instead that there are solutions $r_1 \neq r_2$ of (5) and times $t_A < t_B$ in the common definition interval of r_1 and r_2 such that

$$r_1(t_A) = r_2(t_A) =: r_A, \quad r_1(t_B) = r_2(t_B) =: r_B, \quad r_1(t) < r_2(t) \text{ for every } t \in]t_A, t_B[. \quad (6)$$

Let $\Delta :]0, +\infty[\rightarrow \mathbb{R}$ denote a primitive of δ . Integration in both sides of (5) leads to the equalities

$$\dot{r}_i(t_B) - \dot{r}_i(t_A) = -\Delta(r_B) + \Delta(r_A) - \int_{t_A}^{t_B} \frac{1}{r_i(t)^2} dt, \quad i = 1, 2,$$

implying that

$$\dot{r}_2(t_B) - \dot{r}_2(t_A) > \dot{r}_1(t_B) - \dot{r}_1(t_A),$$

which is not possible since $\dot{r}_1(t_A) \leq \dot{r}_2(t_A)$ and $\dot{r}_1(t_B) \geq \dot{r}_2(t_B)$, by (6). It concludes the proof. \square

Let $r = r(t)$, $t \in]\alpha, 0]$, be a solution of (5). Recalling the arguments at the beginning of Section 2, and setting

$$p(t) := \exp\left(-\int_t^0 \delta(r(s))ds\right), \quad (7)$$

we see that

$$\frac{d}{dt}(p(t)\dot{r}) = -\frac{p(t)}{r^2}, \quad r > 0. \quad (8)$$

On the other hand, denoting by $D_* \geq 0$ an upper bound of δ on $]0, +\infty[$ one checks that inequality (2) still holds in this situation.

Fix numbers $T, r_B > 0$ and consider the set $\mathcal{I} = \mathcal{I}(T, r_B)$ of final speeds $v \in \mathbb{R}$ such that the solution $r = r(t)$ of (5) with $r(0) = r_B$ and $\dot{r}(0) = v$ is defined in the past up to time $t = -T$. The usual smooth dependence theorems state that \mathcal{I} is open and the function

$$\mathfrak{R} : \mathcal{I} \rightarrow \mathbb{R}, \quad v \mapsto r(-T),$$

is continuously differentiable on \mathcal{I} . The main result of this section collects some basic properties of \mathcal{I} and \mathfrak{R} . Notice that, in particular, the second statement implies assertion (a) of Theorem 1.1:

Lemma 3.2. *The following hold:*

(i) *There exists some $\beta \in \mathbb{R}$ such that $\mathcal{I} =]-\infty, \beta[$.*

(ii) *\mathfrak{R} establishes a decreasing diffeomorphism from $] -\infty, \beta[$ into $]0, +\infty[$. With formulas,*

$$\frac{d\mathfrak{R}}{dv}(v) < 0 \text{ for every } v < \beta, \quad \lim_{v \rightarrow -\infty} \mathfrak{R}(v) = +\infty, \quad \lim_{v \rightarrow \beta_-} \mathfrak{R}(v) = 0.$$

Proof. The fact that \mathcal{I} is open and $\mathfrak{R} : \mathcal{I} \rightarrow \mathbb{R}$ is continuously differentiable is a direct consequence of the usual theorems of smooth dependence on initial conditions. Moreover, it follows from Lemmas 2.1 and 3.1 that the set \mathcal{I} is an interval and $\mathfrak{R} : \mathcal{I} \rightarrow \mathbb{R}$ is strictly decreasing.

We claim first that \mathcal{I} is unbounded from below and $\lim_{v \rightarrow -\infty} \mathfrak{R}(v) = +\infty$. It can be done by picking some sequence $\{r_n\}_n$ of solutions of (5) with

$$r_n(0) = r_B \text{ for every } n, \quad 0 > v_n := \dot{r}_n(0) \rightarrow -\infty.$$

Each function r_n is defined on some interval $]\alpha_n, 0]$, maximal to the left. If $\dot{r}_n(t) < 0 \forall t \in]\max(\alpha_n, -T), 0[$ then Lemma 2.1 implies that $\alpha_n < -T$ and we set $a_n := -T$. Otherwise, $\dot{r}_n(t) = 0$ for some $t \in]\max(\alpha_n, -T), 0[$, and we denote by a_n the maximum of such numbers t . In any case, $r_n(t) \geq r_B$ for all $t \in [a_n, 0]$. Setting $p_n : [a_n, 0] \rightarrow \mathbb{R}$ as in (7) for $r = r_n$, and arguing as in (3), equality (8) implies that

$$|p_n(t)\dot{r}_n(t) - \dot{r}_n(0)| \leq \frac{T}{r_B^2}, \quad a_n \leq t \leq 0, \quad n \in \mathbb{N},$$

which, in combination with (2) implies that $\max_{[a_n, 0]} \dot{r}_n \rightarrow -\infty$ as $n \rightarrow +\infty$. In particular, $a_n = -T$ for n big enough and the claim follows.

We observe next that \mathcal{I} is bounded from above. Arguing by contradiction, we assume the existence of a second sequence $\{r_n\}_n$ of solutions of (5) with $v_n := \dot{r}_n(0) \rightarrow +\infty$, all of them defined on $[-T, 0]$ and satisfying $r_n(0) = r_B$. It follows from (8) that

$$\frac{d}{dt}(p_n(t)\dot{r}_n(t)) < 0, \quad t \in [-T, 0],$$

where each $p_n : [-T, 0] \rightarrow \mathbb{R}$ is defined as in (7) for $r = r_n$. It implies that $\dot{r}_n(t) \rightarrow +\infty$ uniformly with respect to $t \in [-T, 0]$, which is not possible since all r_n are positive.

We also need to check that $\lim_{v \rightarrow \beta_-} \mathfrak{R}(v) = 0$. This statement follows from the combination of Lemma 2.2 with the observation that solutions $r = r(t)$ of (5) do not have local minima in open time intervals. Therefore, $\mathcal{I} =]-\infty, \beta[$ for some $\beta \in \mathbb{R}$, thus proving (i).

In order to establish (ii) it remains to show that $\frac{d\mathfrak{R}}{dv}(v) \neq 0$ for every $v < \beta$. We use a contradiction argument and assume instead that $\frac{d\mathfrak{R}}{dv}(v_*) = 0$ for some $v_* < \beta$. It implies the existence of some solution $r_* = r_*(t)$ of (5) such that the linear Dirichlet problem

$$\ddot{u} + \frac{d\delta}{dr}(r_*(t))\dot{r}_*(t)u + \delta(r_*(t))\dot{u} = \frac{2u}{r_*(t)^3}, \quad u(-T) = u(0) = 0, \quad (9)$$

has a nonzero solution $u : [-T, 0] \rightarrow \mathbb{R}$. After possibly replacing u by $-u$ and T by some smaller time there is no loss of generality in further assuming that $u(t) > 0$ for every $t \in]-T, 0[$. Noting that $(d\delta/dr)(r_*(t))\dot{r}_*(t)u(t) + \delta(r_*(t))\dot{u}(t) = (d/dt)[\delta(r_*(t))u(t)]$, integration in (9) leads to

$$\dot{u}(0) - \dot{u}(-T) = 2 \int_{-T}^0 \frac{u(t)}{r_*(t)^3} ds > 0,$$

which is not possible since $\dot{u}(-T) > 0 > \dot{u}(0)$. This contradiction concludes the proof. \square

One immediately arrives to the following reformulation of Theorem 1.1(a), where no traces of assumption $[\mathbf{D}_2]$ are present:

Corollary 3.3. *Let $\delta :]0, +\infty[\rightarrow \mathbb{R}$ be continuously differentiable, nonnegative and bounded. Then for every $r_A, r_B > 0$ and every $T > 0$ there exists a unique solution of (5) with $r(-T) = r_A$ and $r(0) = r_B$.*

We close this subsection by exploring the nondegeneracy of the rectilinear solutions of the Kepler equation (K). Some one-dimensional nondegeneracy was already established in Lemma 3.2(ii), but we would like to show nondegeneracy in the context of the planar Dirichlet problem (K)-(BC). More precisely, we shall adopt the following

Definition. *A solution $x_* : [-T, 0] \rightarrow \mathbb{R}^2 \setminus \{0\}$ of (K) will be called nondegenerate if the combination of the variational equation*

$$\ddot{w} + \langle \nabla D(x_*(t)), w \rangle \dot{x}_*(t) + D(x_*(t))\dot{w} = -\frac{1}{|x_*(t)|^3} w + 3 \frac{\langle x_*(t), w \rangle}{|x_*(t)|^5} x_*(t), \quad w \in \mathbb{R}^2, \quad (10)$$

and the homogeneous Dirichlet boundary conditions $w(-T) = 0 = w(0)$, admits only the trivial solution $w \equiv 0$.

