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An Area Preserving Parametrization for Spherical Rectangles

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MOTIVATION

Computing time

Monte-Carlo renderers for realistic animations reproduce area lights, arbitrary BRDFs, depth-of-field, motion-blur, etc.

→ Need to define estimators for high-dimensional integrals.

→ Usually a **large number of samples** is required, each of them requires two-point visibility evaluation or finding first point in a ray.

→ Leads to a huge number of ray-scene intersection tests.

Even for highly optimized ray-tracing or ray-casting implementations:

→ **Computing time is dominated by ray-scene intersection.**

Sample selection and variance

Time devoted to ray-scene intersections can be lowered by using less rays:

→ This can be done by using PDFs with lower variance, which yield the same quality (noise) with less samples.

→ **Importance sampling** yields lower variance PDFs

→ **Stratified or low-discrepancy point sets** also produce lower variance.

A good solution is to spend extra computing time on more elaborate importance sampling PDFs while also adding stratification or low-discrepancy point sets:

→ **Extra time spent on these methods pays off because ray-casting computation time is reduced.**

Simultaneous stratified and importance sampling

Efficient algorithms for stratified or low-discrepancy sampling have been designed for simple integration domains.

→ Typically $[0,1]^n$, with $n = 2$ or $n = 3$

→ Examples: N-rooks, jittering, best-candidate sampling, QMC sequences, etc.

→ This needs to be combined with importance sampling.

→ In non-simple integration domains (e.g. spherical regions).

In order to combine stratification AND importance sampling,

→ **a map or parametrization of the integration domain can be used.**

Rewriting integrals in parameter space

Assume we need to **compute the integral** I of f in a n-dimensional domain D_n with measure μ. We can find two factors g and p, such that $f = gp$, thus I is:

$$\int_{D_n} f(\mathbf{x})\, \mathrm{d}\mu(\mathbf{x}) = \int_{D_n} g(\mathbf{x})\, p(\mathbf{x})\, \mathrm{d}\mu(\mathbf{x}) = \int_{[0,1]^n} g(M(\mathbf{u}))\, \mathrm{d}\lambda_n(\mathbf{u})$$

here λ_n is the standard Lebesgue measure on $\mathbb{R}^n$ (area)

→ We have done a **change of variable**, from $\mathbf{x}$ to $\mathbf{u} \equiv M^{-1}(\mathbf{x})$

→ M is the **map or parametrization** of domain D_n, it must hold

$$\frac{\mathrm{d}\lambda_n(\mathbf{u})}{\mathrm{d}\mu(\mathbf{x})} = p(\mathbf{x}) \tag{1}$$

Sampling on parameter space

If we assume $S \equiv \{\mathbf{s}_0, \mathbf{s}_1, \ldots \mathbf{s}_{n-1}\}$ is a set of n random sample points taken with uniform probability in $[0,1]^n$, then I can be approximated by the estimator:

$$I \approx \frac{1}{n} \sum_{i=0}^{n-1} g(M(\mathbf{s}_i))$$

then:

- → this is equivalent to **importance sampling** on D_n, by using a PDF proportional to p (w.r.t μ)
- → it can be used with **stratification**, or
- → it can be used with **QMC sequences** in $[0,1]^n$

Integral for reflected radiance

In our case, I is **reflected radiance** from $\mathbf{o}$ at direction $\boldsymbol{\omega}_o$, due to constant unit emitted radiance from a **luminaire** P

$$\begin{aligned} I &\equiv L_r(\mathbf{o}, \boldsymbol{\omega}_o) \\ &= \int_P f_r(\mathbf{o}, \boldsymbol{\omega}_o, \boldsymbol{\omega}_\mathbf{p})\, V(\mathbf{o}, \mathbf{p})\, S(\mathbf{o}, \mathbf{p})\, \cos(\mathbf{n}_\mathbf{o}, \boldsymbol{\omega}_\mathbf{p})\, \mathrm{d}\mathcal{A}(\mathbf{p}) \end{aligned}$$

where V is the visibility term, $\mathcal{A}$ is the area measure and S is the differential solid angle subtended by $\mathbf{p}$ as projected onto $\mathbf{o}$, that is:

$$S(\mathbf{o}, \mathbf{p}) \equiv \frac{\cos(\mathbf{n}_\mathbf{p}, -\boldsymbol{\omega}_\mathbf{p})}{\|\mathbf{p} - \mathbf{o}\|^2}$$