Lemma 3.4. *Every rectilinear solution $x_* : [-T, 0] \rightarrow \mathbb{R}^2 \setminus \{0\}$ of (K) is nondegenerate.*

Proof. Set $x_*(t) := r_*(t)w_0$ where $w_0 \in \mathbb{R}^2$ is unitary and $r_*(t) > 0$ for every $t \in [-T, 0]$. Equation (10) becomes

$$\ddot{w} + \langle \nabla D(r_*(t)w_0), w \rangle \dot{r}_*(t)w_0 + \delta(r_*(t))\dot{w} = -\frac{1}{r_*(t)^3}w + 3\frac{\langle w_0, w \rangle}{r_*(t)^3}w_0, \quad w \in \mathbb{R}^2, \quad (11)$$

where $\delta(r) := D(rw_0)$, $r > 0$. Let $w : [-T, 0] \rightarrow \mathbb{R}^2$ be a solution with $w(-T) = 0 = w(0)$; then $v(t) := \det(w(t), w_0)$ satisfies the linear second-order equation

$$\ddot{v} + \delta(r_*(t))\dot{v} + \frac{v}{r_*(t)^3} = 0, \quad t \in [-T, 0],$$

which also admits the positive solution $r_* : [-T, 0] \rightarrow \mathbb{R}$. We use the method of reduction of order and set $v = r_*(t)v_1$, to obtain

$$\ddot{v}_1 = -\left(2\frac{\dot{r}_*(t)}{r_*(t)} + \delta(r_*(t))\right)\dot{v}_1, \quad v_1(-T) = v_1(0) = 0.$$

The boundary conditions imply the existence of some $t_0 \in]-T, 0[$ such that $\dot{v}_1(t_0) = 0$, subsequently the differential equation implies that $\dot{v}_1 \equiv 0$, and again by the boundary conditions, $v_1 \equiv 0$. Consequently, $\det(w(t), w_0) \equiv 0$, implying the existence of some function $u : [-T, 0] \rightarrow \mathbb{R}$ such that $w(t) = u(t)w_0$ for every $t \in [-T, 0]$. Going back to (11) we see that u must be a solution of (9), and we deduce that $\frac{d\mathfrak{R}}{dv}(\dot{r}_*(0))\dot{u}(0) = u(-T) = 0$ (the function \mathfrak{R} is constructed with respect to the final position $r_B := r_*(0)$). But $\frac{d\mathfrak{R}}{dv}(\dot{r}_*(0)) < 0$, and then, $\dot{u}(0) = 0$, so that $u \equiv 0$ by uniqueness. Therefore $w \equiv 0$, thus concluding the proof. \square

4 Three cornerstones supporting the proof

The purpose of this section is to bring forward three important results, labelled as Propositions 4.1, 4.2 and 4.3, which will hold up the proof of Theorem 1.1(b) in Section 5. In order to keep the pace of the exposition their proofs will be postponed to Sections 6-9, in the second part of the paper.

Proposition 4.1. *Nonrectilinear solutions of (K) are globally defined in the past. With other words, if the nonrectilinear solution $x :]\alpha, 0] \rightarrow \mathbb{R}^2 \setminus \{0\}$ is maximal in the past, then $\alpha = -\infty$.*

Proposition 4.1 was previously proved in [17, Proposition 2.1] assuming that D is constant, and in [18, Proposition 2.1] when $D = D(|x|)$ depends only on the height of the particle. In the proof we shall use assumption $[\mathbf{D}_1]$, but not $[\mathbf{D}_2]$. Nonrectilinear solutions are also globally defined in the future; however, we shall not need this fact in our analysis.

A second ingredient which shall be needed in the next section concerns the existence of a priori bounds for the final speed of solutions joining two given heights in a given flight time. More precisely, assume that $r_A, r_B > 0$ (in addition to $T > 0$) are fixed and consider the boundary conditions

$$|x(-T)| = r_A, \quad |x(0)| = r_B. \quad (12)$$

Proposition 4.2. *There exists some $M > 0$ (depending on r_A, r_B and T but not on x), such that whenever $x : [-T, 0] \rightarrow \mathbb{R}^2 \setminus \{0\}$ is a solution of (K)-(12) then $|\dot{x}(0)| < M$.*

We do not know whether a similar property holds for the initial (in the place of final) speed, and this is the reason why we use time-backwards Poincaré maps instead of their more traditional time-forward cousins. Proposition 4.2 ensures that no families of solutions of (K)-(12) blow up to infinity, thus playing an important role in the application of continuation arguments of topological degree.

We go back now to Proposition 4.1. There is an alternative way to present this result by using the language of Poincaré maps, which, throughout this paper, will be referred to backward time. Thus, given $T > 0$, the associated Poincaré map $\mathcal{P} = \mathcal{P}_T$ maps an initial condition $(x_0, \dot{x}_0) \in (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^2$ into the pair $(x(-T), \dot{x}(-T)) \in \mathbb{R}^2 \times \mathbb{R}^2 \equiv \mathbb{R}^4$. Here $x = x(t)$ stands for the solution of (K) satisfying the *final conditions*:

$$x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0. \quad (FC)$$

The natural domain of \mathcal{P} is the set $\Omega = \Omega_T$ of points $(x_0, \dot{x}_0) \in (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^2$ such that the solution of (K)-(FC) can be extended to $t = -T$. The usual continuous dependence theorems state that Ω is open in \mathbb{R}^4 and $\mathcal{P} : \Omega \rightarrow \mathbb{R}^4$ is continuous. Proposition 4.1 above can be reformulated by saying that

$$\Omega \supset \{(x_0, \dot{x}_0) \in \mathbb{R}^2 \times \mathbb{R}^2 : x_0, \dot{x}_0 \text{ are linearly independent}\}. \quad (13)$$

We denote by $\mathcal{X} : \Omega \rightarrow \mathbb{R}^2$ the first two (position) components of \mathcal{P} . With other words, $\mathcal{X} = \Pi \circ \mathcal{P}$, where $\Pi : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ stands for the projection $(x, \dot{x}) \mapsto x$. Our third postulate will be the following:

Proposition 4.3. *\mathcal{X} admits a continuous extension $\bar{\mathcal{X}} : (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$. Moreover, $\bar{\mathcal{X}}(x_0, \mathbb{R}x_0) \subset [0, +\infty[x_0$, for every $x_0 \in \mathbb{R}^2 \setminus \{0\}$.*

Thus, collinear solutions may collide with the singularity, but admitting that they bounce back at the collision one obtains a flow that is continuous in its position components. This result will be obtained with the help of (Levi-Civita's) regularization theory.

Before closing this section we point out a consequence of the (still unproved) Propositions 4.1 and 4.3 which will play an important role in the proof of Theorem 1.1(b). We go back to the situation described in Lemma 2.2 and suppose that $x_n : [-T, 0] \rightarrow \mathbb{R}^2 \setminus \{0\}$ is a sequence of solutions of (K). For each $n \in \mathbb{N}$ we denote by $\Delta\theta_n$ the angle swept by x_n on the time interval $[-T, 0]$. With formulas,

$$\Delta\theta_n := \theta_n(0) - \theta_n(-T),$$

where $\theta_n(t) := \arg(x_n(t))$ is a continuous choice of the argument on x_n .

Since each solution x_n is defined on $[-T, 0]$, it follows that $(x_n(0), \dot{x}_n(0)) \in \Omega_T$ for every n . One has the following:

Corollary 4.4. *Assume that $(x_n(0), \dot{x}_n(0)) \rightarrow (x_0, \dot{x}_0) \in \partial\Omega_T$. If there are constants $r_A, r_B > 0$ such that $|x_n(-T)| = r_A$ and $|x_n(0)| = r_B$ for every $n \in \mathbb{N}$ then $\liminf_{n \rightarrow +\infty} |\Delta\theta_n| \geq 2\pi$.*

Proof. By assumption, $|x_n(0)| = r_B \forall n \in \mathbb{N}$, and we see that $|x_0| = r_B$. The combination of Proposition 4.1 -in the form given in (13)- and Proposition 4.3, implies that

$$\dot{x}_0 \in \mathbb{R}x_0, \quad x_n(-T) \rightarrow \frac{r_A}{r_B}x_0 \text{ as } n \rightarrow +\infty.$$

Consequently, either $\liminf_{n \rightarrow +\infty} |\Delta\theta_n| \geq 2\pi$ as claimed, or otherwise $\lim_{n \rightarrow +\infty} \Delta\theta_n = 0$. In the latter situation the functions $\xi_n(t) := \langle x_n(t), x_0 \rangle / r_B$ are positive for n big enough, and they satisfy

$$\xi_n(-T) \rightarrow r_A, \quad \xi_n(0) \rightarrow r_B, \quad \min_{-T \leq t \leq 0} \xi_n(t) \rightarrow 0, \quad n \rightarrow +\infty,$$

the last fact following from Lemma 2.2. On the other hand, (K) implies that

$$\ddot{\xi}_n + D(x_n)\dot{\xi}_n = -\frac{\xi_n}{|x_n(t)|^3} < 0,$$

which, when evaluated at the point where ξ_n attains its minimum leads to contradiction. It concludes the reasoning. \square

With the aim of speeding up our main arguments we put off the proofs of Propositions 4.1, 4.2 and 4.3 to Sections 6-9. Nevertheless we shall rely on them in the next section to prove Theorem 1.1(b).

5 Brouwer degree and continuation from the rectilinear problem

In this section we shall prove Theorem 1.1 by relying on Propositions 4.1, 4.2 and 4.3, and using some basic techniques from Brouwer degree theory. We refer the reader to the classical textbook [16] or the recent treatise [7] for comprehensive introductions to Brouwer degree.

An important property of the Brouwer degree is its invariance by homotopies. If two problems are connected by a homotopy in such a way that solutions do not blow up to infinity or escape through the boundary, then they have the same degree. In particular, if the first problem has nonzero degree, then the second has a solution.

In this section we shall need a particular consequence of this property that we describe next. Let $\mathcal{U} \subset \mathbb{R}^N$ be open (but not necessarily bounded), and let

$$\Phi : [0, 1] \times \mathcal{U} \rightarrow \mathbb{R}^N, \quad (\lambda, u) \mapsto \Phi(\lambda, u),$$

be continuous and admit a continuously-defined derivative with respect to the u variables, denoted $\partial_u \Phi : [0, 1] \times \mathcal{U} \rightarrow \mathbb{R}^{N \times N}$. We set $\Sigma := \{(\lambda, u) \in [0, 1] \times \mathcal{U} : \Phi(\lambda, u) = 0\}$ and assume that

- (a) There exists some $u_0 \in \mathcal{U}$ such that $\Sigma \cap (\{0\} \times \mathbb{R}^N) = \{(0, u_0)\}$ and $\det(\partial_u \Phi(0, u_0)) \neq 0$.
- (b) There exists some constant $M > 0$ such that $|u| < M$ for every $(\lambda, u) \in \Sigma$.
- (c) $\bar{\Sigma} \cap ([0, 1] \times (\partial \mathcal{U})) = \emptyset$.