Importance sampling variants

It is possible to decompose the integrand ($f = f_r\,V\,S\,\cos$) in two factors (g and p), in various ways:

Sampling method	$g \equiv$	$p\,$(pdf) $\equiv$
Area sampling	$f_r\,V\,S\,\cos$	1
Solid angle sampling	$f_r\,V\,\cos$	S
Cosine sampling	$f_r\,V$	$S\,\cos$
BRDF sampling	$V\,\cos$	$f_r\,S$
BRDF-cosine sampling	V	$f_r\,S\,\cos$

→ Area sampling leads to very high variance, due to singularity in S.

Single scattering in participating media

Parametrization M can also be used to **compute scattered radiance** $L_s(\mathbf{o}, \boldsymbol{\omega}_o)$ from $\mathbf{o}$ towards $\boldsymbol{\omega}_o$, in a homogeneous **participating media** (accounting only for a single scattering event at $\mathbf{o}$ after emission from a luminaire P).

$$\begin{aligned} I &\equiv L_s(\mathbf{o}, \boldsymbol{\omega}_o) \\ &= \int_P \rho(\mathbf{o}, \boldsymbol{\omega}_o, \boldsymbol{\omega}_\mathbf{p})\, V(\mathbf{o}, \mathbf{p})\, T(\|\mathbf{o} - \mathbf{p}\|)\, S(\mathbf{o}, \mathbf{p})\, \mathrm{d}\mathcal{A}(\mathbf{p}) \end{aligned}$$

where T is transmittance from $\mathbf{p}$ to $\mathbf{o}$:

$$T(d) \equiv e^{-\sigma_t d}$$

and σ_t is the extinction coefficient.

Sampling in participating media

Now it is also possible to decompose the integrand ($f = \rho V T S$) in two factors (g and p), in various ways:

Sampling method	$g \equiv$	p (pdf) $\equiv$
Area sampling	$\rho V T S$	1
Solid angle sampling	$\rho V T$	S
Phase function sampling	$V T$	ρS

$\rightarrow$ Again, area sampling yields high variance due to singularity in S.

Parametrizations of planar polygons

In rendering systems, P is usually a planar polygon. Thus, for each of these P we must find a map M from $[0,1]^2$ to P, such that:

- → M can be computed in constant time.
- → f can be decomposed as g times p, where p is the PDF used for importance sampling. Typically $p = S$ or $p = S \cdot \cos$
- → M obeys this relation:

$$p(\mathbf{p}) \;=\; \frac{\mathrm{d}\mathcal{A}(\mathbf{u})}{\mathrm{d}\mathcal{A}(\mathbf{p})} \;=\; \begin{cases} S(\mathbf{o},\mathbf{p}) \\ S(\mathbf{o},\mathbf{p})\cos(\mathbf{n_o},\boldsymbol{\omega}_\mathbf{p}) \end{cases}$$

where $\mathbf{p} = M(\mathbf{u})$.

Evaluation of parametrizations

→ Area sampling has a very low performance, as its variance grows unbounded when point $\mathbf{p}$ approaches the luminaire P.

→ Cosine sampling is preferable (has lower variance) to solid angle sampling.

→ Cosine sampling requires more complex mappings than solid angle sampling.

→ BRDF sampling (for non-constant BRDFs) depends on the BRDF. Designing such a map is quite complex. However, the use of Multiple Importance Sampling helps a lot here.

As a consequence, we look for **parametrizations for either solid angle sampling or cosine sampling** in simple planar polygons.

PREVIOUS WORK

Solid angle parametrization for triangles

James Arvo (1995) proposed an analytical map for **solid angle sampling of triangles** (PDF $\propto S$).

Pros:

→ Simple analytical mapping, easy to evaluate.

→ Can be extended to any polygon, by decomposition into triangles.

Cons:

→ It has a highly deformed Jacobian, which can degrade stratification properties of sample sets (see results section).

Cosine sampling of triangles

Carlos Ureña (2000) proposed a method to **generate samples in a triangle** with probability proportional to $S \cdot \cos$ (cosine sampling).

Pros:

→ Has lower variance than solid angle sampling.

→ Can be extended to any polygon, by decomposition into triangles.

Cons: A sample generation algorithm, **not a map**, therefore

→ Does not allow stratified points.