Lemma 5.1. *Under the above, for each $\lambda \in [0, 1]$ there exists some $u_\lambda \in \mathcal{U}$ such that $(\lambda, u_\lambda) \in \Sigma$.*

Proof. (See Fig. 1(a)). Assumptions (b)-(c) guarantee the existence of some $\rho > 0$ such that $u + \bar{B}(\rho) \subset \mathcal{U}$ for any $(\lambda, u) \in \Sigma$. Here, $\bar{B}(\rho)$ denotes the closed ball of radius ρ in \mathbb{R}^N . Set

$$\mathcal{V} := \{u \in \mathbb{R}^N \text{ such that } |u| < M \text{ and } u + \bar{B}(\rho) \subset \mathcal{U}\},$$

which is open and bounded. Moreover, $\bar{\mathcal{V}} \subset \mathcal{U}$ and $\Sigma \subset [0, 1] \times \mathcal{V}$. Therefore, the Brouwer degree $\deg_B(\Phi(\lambda, \cdot), \mathcal{V})$ does not depend on $\lambda \in [0, 1]$. For $\lambda = 0$, $\deg_B(\Phi(0, \cdot), \mathcal{V}) = \text{sign}(\det \partial_u \Phi(0, u_0)) = \pm 1$, and thus,

$$\deg_B(\Phi(\lambda, \cdot), \mathcal{V}) \neq 0 \text{ for every } \lambda \in [0, 1].$$

The result follows. \square

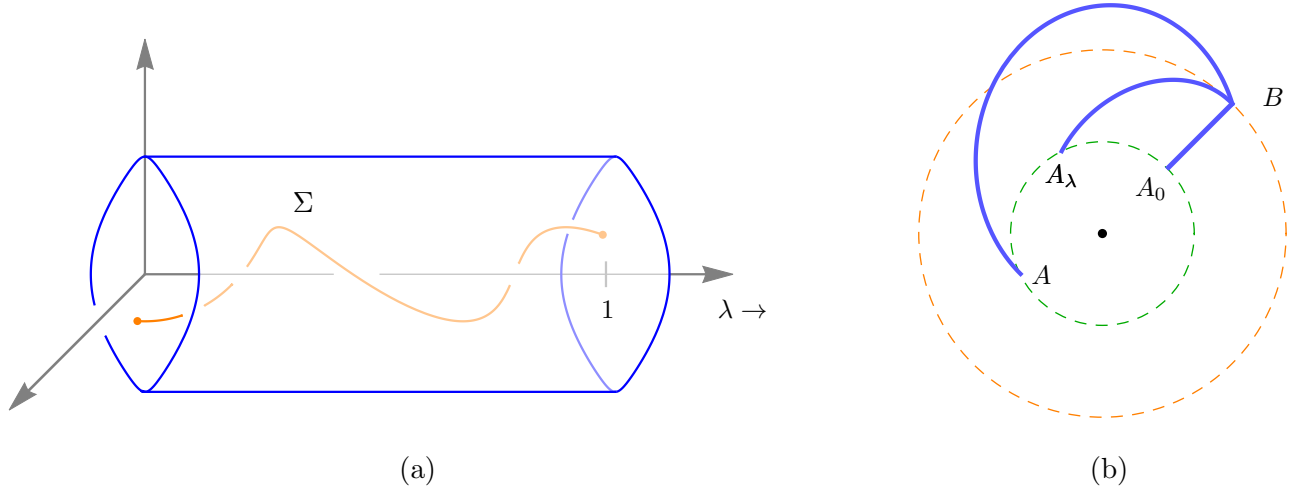


Figure 1: (a): The projection of Σ on the λ -variable is the full interval $[0, 1]$. (b): Rotating the starting point on a circumference centered at the origin we obtain a homotopy from a rectilinear problem to the original one.

Remark 5.2. *Well-known arguments going back to Leray-Schauder ([15, Théorème Fondamental, p. 63]) show, under the assumptions above, the existence of a connected set of solutions sweeping all values of λ . This fact will not be used in our argument.*

Proof of Theorem 1.1(b). Write $A = r_A(\cos \theta_A, \sin \theta_A)$ and $B = r_B(\cos \theta_B, \sin \theta_B) = r_B w_0$, where $r_A, r_B > 0$, $|\theta_B - \theta_A| < 2\pi$, and $w_0 := (\cos \theta_B, \sin \theta_B)$. We consider the map Ψ sending each final speed $v_0 \in \mathbb{R}^2$ into the polar coordinates $(r(-T), \theta(-T))$ of the position $x(-T) = \mathcal{X}(B, v_0)$, of the solution x of (K) satisfying $x(0) = B$, $\dot{x}(0) = v_0$. The lifting of the angle $\theta = \theta(t)$ is chosen so that $\theta(0) = \theta_B$. In view of Proposition 4.1 and Lemma 3.2(i), the map Ψ is naturally defined on the open set

$$\mathcal{U} := \mathbb{R}^2 \setminus ([\beta, +\infty[w_0),$$

for some $\beta \in \mathbb{R}$. We also set

$$\Phi : [0, 1] \times \mathcal{U} \rightarrow \mathbb{R}^2, \quad \Phi(\lambda, v_0) := \Psi(v_0) - (r_A, \vartheta(\lambda)),$$

where $\vartheta(\lambda) := (1 - \lambda)\theta_B + \lambda\theta_A$. Denoting $A_\lambda := r_A(\cos \vartheta(\lambda), \sin \vartheta(\lambda))$ we see that, for $\lambda = 0$, $A_0 = (r_A/r_B)B$ lies on the same ray as B , and thus $\Phi(0, v_0) = 0$ if and only if v_0 is the final velocity of a rectilinear arc from A_0 to B in the flight time T . In the case $\lambda = 1$ it is clear that $\Phi(1, v_0) = 0$ if and only if v_0 is the final velocity of an arc from $A_1 = A$ to B in the flight time T . With an additional remark: the direction of rotation must be set according to the sign of $\theta_B - \theta_A$, clockwise for $\theta_B - \theta_A < 0$ and counterclockwise for $\theta_B - \theta_A > 0$. Finally, for $0 < \lambda < 1$ one checks that $\Phi(\lambda, v_0) = 0$ if and only if v_0 is the final velocity of an arc from A_λ to B in the flight time T (the previous comments on the direction of the rotation still apply). See Fig. 1(b).

Theorem 1.1(b) follows by applying Lemma 5.1 to this function. In order to check assumption (a) we observe that if $v_0 \in \mathcal{U}$ satisfies that $\Phi(0, v_0) = 0$ then $v_0 = \mu w_0$ for some $\mu \in \mathbb{R}$ and our solution is rectilinear. Corollary 3.3 then implies that there exists a unique solution of this type, which is nondegenerate by Lemma 3.4. Assumption (b) is ensured by Proposition 4.2, while (c) is actually a consequence of Corollary 4.4. The proof is complete. \square

6 Nonrectilinear solutions are defined for all (past) time

The goal of this section is to prove Proposition 4.1; with this purpose we assume $[\mathbf{D}_1]$ but not $[\mathbf{D}_2]$. Let us go to polar coordinates and replace the cartesian dependent variables $x \in \mathbb{R}^2 \setminus \{0\}$ by $r := |x| > 0$ and $\theta := \arg(x) \in \mathbb{R}/2\pi\mathbb{Z}$. The equations of motion are simpler if one introduces the angular momentum $c := \det(x, \dot{x}) = r^2\dot{\theta}$, which is another function of time. System (K) yields

$$\ddot{r} + D(x)\dot{r} + \frac{1}{r^2} - \frac{c^2}{r^3} = 0, \quad \dot{c} + D(x)c = 0,$$

or, equivalently (assuming that $x = x(t)$ is defined on the time interval $] \alpha, 0]$ for some $\alpha < 0$),

$$\frac{d}{dt}(p\dot{r}) = p \left(\frac{c^2}{r^3} - \frac{1}{r^2} \right), \quad c(t) = \frac{c(0)}{p(t)}, \quad (14)$$

the function $p = p(t) > 0$ being given by (1). We shall also introduce the *amended potential and total energies*

$$v := -\frac{1}{r} + \frac{c^2}{2r^2}, \quad h := \frac{\dot{r}^2}{2} + v, \quad (15)$$

which are again functions of t . A direct computation shows that $\dot{h} = -D(x)(\dot{r}^2 + c^2/r^2) \leq 0$, and thus, h is decreasing. Through Sections 6-7, *we shall think of the six functions r, θ, c, p, v, h as associated to the solution x .*

Proof of Proposition 4.1. We follow along the lines of [17, Proposition 2.1], which deals with the special case $D \equiv \text{const}$. See also [18, Proposition 2.1], where solutions are shown to be globally-defined in the future under the assumption that $D = D(|x|)$ depends only on the height of the particle.