→ Does not allow QMC sequences.

Cosine parametrization for arbitrary polygons

Jim Arvo (2001) proposed a map for **cosine sampling of general polygons** by using a polynomial approximation to the exact cosine parametrization.

Pros:

→ Low variance **(approximate)** cosine sampling.

→ Handles any planar polygon, even non-convex.

Cons: based on decomposition of polygon by using hemispherical sectors determined by vertex positions.

→ Complex implementation.

→ Slower to evaluate than other simpler maps.

THE NEW PARAMETRIZATION

Parametrization of a rectangle

We have designed a parametrization or map M which allows **solid-angle sampling of planar rectangles**:

→ Planar rectangles are often used as luminaires in rendering, also to insert fake luminaires (a window or a door)

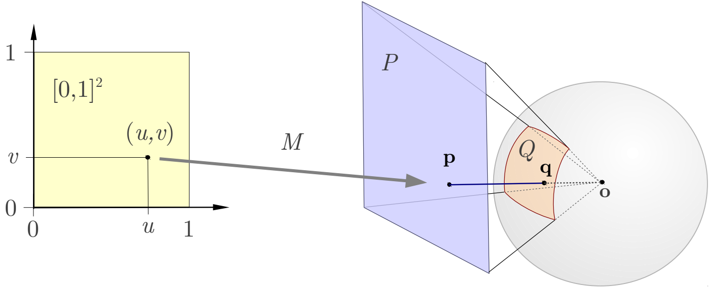
→ The map is exact (analytic), simple to implement and fast to evaluate.

→ It is inspired by Arvo's map for spherical triangles.

Under this map, **area measure in parameter space is proportional to subtended solid angle measure in the planar rectangle** (as projected onto $\mathbf{o}$).

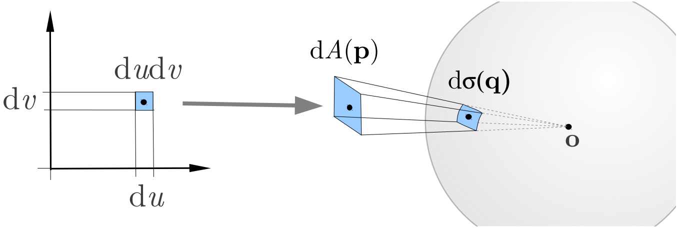
A view of map M

For any point (u, v) in parameter space, its image is $\mathbf{p} = M(u, v)$ (which is in P). Point $\mathbf{q}$ is the projection of $\mathbf{p}$ onto the unit radius sphere centred at $\mathbf{o}$, that is $\mathbf{q} = (\mathbf{o} - \mathbf{p})/\|\mathbf{o} - \mathbf{p}\|$



Basic property of M

The two (differential) areas $\mathrm{d}u\mathrm{d}v$ and $\mathrm{d}\sigma(\mathbf{q})$ are proportional:



that is, map M obeys:

$$\mathrm{d}u\mathrm{d}v \;=\; \frac{1}{\sigma(Q)}\,\mathrm{d}\sigma(\mathbf{q}) \;=\; \frac{1}{\sigma(Q)}\,S(\mathbf{o},\mathbf{p})\,\mathrm{d}\mathcal{A}(\mathbf{p}) \qquad (2)$$

Two-step approach

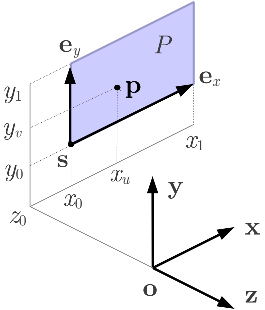
Map M is computed in two steps, yielding $\mathbf{p} = (x_u, y_v, z)$ as a function of (u, v):

1. coordinate x_u is obtained, as a function of u only.
2. coordinate y_v is obtained, as a function of both v and x_u.

Note that:

→ both functions are defined so that map M obeys equality (2).

→ all coordinates are relative to local reference system $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ aligned with rectangle P.



The vector $\mathbf{m}_u$

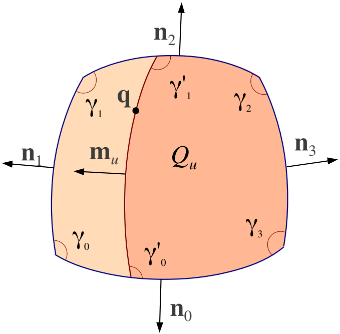
The vector $\mathbf{m}_u$ is key for designing map M. It determines spherical sub-rectangle Q_u contained in Q.