We shall argue by contradiction and assume that $\alpha > -\infty$. Lemma 2.1 then states that $\liminf_{t \downarrow \alpha} r(t) = 0$. In view of (14) one has the inequality

$$\frac{d}{dt} \left(p(t)\dot{r}(t) \right) \geq p(t) \left(\frac{c(0)^2}{r(t)^3} - \frac{1}{r(t)^2} \right), \quad t \in] \alpha, 0]. \quad (16)$$

The function $\varrho \mapsto c(0)^2/\varrho^3 - 1/\varrho^2$ is bounded from below on $]0, +\infty[$. Together with (2) we see that there exists some constant $M > 1$ such that

$$\frac{d}{dt} \left(p(t)\dot{r}(t) \right) \geq -M, \quad t \in] \alpha, 0]. \quad (17)$$

On the other hand, r is positive and $\liminf_{t \downarrow \alpha} r(t) = 0$, and we deduce that

$$\limsup_{t \downarrow \alpha} \left(p(t)\dot{r}(t) \right) \geq 0. \quad (18)$$

The combination of (17)-(18) implies that

$$p(t)\dot{r}(t) \geq -M(t - \alpha), \quad t \in] \alpha, 0],$$

and consequently (by (2)),

$$\dot{r}(t) \geq -\frac{M(t - \alpha)}{p(t)} \geq M\alpha e^{-D_*\alpha}, \quad t \in] \alpha, 0].$$

We have shown that \dot{r} is bounded from below on $] \alpha, 0]$. It implies that the limit $\lim_{t \downarrow \alpha} r(t)$ exists, and since $\liminf_{t \downarrow \alpha} r(t) = 0$ it follows that $\lim_{t \downarrow \alpha} r(t) = 0$. Therefore, (16) gives

$$\lim_{t \downarrow \alpha} \frac{d}{dt} (p(t) \dot{r}(t)) = +\infty,$$

and so, the function $t \mapsto p(t) \dot{r}(t)$ is strictly increasing for t near α . Together with (18) we deduce that $\dot{r}(t) > 0$ for t sufficiently close to α . Let now $\hat{v}, \hat{h} :] \alpha, 0] \rightarrow \mathbb{R}$ be defined by

$$\hat{v}(t) := \frac{c(0)^2}{2r(t)^2} - \frac{1}{r(t)}, \quad \hat{h}(t) := \frac{1}{2} \dot{r}(t)^2 + \hat{v}(t),$$

and observe that

$$\lim_{t \downarrow \alpha} \hat{v}(t) = \lim_{t \downarrow \alpha} \hat{h}(t) = +\infty.$$

In particular, $\hat{v}(t) > 0$ for t close to α , and one has

$$\frac{d\hat{h}}{dt} = \dot{r}(t) \left[-D(x(t)) \dot{r}(t) + \frac{c(t)^2 - c(0)^2}{r(t)^3} \right] \geq -D(x(t)) \dot{r}(t)^2 \geq -2D_* \hat{h}(t),$$

or, what is the same, $\frac{d}{dt} \left[e^{2D_* t} \hat{h}(t) \right] \geq 0$, contradicting the previous observation that $\hat{h}(t) \rightarrow +\infty$ as $t \downarrow \alpha$. This contradiction concludes the proof. \square

7 Bounds for the arrival velocity when connecting two given heights in a given time

The goal of this section is to prove Proposition 4.2, which ensures the boundedness of the set of arrival velocities of solutions travelling between two given heights in a given time lapse. As in the previous section, we assume $[\mathbf{D}_1]$, but not $[\mathbf{D}_2]$.

As a preliminary remark we point out that in the case of rectilinear solutions this statement was already proved in Corollary 3.3. Therefore, it only remains to check Proposition 4.2 in the nonrectilinear situation. Remembering Proposition 4.1, nonrectilinear solutions are globally defined in the past.

We may divide our task into smaller steps using polar coordinates $x = r e^{i\theta}$. Since

$$|\dot{x}(0)| = \sqrt{\dot{r}(0)^2 + r(0)^2 \dot{\theta}(0)^2} = \sqrt{\dot{r}(0)^2 + c(0)^2 / r_B^2},$$

we see that the nonrectilinear case of Proposition 4.2 can be equivalently reformulated as follows:

Proposition 4.2*. *For any $r_A, r_B, T > 0$ there exists some $M > 0$ such that whenever $x = x(t)$ is a nonrectilinear solution of (K)-(12) then*

$$(i) \quad \dot{r}(0) \geq -M, \quad (ii) \quad \dot{r}(0) \leq M, \quad (iii) \quad |c(0)| \leq M.$$

The goal of the remaining of this section is to prove the three assertions of Proposition 4.2*. We begin with a result that clearly implies Proposition 4.2*(i).

Lemma 7.1. *Let $r_B, T > 0$ be given, and let $x_n = x_n(t)$, $t \in [-T, 0]$, be a sequence of solutions of (K) with*

$$r_n(0) = r_B, \quad \dot{r}_n(0) \rightarrow -\infty.$$

Then, $\dot{r}_n(t) \rightarrow -\infty$ as $n \rightarrow +\infty$, uniformly with respect to $t \in [-T, 0]$.

Proof. After possibly discarding a finite number of terms there is no loss of generality in assuming that $\dot{r}_n(0) < 0$ for every $n \in \mathbb{N}$. We set

$$\tau_n := \min \{t \in [-T, 0] : \dot{r}_n(s) \leq 0 \ \forall s \in [t, 0]\},$$

and consider the functions $p_n : [\tau_n, 0] \rightarrow \mathbb{R}$ defined as in (1) for $x = x_n$. Recalling (14), for each $n \in \mathbb{N}$ one has

$$\frac{d}{dt}(p_n(t)\dot{r}_n) \geq -\frac{p_n(t)}{r_n^2} \geq -\frac{1}{r_B^2}, \quad \tau_n \leq t \leq 0,$$

and integration gives

$$\dot{r}_n(t) \leq p_n(t)\dot{r}_n(t) \leq \dot{r}_n(0) + \frac{T}{r_B^2}, \quad \tau_n \leq t \leq 0.$$

In particular, for n big enough $\dot{r}_n(\tau_n) < 0$ and we deduce that $\tau_n = -T$. Thus, $\dot{r}_n(t) \rightarrow -\infty$, uniformly with respect to $t \in [-T, 0]$. It concludes the proof. \square

Our next task will consist in establishing the upper bounds announced in assertion (ii) of Proposition 4.2*. This fact will arise from Lemma 7.1 and statement (iii) of the following

Lemma 7.2. *Let $x_n :]-\infty, 0] \rightarrow \mathbb{R}^2 \setminus \{0\}$ be a sequence of nonrectilinear solutions of (K) satisfying, for some $r_* > 0$,*

$$r_n(0) = r_*, \quad \dot{r}_n(0) \rightarrow +\infty.$$

Then, for n big enough the following hold:

- (i) *There exists some $\mathfrak{t}_n < 0$ such that $\dot{r}_n(\mathfrak{t}_n) = 0$. Moreover, $\lim_{n \rightarrow +\infty} \mathfrak{t}_n = 0$.*
- (ii) *$\dot{r}_n(t) > 0$ for all $t \in]\mathfrak{t}_n, 0[$, and $\dot{r}_n(t) < 0$ for all $t < \mathfrak{t}_n$.*
- (iii) *There exists some $\tau_n < \mathfrak{t}_n$ such that $r_n(\tau_n) = r_*$. Moreover, $\lim_{n \rightarrow +\infty} \tau_n = 0$ and $\lim_{n \rightarrow +\infty} \dot{r}_n(\tau_n) = -\infty$.*

In the proof of Lemma 7.2 we shall need the following result. Here, the potential and total energies $v = v(t)$, $h = h(t)$ are defined as in (15).

Lemma 7.3. *Let $a < b$ be given numbers and let $x : [a, b] \rightarrow \mathbb{R} \setminus \{0\}$ be a nonrectilinear solution of (K) satisfying*

$$\dot{r}(t) \geq 0 \ \forall t \in [a, b], \quad v(b) \geq 0.$$

Then $\dot{v} \leq 0$ on $[a, b]$.

Proof. A direct computation gives

$$\begin{aligned} \dot{v}(t) &= \dot{r}(t) \left(\frac{1}{r(t)^2} - \frac{c(t)^2}{r(t)^3} \right) + \frac{c(t)\dot{c}(t)}{r(t)^2} \leq \dot{r}(t) \left(\frac{1}{r(t)^2} - \frac{c(t)^2}{r(t)^3} \right) \leq \frac{\dot{r}(t)}{r(t)} \left(\frac{1}{r(t)} - \frac{c(b)^2}{r(t)^2} \right) \leq \\ &\leq -\frac{\dot{r}(t)}{r(t)} \left(\frac{c(b)^2}{2r(t)^2} - \frac{1}{r(t)} \right) \end{aligned}$$

The function $\varrho > 0 \mapsto c(b)^2/(2\varrho^2) - 1/\varrho$ vanishes only at $\varrho = c(b)^2/2$, is positive for smaller ϱ and negative for bigger ϱ . By assumption it is nonnegative at $\varrho = r(b)$ and hence

$$0 < r(t) \leq r(b) \leq \frac{1}{2c(b)^2}, \quad \frac{c(b)^2}{2r(t)^2} - \frac{1}{r(t)} \geq 0, \quad \text{for any } t \in]a, b[,$$

thus concluding the proof. \square

Proof of Lemma 7.2. (i): We use a contradiction argument: should this statement fail to hold, after possibly passing to a subsequence it would be possible to find some $\epsilon > 0$ such that $\dot{r}_n(t) > 0$ for every $t \in [-\epsilon, 0]$. Motivated by Lemma 7.3 we distinguish three possibilities:

- (a) $v_n(0) \rightarrow +\infty$. By Lemma 7.3, $v_n(t) \rightarrow +\infty$ as $n \rightarrow +\infty$, uniformly with respect to $t \in [-\epsilon, 0]$, and in particular, $v_n(t) > 0$ for all $t \in [-\epsilon, 0]$ and n big enough. We define $p_n : [-\epsilon, 0] \rightarrow \mathbb{R}$ as in (1) for $x = x_n$ and observe that, by (14),

$$\frac{d}{dt} \left(p_n(t) \dot{r}_n(t) \right) = p_n(t) \left[\frac{c_n(t)^2}{r_n(t)^3} - \frac{1}{r_n(t)^2} \right] \geq \frac{p_n(t)}{r_n(t)} v_n(t) \geq \frac{e^{-D_* \epsilon}}{r_*} v_n(t) \rightarrow +\infty \text{ as } n \rightarrow +\infty, \quad (19)$$

uniformly with respect to $t \in [-\epsilon, 0]$. Since $p_n(-\epsilon) \dot{r}_n(-\epsilon) > 0$ we see that $p_n(t) \dot{r}_n(t) \rightarrow +\infty$ as $n \rightarrow +\infty$, uniformly with respect to $t \in [-\epsilon/2, 0]$. It follows that $\dot{r}_n(t) \rightarrow +\infty$ uniformly with respect to $t \in [-\epsilon/2, 0]$, and so

$$r_* = r_n(0) = r_n(-\epsilon/2) + \int_{-\epsilon/2}^0 \dot{r}_n(s) ds \rightarrow +\infty,$$

a contradiction.

- (b) $\{v_n(-\epsilon)\}$ bounded from above. By Lemma 7.3, the sequence $\{v_n(t)\}$ is bounded from above, uniformly with respect to $t \in [-\epsilon, 0]$. On the other hand, for each $n \in \mathbb{N}$ we have

$$\frac{1}{2} \dot{r}_n(t)^2 = h_n(t) - v_n(t) \geq h_n(0) - v_n(t) \geq \frac{1}{2} \dot{r}_n(0)^2 - v_n(t), \quad t \in [-\epsilon, 0],$$

and we deduce that $\dot{r}_n(t) \rightarrow +\infty$ as $n \rightarrow +\infty$, uniformly with respect to $t \in [-\epsilon, 0]$. As before, it implies that $r_* = r_n(0) \rightarrow +\infty$ as $n \rightarrow +\infty$, a contradiction.

- (c) $v_n(-\epsilon) \rightarrow +\infty$ but $\{v_n(0)\}$ is bounded from above. Since, moreover, $v_n(0) \geq -1/r_*$, we see that $h_n(0) = \dot{r}_n(0)^2/2 + v_n(0) \rightarrow +\infty$. Then, for n big enough

$$v_n(0) < \frac{1}{2} \min(v_n(-\epsilon), h_n(0)) < v_n(-\epsilon),$$

and there exists some $s_n \in]-\epsilon, 0[$ such that $v_n(s_n) = \min(v_n(-\epsilon), h_n(0))/2$. Thus, $v_n(s_n) \rightarrow +\infty$ as $n \rightarrow +\infty$ and $v_n(s_n) > 0$ for n big enough. Lemma 7.3 implies that

$$v_n(t) \geq v_n(s_n) \text{ if } -\epsilon \leq t \leq s_n; \quad v_n(t) \leq v_n(s_n) \text{ if } s_n \leq t \leq 0.$$

Now, arguing as in case (a), for $t \in [-\epsilon, s_n]$ we have

$$\frac{d}{dt} \left(p_n(t) \dot{r}_n(t) \right) \geq \frac{p_n(t)}{r_n(t)} v_n(t) \geq \frac{e^{-D_* \epsilon}}{r_*} v_n(s_n) \rightarrow +\infty \text{ as } n \rightarrow +\infty,$$

from where it follows that $s_n \rightarrow -\epsilon$ as $n \rightarrow +\infty$. On the other hand, repeating the argument of case (b), for $t \in [s_n, 0]$ one has

$$\frac{1}{2} \dot{r}_n(t)^2 = h_n(t) - v_n(t) \geq h_n(0) - v_n(s_n) \geq \frac{1}{2} h_n(0) \rightarrow +\infty \text{ as } n \rightarrow +\infty,$$

and we deduce that $s_n \rightarrow 0$, a new contradiction. These three contradictions prove (i).

(ii): Let the sequence $\{\mathbf{t}_n\}$ be as given by (i). After possibly replacing \mathbf{t}_n with

$$\mathbf{t}_n^* := \max\{t < 0 : \dot{r}_n(t) = 0\},$$

there is no loss of generality in assuming that $\dot{r}_n(t) > 0$ for all $t \in]\mathbf{t}_n, 0[$. Since each $h_n = h_n(t)$ is a decreasing function of t , we see that

$$v_n(\mathbf{t}_n) = h_n(\mathbf{t}_n) \geq h_n(0) \geq \frac{1}{2}\dot{r}_n(0)^2 - \frac{1}{r_*} \rightarrow +\infty \quad \text{as } n \rightarrow +\infty.$$

Thus, $v_n(\mathbf{t}_n) > 0$ for n big enough, and we see that

$$\ddot{r}_n(\mathbf{t}_n) = -D(x_n(\mathbf{t}_n))\dot{r}_n(\mathbf{t}_n) + \frac{c_n(\mathbf{t}_n)^2}{r_n(\mathbf{t}_n)^3} - \frac{1}{r_n(\mathbf{t}_n)^2} \geq \frac{v_n(\mathbf{t}_n)}{r_n(\mathbf{t}_n)} > 0,$$

and we deduce that $\dot{r}_n(t) < 0$ for $0 < \mathbf{t}_n - t$ small. If there were some $\mathbf{s}_n < \mathbf{t}_n$ such that

$$\dot{r}_n(\mathbf{s}_n) = 0, \quad \dot{r}_n(t) < 0 \text{ for every } t \in]\mathbf{s}_n, \mathbf{t}_n[,$$

then Lemma 7.3 would imply that $v_n(\mathbf{s}_n) \geq v_n(\mathbf{t}_n) > 0$, and arguing as above, $\ddot{r}_n(\mathbf{s}_n) > 0$. This is a contradiction and concludes the proof.

(iii) Notice that $r_n(\mathbf{t}_n) < r_*$. We set

$$\tau_n := \inf\{t \in]-\infty, \mathbf{t}_n[: r_n(t) < r_*\} \in]-\infty, \mathbf{t}_n[, \quad \mathbf{t}_n := \lim_{t \downarrow \tau_n} r_n(t) \leq r_*,$$

and write

$$\dot{r}_n(t) := \begin{cases} -\varphi_n(r_n(t)) & \text{if } \tau_n < t < \mathbf{t}_n, \\ \psi_n(r_n(t)) & \text{if } \mathbf{t}_n < t < 0. \end{cases}$$

The functions $\varphi_n :]r_n(\mathbf{t}_n), \mathbf{t}_n[\rightarrow \mathbb{R}$ and $\psi_n :]r_n(\mathbf{t}_n), r_*[\rightarrow \mathbb{R}$ are continuous, and since all energy functions $h_n = h_n(t)$ are decreasing we see that $0 < \psi_n(r) \leq \varphi_n(r)$ for every $r \in]r_n(\mathbf{t}_n), \mathbf{t}_n[$. Thus,

$$\mathbf{t}_n - \tau_n = \int_{\tau_n}^{\mathbf{t}_n} dt = - \int_{\tau_n}^{\mathbf{t}_n} \frac{\dot{r}_n(t)}{\varphi_n(r_n(t))} dt = \int_{r_n(\mathbf{t}_n)}^{\mathbf{t}_n} \frac{1}{\varphi_n(r)} dr \leq \int_{r_n(\mathbf{t}_n)}^{r_*} \frac{1}{\psi_n(r)} dr = -\mathbf{t}_n,$$

i.e., $2\mathbf{t}_n \leq \tau_n < 0$. In particular, $-\infty < \tau_n \rightarrow 0$ and we see that $\mathbf{t}_n = r_n(\mathbf{t}_n) = r_*$. The result follows. \square

We conclude this section by checking assertion (iii) of Proposition 4.2*. Having already shown the previous statements (i)-(ii), the remaining work is collected in the following

Lemma 7.4. *Let $T, r_A, r_B > 0$ be given, and let $x_n : [-T, 0] \rightarrow \mathbb{R}^2 \setminus \{0\}$ be a sequence of nonreciprocal solutions of (K)-(12) such that $\{\dot{r}_n(0)\}$ is bounded. Then $\{c_n(0)\}$ is bounded.*

Proof. We distinguish two cases:

Assume firstly that $\max_{-T/2 \leq t \leq 0} r_n(t) \rightarrow +\infty$. Since $r_n(0) = r_B$ for every n , there exists a sequence $\{t_n\}_n \subset [-T/2, 0]$ with $r_n(t_n) \geq 1$ for every n and $\dot{r}_n(t_n) \rightarrow -\infty$. Thus, Lemma 7.1 states that $\max_{t \in [-T, t_n]} \dot{r}_n(t) \rightarrow -\infty$, implying that $r_A = r_n(-T) \rightarrow +\infty$, a contradiction.

The other possibility is that, after possibly passing to a subsequence, there exists some constant $\varrho > 0$ such that $r_n(t) \leq \varrho$ for every $t \in [-T/2, 0]$ and for every $n \in \mathbb{N}$. The left hand side of (14) gives

$$\frac{d}{dt}(p_n(t)\dot{r}_n(t)) = p_n(t) \left(\frac{c_n(t)^2 - r_n(t)}{r_n(t)^3} \right) \geq p_n(t) \left(\frac{c_n(t)^2 - \varrho}{r_n(t)^3} \right), \quad t \in [-T/2, 0], \quad n \in \mathbb{N}.$$

Using a contradiction argument we assume that $|c_n(0)| \rightarrow +\infty$. The right hand side of (14) implies that $|c_n(t)| \geq |c_n(0)| \rightarrow +\infty$ for any $t \in [-T/2, 0]$. In particular, for n big enough, $c_n(t)^2 - \varrho \geq c_n(0)^2 - \varrho > 0$ for all $t \in [-T/2, 0]$ and one has:

$$\gamma_n := \min_{t \in [-T/2, 0]} \left[\frac{d}{dt}(p_n(t)\dot{r}_n(t)) \right] \geq \min_{t \in [-T/2, 0]} \left[p_n(t) \left(\frac{c_n(0)^2 - \varrho}{\varrho^3} \right) \right] \rightarrow +\infty \text{ as } n \rightarrow +\infty.$$

On the other hand, integration gives

$$p_n(t)\dot{r}_n(t) \leq \dot{r}_n(0) + \gamma_n t \leq |\dot{r}_n(0)| + \gamma_n t, \quad t \in \left[-\frac{T}{2}, 0 \right],$$

and thus (by (2)), $\dot{r}_n(t) \leq |\dot{r}_n(0)|e^{D_*T/2} + \gamma_n e^{D_*T/2}t$ for every $t \in [-T/2, 0]$. Integrating again we find that $r_n(-T/2) \rightarrow +\infty$, contradicting our assumption that $\{r_n\}$ was uniformly bounded on $[-T/2, 0]$. It proves the result. \square

8 Colliding solutions of the Kepler problem

In this section we continue the study, initiated in Section 3, of the rectilinear solutions of the frictional Kepler problem (K). It is divided into two subsections. In the first one we explore the behavior of rectilinear solutions in the vicinity of a collision, and in the second we introduce the Levi-Civita regularization for the rectilinear problem and prove the divergence of some Sundman integrals. Together they will pave the way to complete the proof of Proposition 4.3 in Section 9.