→ $\mathbf{m}_u$ is a function of u and determines x_u. It is chosen so that:

$$\sigma(Q_u) \;=\; \sigma(Q)u$$

→ $\mathbf{m}_u$ is in the plane XZ, thus determined by the angle φ_u:

$$\cos\varphi_u \;=\; \mathbf{m}_u \cdot \mathbf{z}$$



Solid angle covered by Q and Q_u

The solid angle of both Q and Q_u can be written in terms of their internal angles (Girard's formula):

$$
\begin{aligned}
\sigma(Q) &= \gamma_0 + \gamma_1 + \gamma_2 + \gamma_3 - 2\pi \\
\sigma(Q_u) &= \gamma_0' + \gamma_1' + \gamma_2 + \gamma_3 - 2\pi
\end{aligned}
$$

The internal angles γ_i can be obtained from the normals:

$$
\begin{aligned}
\gamma_i &= \arccos(-\mathbf{n}_i \cdot \mathbf{n}_{i\oplus 1}) \\
\gamma_0' &= \arccos(-\mathbf{n}_0 \cdot \mathbf{m}_u) \\
\gamma_1' &= \arccos(-\mathbf{n}_2 \cdot \mathbf{m}_u)
\end{aligned}
$$

Expression for coordinate x_u

All previous equalities **can be used to compute x_u as**:

$$x_u = ((\mathbf{o}-\mathbf{s})\cdot\mathbf{z})\,\frac{\cos\varphi_u}{\sin\varphi_u}$$

where

$$\cos\varphi_u = \frac{\operatorname{sign}(f(u))}{\sqrt{f^2(u)+(\mathbf{n}_0\cdot\mathbf{z})^2}} \qquad \sin\varphi_u = \sqrt{1-\cos^2\varphi_u}$$

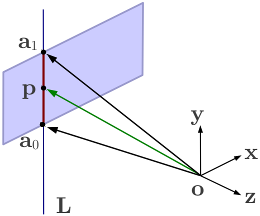
and

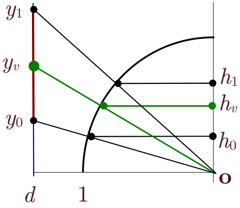
$$f(u) \equiv \frac{\cos\phi(u)(\mathbf{n}_0\cdot\mathbf{z})-(\mathbf{n}_2\cdot\mathbf{z})}{\sin\phi(u)} \qquad \phi(u) \equiv \sigma(Q)u-\gamma_2-\gamma_3+2\pi$$

(See Appendix A in the paper for a derivation)

Expression for coordinate y_v

y_v fixes position of $\mathbf{p}$ in the segment between $\mathbf{a}_0$ and $\mathbf{a}_1$:





$$y_v = \frac{h_v\, d}{\sqrt{1 - h_v^2}}$$

where $\mathbf{a}_0 \equiv (x_u, y_0, z_0)$, $\mathbf{a}_1 \equiv (x_u, y_1, z_0)$ and:

$$h_0 \equiv y_0/\|\mathbf{a}_0\| \qquad h_1 \equiv y_1/\|\mathbf{a}_1\| \qquad d \equiv \sqrt{x_u^2 + z_0^2}$$

Value h_v (Y coordinate of $\mathbf{q}$) is **obtained linearly from v**:

$$h_v = h_0 + v(h_1 - h_0)$$

Implementation of map M

We present a **simple implementation** in C (see the paper).

→ Several **constants** (values independent of u and v but dependent of $\mathbf{o}$ or P) **are precomputed and reused** for several samples from the same spherical rectangle.

→ The implementation is **robust**, avoiding divide-by-zero when distance (from shading point $\mathbf{o}$ to plane containing P) is small (as compared to size of P).

→ **Validated** by comparison with Arvo's triangle mapping and simple area sampling (produces unbiased results).

→ Included in a production renderer (Arnold).