Thus, throughout this section we go back to equation (5). As in Section 3 it will be assumed that $\delta :]0, +\infty[\rightarrow \mathbb{R}$ is *nonnegative, bounded and continuously differentiable*. In addition, in the second subsection we shall further introduce some growth conditions on $d\delta/dr$ near the origin (corresponding to $[\mathbf{D}_2]$).

8.1 Asymptotics near collisions

Let $r :]\alpha, \omega[\rightarrow \mathbb{R}$ be a solution of (5), assumed maximal *both to the left and to the right*. Its associated energy is the function $h :]\alpha, \omega[\rightarrow \mathbb{R}$ defined by

$$h(t) := \frac{1}{2}\dot{r}(t)^2 - \frac{1}{r(t)}.$$

Notice that h is decreasing. In fact, a direct computation gives

$$\dot{h}(t) = -\delta(r(t))\dot{r}(t)^2 \leq 0. \tag{20}$$

Lemma 8.1. *If $\alpha > -\infty$, then the following hold:*

(i) $\lim_{t \downarrow \alpha} r(t) = 0.$

(ii) $h(\alpha) := \lim_{t \downarrow \alpha} h(t) < +\infty$.

(iii) $\lim_{t \downarrow \alpha} \frac{r(t)}{(t - \alpha)^{2/3}} = \sqrt[3]{\frac{9}{2}}$.

Proof. (i): It follows from equation (5) that at a critical point t_0 , $\ddot{r}(t_0) < 0$ and so r attains a strict local maximum (in particular, r has at most a critical point). In combination with Lemma 2.1 it implies that $\lim_{t \downarrow \alpha} r(t) = 0$, as claimed.

(ii) The previous argument actually gives some further information: there exists some $\alpha < t_0 < \omega$ such that $\dot{r}(t) > 0$ for all $t \in]\alpha, t_0[$. Using a contradiction argument we assume that $\lim_{t \downarrow \alpha} h(t) = +\infty$; then, after possibly replacing t_0 by a smaller number there is no loss of generality in further assuming that $h(t) > 1$ for all $t \in]\alpha, t_0[$. Setting $r_0 := r(t_0) > 0$ we see that there exists a C^1 function $\varphi :]0, r_0[\rightarrow]1, +\infty[$, $\varphi = \varphi(r)$, such that $h(t) = \varphi(r(t))$ for any $t \in]\alpha, t_0[$. Since

$$\dot{r}(t) = \sqrt{2} \sqrt{\varphi(r(t)) + \frac{1}{r(t)}}, \quad \alpha < t < t_0, \quad (21)$$

differentiation and comparison with (20) yields

$$0 \geq \frac{d\varphi}{dr}(r(t)) = -\delta(r(t))\dot{r}(t) \geq -D_*\dot{r}(t) = -\sqrt{2}D_*\sqrt{\varphi(r(t)) + \frac{1}{r(t)}}, \quad \alpha < t < t_0,$$

where $D_* \geq 0$ is an upper bound of δ . Therefore,

$$\frac{d\varphi}{dr}(r) \geq -\sqrt{2}D_*\sqrt{\varphi(r) + \frac{1}{r}} \geq -\sqrt{2}D_*\left(\sqrt{\varphi(r)} + \sqrt{\frac{1}{r}}\right) \geq -\sqrt{2}D_*\left(\varphi(r) + \frac{1}{\sqrt{r}}\right),$$

for any $0 < r < r_0$. Thus,

$$\frac{d}{dr}\left(e^{\sqrt{2}D_*r}\varphi(r)\right) \geq -\frac{\sqrt{2}D_*e^{\sqrt{2}D_*r}}{\sqrt{r}}, \quad 0 < r < r_0,$$

which is not possible since $\lim_{r \downarrow 0} e^{\sqrt{2}D_*r}\varphi(r) = +\infty$ but the right hand side of the inequality is integrable on $]0, r_0[$. This contradiction concludes the proof.

(iii) By combining (21) and (ii) we see that

$$\lim_{t \downarrow \alpha} \sqrt{r(t)} \dot{r}(t) = \sqrt{2},$$

so that, by L'Hopital rule,

$$\lim_{t \downarrow \alpha} \frac{r(t)^{3/2}}{t - \alpha} = \frac{3}{\sqrt{2}},$$

implying the statement. □

Asymptotics of type (iii) were already obtained by Sperling [23] (see also [19, pp. 152-153]) for the forced Kepler problem; however Sperling's results do not apply here directly since the friction force $-\delta(r)\dot{r}$ may not be bounded near the collision. We also remark that the corresponding version of Lemma 8.1 when $t \uparrow \omega$ still holds if $\omega < +\infty$. In fact, statements (i) and (iii) can be readily translated to this situation with the same proofs. The situation in case (ii) is different: the result is still true but it needs a new proof. We shall skip the details since this is not needed in this paper; nevertheless, we emphasize the following consequence of the proofs of statements (i) and (iii) of Lemma 8.1:

Corollary 8.2. *If $\omega < +\infty$ and $h(\omega) := \lim_{t \uparrow \omega} h(t) > -\infty$, then $\lim_{t \uparrow \omega} \frac{r(t)}{(\omega - t)^{2/3}} = \sqrt[3]{\frac{9}{2}}$.*

We close this subsection with a result that estimates the length of the maximal definition interval of a solution from the final value (assumed finite) of its energy:

Lemma 8.3. *Assume that $-\infty < \alpha < \omega < +\infty$ and $h(\omega) > -\infty$. Then, $h(\omega) < 0$ and $\omega - \alpha > \frac{1}{2\sqrt{-h(\omega)^3}}$.*

Proof. Corollary 8.2 implies in particular that $\lim_{t \uparrow \omega} r(t) = 0$. Together with Lemma 8.1(i) we see that there exists some point $t_0 \in]\alpha, \omega[$ such that $\dot{r}(t_0) = 0$. Since h is decreasing, $h(t_0) = -1/r(t_0) \geq h(\omega)$; thus, $h(\omega) < 0$ and $r_0 := r(t_0) \geq -1/h(\omega)$.

The proof of Lemma 8.1(i) implies that $\dot{r}(t) < 0$ for every $t \in]t_0, \omega[$. It follows that there exists an unique point $t_1 \in]t_0, \omega[$ such that $r(t_1) = r_0/2$, and we see that

$$\ddot{r}(t) \geq -\frac{1}{r(t)^2} > -\frac{4}{r_0^2}, \quad t_0 < t < t_1,$$

and integration gives

$$\frac{r_0}{2} = r(t_1) = r_0 + \int_{t_0}^{t_1} \dot{r}(s) ds = r_0 + \int_{t_0}^{t_1} (t_1 - s) \ddot{r}(s) ds > r_0 - \frac{4}{r_0^2} \int_{t_0}^{t_1} (t_1 - s) ds = r_0 - \frac{2}{r_0^2} (t_1 - t_0)^2,$$

implying that

$$\omega - \alpha > t_1 - t_0 > \frac{\sqrt{r_0^3}}{2} \geq \frac{1}{2\sqrt{-h(\omega)^3}},$$

and thus concluding the proof. \square

8.2 The Levi-Civita regularization for the rectilinear Kepler problem

In this third subsection we continue to assume that $\delta :]0, +\infty[\rightarrow \mathbb{R}$ is nonnegative, bounded and continuously differentiable, but in addition, mimicking $[\mathbf{D}_2]$ we introduce the hypothesis

$$\lim_{r \downarrow 0} \sqrt{r} \frac{d\delta}{dr}(r) = 0. \quad (22)$$

Let $r :]\alpha, 0] \rightarrow]0, +\infty[$ be a solution of (5) and set

$$s(t) := - \int_t^0 \frac{d\tau}{r(\tau)}, \quad \alpha < t \leq 0, \quad A := \lim_{t \downarrow \alpha} s(t) < 0.$$

Then, the pair of functions $(u, E) :]A, 0] \rightarrow]0, +\infty[\times \mathbb{R}$ defined by

$$u(s(t)) := \sqrt{r(t)}, \quad E(s(t)) := h(t), \quad \alpha < t \leq 0,$$

solves the system

$$\begin{cases} u'' + \delta(u^2)u^2u' = \frac{Eu}{2}, \\ E' = -4\delta(u^2)(u')^2, \end{cases} \quad (LC)$$

on the invariant manifold

$$\mathcal{M} := \left\{ (u, u', E) \in]0, +\infty[\times \mathbb{R} \times \mathbb{R} : 2(u')^2 - Eu^2 = 1 \right\},$$

(see [17]). Here and in what follows we denote by primes the derivatives with respect to the Levi-Civita time s ; derivatives with respect to t are denoted by dots.

Conversely, given a solution $(u, E) :]A, 0] \rightarrow]0, +\infty[\times \mathbb{R}$ of (LC) on \mathcal{M} we set

$$t(s) := \int_0^s u(\sigma)^2 d\sigma, \quad A < s \leq 0, \quad \alpha := \lim_{s \downarrow A} t(s) < 0,$$

and we see that the function $r = r(t)$ defined on the time interval $] \alpha, 0]$ by

$$r(t(s)) := u(s)^2, \quad A < s \leq 0,$$

is a solution of (5) with energy $h :] \alpha, 0] \rightarrow \mathbb{R}$ given by $h(t(s)) = E(s)$ for any $s \in]A, 0]$.

In this situation we shall say that (u, E) is the Goursat transform of r , and r is the Levi-Civita transform of (u, E) . The Goursat and Levi-Civita transforms define mutually-inverse, bijective correspondences from the set of solutions $r = r(t) > 0$ of (5) defined on $] \alpha, 0]$ for some $-\infty \leq \alpha < 0$, into the set of solutions $(u, E) = (u(s), E(s)) \in]0, +\infty[\times \mathbb{R}$ of (LC) on \mathcal{M} defined on $]A, 0]$ for some $-\infty \leq A < 0$.

Notice now that (LC) is naturally defined for $(u, E) \in \mathbb{R}^2$ and does not require that u be positive. In fact, the map $u \in \mathbb{R} \mapsto \delta(u^2)$ is continuously differentiable by assumption (22). This is the motivation behind the following statement. Here $r :] \alpha, 0] \rightarrow]0, +\infty[$ is an arbitrary solution of (5) and $(u, E) :]A, 0] \rightarrow \mathbb{R}^2$ stands for its Goursat transform.

If (u, E) is maximal to the left as a solution of (LC) then r is also maximal to the left as a solution of (5). The converse is not true in general.

The first part of this assertion is clear. In order to support the last part we shall prove the following:

Lemma 8.4. *If $\dot{r}(t) > 0$ for every $t \in] \alpha, 0]$, then, $A > -\infty$ and (u, E) is not maximal to the left as a solution of (LC) .*

Proof. If the interval $] \alpha, 0]$ is not maximal to the left for the solution r of (5) then the result is clear. Thus, we may henceforth assume that $] \alpha, 0]$ is maximal to the left. Notice that

$$\ddot{r}(t) = -\delta(r(t))\dot{r}(t) - \frac{1}{r(t)^2} \leq 0, \quad t \in] \alpha, 0],$$

so that r is concave on $] \alpha, 0]$. Consequently, its graph stays below that of its tangent line at $t = 0$, i.e.,

$$r(t) \leq r(0) + \dot{r}(0)t, \quad t \in] \alpha, 0],$$

and since r is positive on $] \alpha, 0]$ we see that $\alpha > -\infty$. Lemma 8.1 then implies that $A > -\infty$, $\lim_{s \downarrow A} E(s) < +\infty$, and $\lim_{s \downarrow A} u(s) = 0$. Since (u, E) stays in \mathcal{M} we see that $\lim_{s \downarrow A} u'(s) = 1/\sqrt{2}$, and thus, the solution (u, E) can be extended to the left of A . It proves the result. \square

Lemma 8.5. *Let $(u, E) :]A, 0] \rightarrow \mathbb{R}^2$ be a solution of (LC) on \mathcal{M} , maximal to the left. If $A > -\infty$ then there exists some $s_1 \in]A, 0]$ such that $u'(s)u(s) < 0$ for all $s \in]A, s_1]$.*

Proof. We distinguish two cases depending on the sign of E .

Case I: There exists some $s_0 \in]A, 0]$ such that $E(s_0) \geq 0$. The second equation of (LC) implies that E is decreasing, and we deduce that $E(s) \geq 0$ for all $s \in]A, s_0]$. Since our solution stays on \mathcal{M} we see that $u'(s) \neq 0$ for all $s \in]A, s_0]$, and we deduce that there exists some $s_1 \in]A, s_0]$ such that $u(s) \neq 0$ for all $s \in]A, s_1]$.

If $u'(s)u(s) < 0$ for all $s \in]A, s_1]$ we are done; thus, let us assume that $u'(s)u(s) > 0$ for all $s \in]A, s_1]$. By introducing a translation in the time variable s there is no loss of generality in assuming that $s_1 = 0$, and after possibly replacing u by $-u$ we may assume that $u(s), u'(s) > 0$ for all $s \in]A, 0]$. The Levi-Civita transform $r :]\alpha, 0] \rightarrow]0, +\infty[$ of (u, E) then satisfies that $\dot{r}(t) > 0$ for all $t \in]\alpha, 0]$, and Lemma 8.4 implies that (u, E) is not maximal, a contradiction. It proves the result in this case.

Case II: $E(s) < 0$ for all $s \in]A, 0]$. Since E is decreasing on $]A, 0]$ it implies that E is bounded on $]A, 0]$. Since our solution stays in \mathcal{M} we see that $2u'(s)^2 \leq 1$ for all $s \in]A, 0]$, and thus, u' is also bounded on $]A, 0]$. Finally, integration implies that u is again bounded on $]A, 0]$, and system (LC) implies that u'' is also bounded on $]A, 0]$. The usual prolongation theory for ordinary differential equations implies that (u, E) is not maximal, a contradiction. It concludes the proof. \square

Lemma 8.6. *Let $(u, E) :]A, 0] \rightarrow \mathbb{R}^2$ be a solution of (LC) on the invariant manifold \mathcal{M} , maximal to the left. Then $\int_A^0 u(s)^2 ds = +\infty$.*

Proof. Let the strictly increasing function $t :]A, 0] \rightarrow \mathbb{R}$ be defined by $t(s) := \int_0^s u(\sigma)^2 d\sigma$, assume, by a contradiction argument, that $\alpha := \lim_{s \downarrow A} t(s) > -\infty$, and let the function $r :]\alpha, 0] \rightarrow \mathbb{R}$ be defined by $r(t(s)) := u(s)^2$. We shall distinguish three cases and find a contradiction in each of them.

(i) $A > -\infty$. Then, Lemma 8.5 states the existence of some $s_1 \in]A, 0]$ such that $u'(s)u(s) < 0$ for every $s \in]A, s_1]$. After a translation in the independent variable s and possibly replacing u by $-u$ we may assume that $s_1 = 0$ and $u'(s) < 0 < u(s)$ for all $s \in]A, 0]$. Thus, $r :]\alpha, 0] \rightarrow]0, +\infty[$ is the Levi-Civita transform of (u, E) , and in particular, it is a solution of (5), maximal to the left. It satisfies $\dot{r}(t) < 0$ for every $t \in]\alpha, 0]$, and Lemma 8.1(i) implies that $\alpha = -\infty$. It concludes the proof in this case.

(ii) $A = -\infty$ and u has infinitely many zeroes. The definition of the manifold \mathcal{M} implies that these zeroes are nondegenerate, and in particular isolated; thus, they make up an ordered sequence $\dots < s_2 < s_1 < s_0 \leq 0$. The sequence $t_i := t(s_i)$ is strictly decreasing and convergent, and for each $i \geq 1$ the restriction of r to $]t_i, t_{i-1}[$ is a maximal solution of (5) satisfying $h(t_{i-1}) > -\infty$. Then, Lemma 8.3 states that the sequence of energies $h(t_i) := \lim_{t \downarrow t_i} h(t)$ satisfies $h(t_i) \rightarrow -\infty$. On the other hand $h(t(s)) = E(s)$, and the second equation of system (LC) implies that $\{h(t_i)\}$ is increasing, a contradiction.

(iii) $A = -\infty$ and the set Z of zeroes of u in $] -\infty, 0]$ is finite. Write

$$\{-T, 0\} \cup t(Z) = \{-T = t_p < t_{p-1} < \dots < t_0 = 0\},$$

and observe that the restriction of r to each interval $]t_i, t_{i-1}[$ is a solution of (5) satisfying $h(t_{i-1}) > -\infty$. The combination of Lemma 8.1(iii) and Corollary 8.2 then implies that $\int_{t_i}^{t_{i-1}} \frac{1}{r(t)} dt < +\infty$ for each i , and therefore $\int_{-T}^0 \frac{1}{r(t)} dt < +\infty$, a contradiction. The proof is complete. \square

9 The Levi-Civita regularization for the planar Kepler problem

In this section we prove Proposition 4.3, which was key in our proof of Theorem 1.1. Henceforth we assume both $[\mathbf{D}_{1-2}]$.

The Levi-Civita regularization applies not only to the 1-dimensional Kepler problem (5) but also to the more general planar problem (K). The well-known procedure goes as follows. Let $x :]\alpha, 0] \rightarrow \mathbb{R}^2 \setminus \{0\} \equiv \mathbb{C} \setminus \{0\}$ be a given solution of (K), and let $w_0 \in \mathbb{C}$ be such that $w_0^2 = x(0)$. The transformation from $\mathbb{C} \setminus \{0\}$ to itself given by $z \mapsto z^2$ is a covering map, and thus, there exists a (unique) continuous lifting $z :]\alpha, 0] \rightarrow \mathbb{C} \setminus \{0\}$ with $z(0) = w_0$ and $z(t)^2 = x(t)$ for every $\alpha < t \leq 0$. Let $h, s :]\alpha, 0] \rightarrow \mathbb{R}$ (energy and new time) be defined by

$$h(t) := \frac{1}{2} |\dot{x}(t)|^2 - \frac{1}{|x(t)|}, \quad s(t) := \int_0^t \frac{1}{|x(\tau)|} d\tau.$$

Then, setting $A := \lim_{t \downarrow \alpha} s(t)$ we see that the pair of functions $(w, E) :]A, 0] \rightarrow (\mathbb{C} \setminus \{0\}) \times \mathbb{R}$ defined by

$$w(s(t)) := z(t), \quad E(s(t)) := h(t),$$

solves the system

$$(\widehat{LC}) \begin{cases} w'' + D(w^2)|w|^2 w' = \frac{Ew}{2} \\ E' = -4D(w^2)|w'|^2 \end{cases}, \quad w \in \mathbb{C}, E \in \mathbb{R},$$

on the invariant manifold

$$\widehat{\mathcal{M}} := \left\{ (w, w', E) \in (\mathbb{C} \setminus \{0\}) \times \mathbb{C} \times \mathbb{R} : 2|w'|^2 - E|w|^2 = 1 \right\}.$$

In addition, $w(0) = w_0$. Notice that the choice $-w_0$ instead of w_0 leads to the pair $(-w, E)$ in the place of (w, E) .