RESULTS

Results

We have used our implementation of the map for a rectangular light source. We have computed images with **just direct lighting**. We have compared the map to:

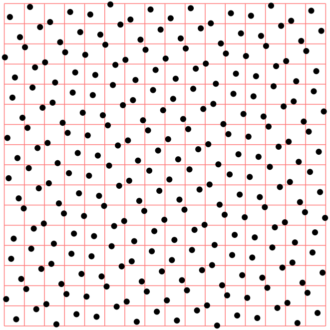
→ **Raw area** sampling

→ **Solid angle sampling of two triangles**, by using Arvo's mapping for each triangle. By sharing an edge, these two triangles cover the whole rectangle.

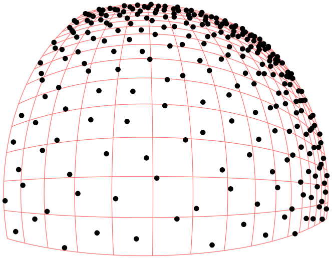
We show rendered images, numerical comparisons and visualizations of points warped by the map.

Linear area parametrization

Hammersley point set (16×16 points) and isoparametric curves (one point per cell). Regions far away from the center receive more samples than necessary.



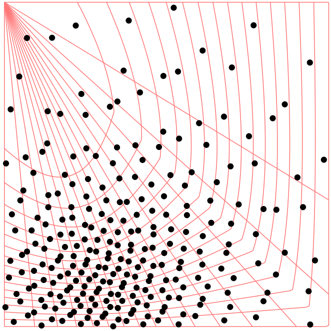
Planar rectangle P



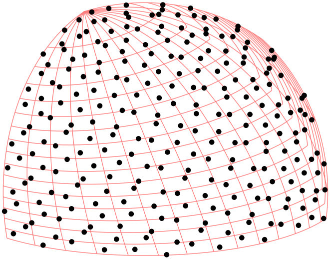
Spherical rectangle Q

Solid angle parametrization (two Arvo triangles)

Same point set and isoparametric curves, but now using Arvo's parametrization for triangles. Note both triangular and L-shaped cells.



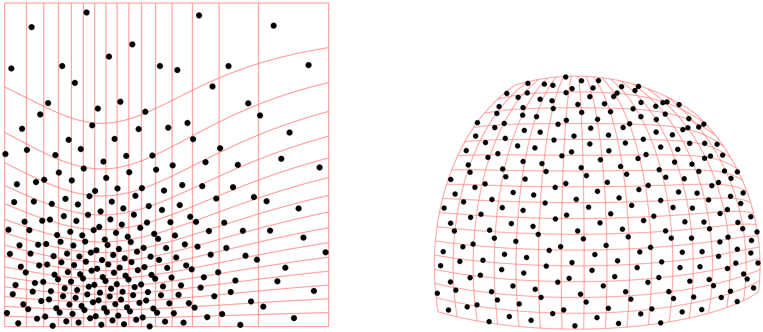
Planar rectangle P



Spherical rectangle Q

Solid angle parametrization (rectangle)

Same point set, using our proposed rectangle parametrization. Note that isoparams with constant u are now vertical, as x_u depends on u only.

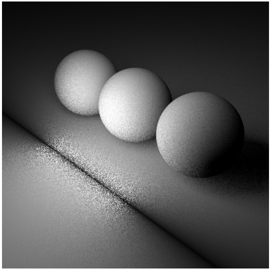


Planar rectangle P | Spherical rectangle Q

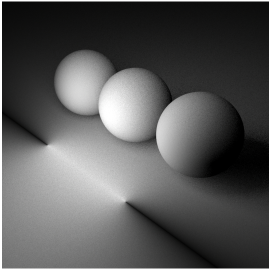
Light source close to ground plane

Arnold renderings, direct lighting only, 9 paths per pixel. Lambertian materials. Two-sided rectangular light source (made invisible) perpendicular to the ground.

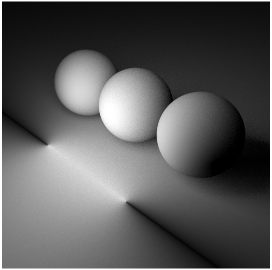
Naive area sampling yields very high variance.



area



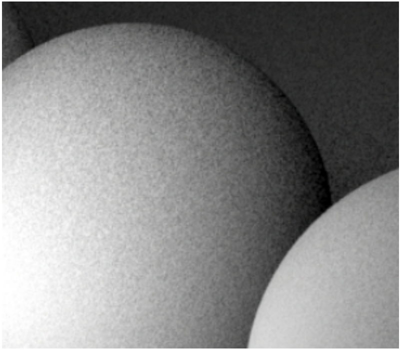
solid angle (two triangles)