Conversely, given a solution $(w, E) :]A, 0] \rightarrow (\mathbb{C} \setminus \{0\}) \times \mathbb{R}$ of (\widehat{LC}) on $\widehat{\mathcal{M}}$, letting $t :]A, 0] \rightarrow \mathbb{R}$ be defined by

$$t(s) := \int_0^s |w(\sigma)|^2 d\sigma, \quad A < s \leq 0,$$

and setting $\alpha := \lim_{s \downarrow A} t(s)$, we see that the function $x = x(t)$ defined on $] \alpha, 0]$ by $x(t(s)) := w(s)^2$ is a solution of (K) with energy

$$h :]\alpha, 0] \rightarrow \mathbb{R}, \quad h(t(s)) = E(s) \text{ for any } A < s \leq 0,$$

and moreover, $x(0) = w_0^2$ for $w_0 := w(0)$.

Under these circumstances we shall say that (w, E) is the Goursat transform of (x, w_0) and (x, w_0) is the Levi-Civita transform of (w, E) . With this in mind, the Goursat and Levi-Civita transforms define mutually-inverse, bijective correspondences between the set of couples (x, w_0) , where $x = x(t)$ is a solution of (K) defined on $] \alpha, 0]$ for some $-\infty \leq \alpha < 0$ and w_0 is a choice of the square-root of $x(0)$, and the set of pairs (w, E) , where $(w, E) = (w(s), E(s)) \in (\mathbb{C} \setminus \{0\}) \times \mathbb{R}$ is a solution of (\widehat{LC}) on $\widehat{\mathcal{M}}$ defined on $]A, 0]$ for some $-\infty \leq A < 0$.

Observe also that the values of w, w', E at time $s = 0$ are related to those of x, \dot{x} at $t = 0$ as follows: $w(0)$ is a square root of $x(0)$, $w'(0) = \frac{|x(0)| \dot{x}(0)}{2w(0)}$, and $E(0) = \frac{|\dot{x}(0)|^2}{2} - \frac{1}{|x(0)|}$. This fact will be used later.

At this moment one might ask about the connections between the Goursat or Levi-Civita transforms (as defined in the previous section in the one dimensional situation), and the newly-defined notions of Goursat/Levi-Civita transforms for the planar problem. With this goal let $x(t) = r(t)v_0$, $t \in]\alpha, 0]$, be a rectilinear solution of (K) . We assume that $|v_0| = 1$ and r is positive, so that r must be a solution of (5) for $\delta(r) := D(rv_0)$. Then, letting (u, E) be the Goursat transform of r , and assuming that $v_1 \in \mathbb{C}$ is a square root of v_0 , the Goursat transform (w, E) of $(x, \sqrt{r(0)}v_1)$ is given by $w(s) = u(s)v_1$. Similarly, if $(w, E) :]-A, 0] \rightarrow (\mathbb{C} \setminus \{0\}) \times \mathbb{R}$ is a solution of (\widehat{LC}) on $\widehat{\mathcal{M}}$, and this solution is rectilinear in the sense that $w(s) = u(s)v_1$ for some function $u :]A, 0] \rightarrow]0, +\infty[$ and some $v_1 \in \mathbb{C}$ with $|v_1| = 1$, then (u, E) must be a solution of (LC) in \mathcal{M} for $\delta(u) := D(uv_1^2)$, and the Levi-Civita transform (x, w_0) of (w, E) is given by $x(t) = r(t)v_1^2$, where $r = r(t)$ is the Levi-Civita transform of (u, E) .

On the other hand, the initial conditions in the rectilinear case are related as follows: if $x(0) = x_0$ and $\dot{x}(0) = \lambda x_0$ for some $x_0 \in \mathbb{C} \setminus \{0\}$ and some $\lambda \in \mathbb{R}$, then $u(0) = |x_0|$, $u'(0) = |x_0|^{3/2}\lambda/2$, and $E(0) = \lambda^2|x_0|^2/2 - 1/|x_0|$.

We finally notice that, as a consequence of assumption $[\mathbf{D}_2]$, the map $\mathbb{C} \rightarrow \mathbb{R}$, $w \mapsto D(w^2)$ is continuously differentiable in the real sense (i.e., regarded as a map from \mathbb{R}^2 to \mathbb{R}). Thus, we can see (\widehat{LC}) as a C^1 , autonomous system on $\mathbb{C} \times \mathbb{R} \equiv \mathbb{R}^2 \times \mathbb{R}$.

Proof of Proposition 4.3. It suffices to check that given a converging sequence

$$\{(x_0^{(n)}, \dot{x}_0^{(n)})\}_n \rightarrow (x_0^*, \dot{x}_0^*),$$

with $(x_0^{(n)}, \dot{x}_0^{(n)}) \in \Omega$ for every $n \in \mathbb{N}$, $x_0^* \neq 0$, $\dot{x}_0^* = \lambda x_0^*$ for some $\lambda \in \mathbb{R}$, and $(x_0^*, \dot{x}_0^*) \in \partial\Omega$, then $\{\mathcal{X}(x_0^{(n)}, \dot{x}_0^{(n)})\}$ is convergent.

With this goal, for each natural index n we denote by $x_n :]\alpha_n, 0] \rightarrow \mathbb{R}^2 \setminus \{0\} \equiv \mathbb{C} \setminus \{0\}$ the solution of (K) with $x_n(0) = x_0^{(n)}$ and $\dot{x}^{(n)}(0) = \dot{x}_0^{(n)}$. The definition interval $]\alpha_n, 0]$ is chosen maximal to the left. The points $(x_0^{(n)}, \dot{x}_0^{(n)})$ being in Ω we see that $\alpha_n < -T$ for every $n \in \mathbb{N}$.

Similarly, we call $x_* :]\alpha_*, 0] \rightarrow \mathbb{C} \setminus \{0\}$ the solution (maximal to the left) of (K) with $x_*(0) = x_0^*$ and $\dot{x}_*(0) = \dot{x}_0^* = \lambda x_0^*$. The fact that $(x_0^*, \lambda x_0^*) \notin \Omega$ can be equivalently rewritten as $\alpha_* \geq -T$.

For each $n \in \mathbb{N}$ we choose some $w_0^{(n)} \in \mathbb{C}$ with $(w_0^{(n)})^2 = x_0^{(n)}$, and pick some $w_0^* \in \mathbb{C}$ such that $(w_0^*)^2 = x_0^*$. These choices are to be made in such a way that

$$w_0^{(n)} \rightarrow w_0^* \text{ as } n \rightarrow +\infty.$$

For each natural index $n \in \mathbb{N}$ we also denote by $(w_n, E_n) :]A_n, 0] \rightarrow \mathbb{C} \times \mathbb{R}$ the Goursat transform of $(x_n, w_0^{(n)})$. It is a solution of (\widehat{LC}) , and therefore, it can be extended to a possibly bigger interval $]\hat{A}_n, 0]$, maximal to the left. Similarly, the Goursat transform $(w_*, E_*) :]A_*, 0] \rightarrow \mathbb{C} \times \mathbb{R}$ of (x_*, w_0^*) will be extended to some (perhaps greater) interval $]\hat{A}_*, 0]$, maximal to the left. Notice that

$$w_*(s) = u_*(s) \frac{w_0^*}{|w_0^*|}, \quad s \in]\hat{A}_*, 0],$$

where $(u_*, E_*) :]\hat{A}_*, 0] \rightarrow \mathbb{R}^2$ is a maximal solution of (LC) for $\delta(u) := D(u x_0^*/|x_0^*|)$. Observe also that it stays on the invariant manifold \mathcal{M} .

Lemma 8.6 states that $\int_{\hat{A}_*}^0 |w_*(s)|^2 ds = \int_{\hat{A}_*}^0 u_*(s)^2 ds = +\infty$. Therefore, there exists a unique point $S_* \in]A_*, 0[$ such that $\int_{S_*}^0 u_*(s)^2 ds = T$. The remaining of this proof is devoted to show that

$$\mathcal{X}(x_0^{(n)}, \dot{x}_0^{(n)}) \rightarrow u_*(S_*)^2 \frac{x_0^*}{|x_0^*|} \quad \text{as } n \rightarrow +\infty.$$

With this goal we observe that

$$w_n(0) = w_0^{(n)} \rightarrow w_0^* = w_*(0), \quad \dot{w}_n(0) = \frac{|x_0^{(n)}| \dot{x}_0^{(n)}}{2w_0^{(n)}} \rightarrow \frac{|x_0^*| \dot{x}_0^*}{2w_0^*}, \quad \text{as } n \rightarrow +\infty,$$

and similarly, $E_n(0) \rightarrow E_*(0)$. Therefore, continuous dependence implies that $w_n(s) \rightarrow w_*(s)$ uniformly with respect to $s \in [S_* - 1, 0]$. In particular, for n big enough we see that $\int_{S_* - 1}^0 |w_n(s)|^2 ds > T$ and there exists some $S_n \in]S_* - 1, 0[$ such that $\int_{S_n}^0 |w_n(s)|^2 ds = T$. After possibly passing to a subsequence we may assume that $\{S_n\} \rightarrow S_{**} \in [S_* - 1, 0]$, and since $w_n \rightarrow w_*$ uniformly on $[S_* - 1, 0]$ we see that $T = \int_{S_n}^0 |w_n(s)|^2 ds \rightarrow \int_{S_{**}}^0 |w_*(s)|^2 ds$. Therefore, $\int_{S_{**}}^0 |w_*(s)|^2 ds = T$ and we deduce that $S_{**} = S_*$. Consequently, $\mathcal{X}(x_0^{(n)}, \dot{x}_0^{(n)}) = w_n(S_n)^2 \rightarrow w_*(S_*)^2$, thus concluding the proof. \square

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