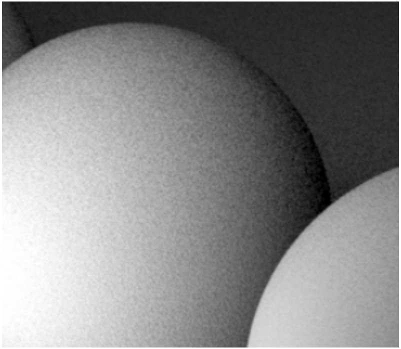
solid angle (rectangle)

Spherical rectangle versus two spherical triangles

Rectangle yields less noise (similar computing time).



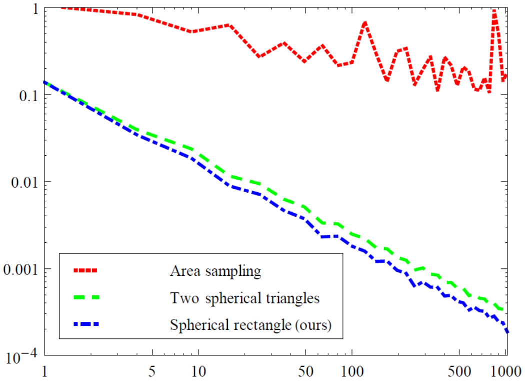
two Arvo triangles



rectangle

Error comparison

RMSE as a function of the number of light samples per camera ray for the three methods.



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Using Multiple Importance Sampling (MIS)

The use of **Multiple Importance Sampling** allows to combine solid angle sampling with BRDF sampling.

→ In our test scene, surface material is diffuse (constant BRDF), thus BRDF sampling is in fact equivalent to perfect importance sampling of the cosine term (PDF is proportional to $\cos$).

→ This is combined with area or solid angle sampling (constant PDF or PDF proportional to solid angle).

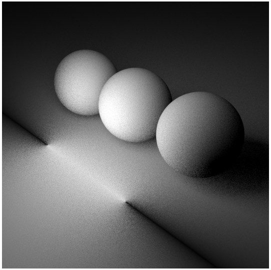
→ The balance heuristic is used to combine the two PDFs.

→ This gives further reduction in variance for all methods.

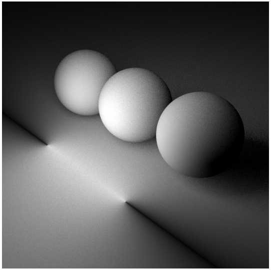
Comparison with MIS

Same test scene and render settings.

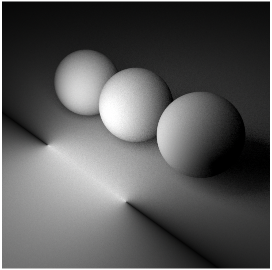
Far less noise, but area sampling still the worst.



area + BRDF



solid angle + BRDF
(two Arvo triangles)



solid angle + BRDF
(rectangle)

Spherical rectangle versus two spherical triangles (w/ MIS)

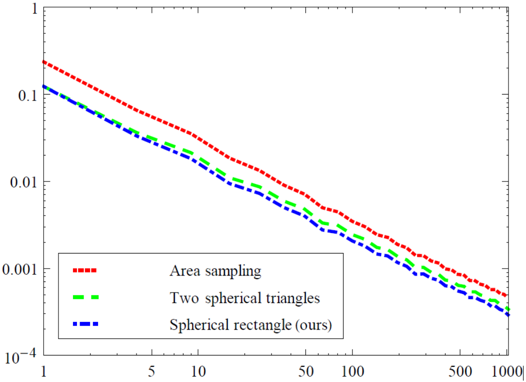
Rectangle yields slightly less noise.

two Arvo triangles + BRDF

rectangle + BRDF

Error comparison with MIS

RMSE when each method is combined with BRDF sampling.



CONCLUSION AND FUTURE WORK

Conclusions

→ New simple, analytical parametrization for spherical rectangles.

→ We provide a robust implementation in C.

→ Better than existing alternatives (higher performance).

Future work

→ Explore analytical or approximated cosine sampling for rectangles.

→ Include importance sampling of emissive texture and solid angle.

→ Explore analytical or approximated solid angle sampling for other shapes (discs, arbitrary quads, etc).

Thanks!