Construction of free boundary CMC annuli in geodesic balls

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Introduction

Let $\mathbb{B}^3$ be a **geodesic ball** in a space form $\mathbb{M}^3$, that is, $\mathbb{M}^3 = \mathbb{R}^3, \mathbb{S}^3, \mathbb{H}^3$. We say that a constant mean curvature (CMC) surface $\Sigma \subset \mathbb{B}^3$ is **free boundary** in $\mathbb{B}^3$ if Σ meets the sphere $\partial\mathbb{B}^3$ **orthogonally**.

Free boundary CMC surfaces appear naturally as solutions of the **partitioning problem**, which asks to find the critical points of the area functional among all surfaces dividing $\mathbb{B}^3$ into two pieces of **prescribed volumes**.

In recent years, free boundary problems have received significant attention, specially in the study of **minimal** surfaces ($H = 0$) in the unit ball of Euclidean space. However, even in this particular situation, important questions remain open, such as the **critical catenoid conjecture**, which asks whether an embedded free boundary minimal annulus must be a piece of a catenoid. Our results show that the conjecture does not hold when we relax the minimal hypothesis to CMC.

Previous results & questions

Theorem (Nitsche, 1985) [N]: any free boundary CMC **disk** must be totally umbilic. In particular, it must be **rotational**.

Question 1 **(Nitsche, 1985)**: are **Delaunay surfaces** the only free boundary CMC **annuli** in $\mathbb{B}^3$?

Theorem (Wente, 1995) [W]: In $\mathbb{R}^3$, there exist **immersed** free boundary CMC ($H \neq 0$) annuli which are **not rotational**. In particular, this gives a negative answer to Question 1 in the Euclidean case.

Question 2 **(Wente, 1995):** are **Delaunay surfaces** the only **embedded** free boundary CMC annuli in $\mathbb{B}^3$?

Theorem (Fernández-Hauswirth-Mira, 2022) [FHM]: In $\mathbb{R}^3$, there exist **immersed** free boundary minimal annuli which are **not rotational**.

Our results

We prove the following existence theorems:

Theorem [CFM1]: in $\mathbb{R}^3$, there exist **embedded** free boundary CMC ($H \neq 0$) annuli which are **not rotational**.

Theorem [CFM2]: in $\mathbb{S}^3$, for any $H \geq 0$ there exist **non-rotational** CMC-H annuli which are free boundary in some geodesic ball. If $H \geq 1/\sqrt{3}$, they are **embedded**.

Theorem [CFM2]: in $\mathbb{H}^3$, for any $H > 1$ there exist **non-rotational**, **embedded** CMC-H annuli which are free boundary in some geodesic ball.

In particular, we give a negative answer to Question 2 posed by Wente. Moreover, we extend the answer to Question 1 by generalizing the results in [FHM,W] to the case of geodesic balls in space forms.

All these annuli appear as 1-parameter **deformations** of a free boundary piece of a Delaunay surface. Moreover, their symmetry group is **prismatic** and they are foliated by spherical curvature lines.

We summarize the known examples of non-rotational free boundary annuli in the table below. Our constructions are highlighted in green:

	$\mathbb{S}^3$	$\mathbb{R}^3$	$\mathbb{H}^3$
Minimal	Immersed [CFM2]	Immersed [FHM]	Work in progress!
CMC	Embedded [CFM2]	Embedded [CFM1]	Embedded [CFM2]

Idea of the construction for the Euclidean case

Let $\mathcal{F}$ be the set of **conformal** CMC immersions $\psi(u, v) : \mathbb{R}^2 \to \mathbb{R}^3$ **parametrized by curvature lines** such that each v-line $v \mapsto \psi(u_0, v)$ is contained in some **sphere** $\mathbb{S}(u_0)$. $\mathcal{F}$ constitutes a 4-parameter family of CMC surfaces. One can check that (a) the intersection angle of the surfaces with each sphere $\mathbb{S}(u)$ is **constant** (Joachimsthal's Theorem), and (b) the centers of the spheres $\mathbb{S}(u)$ lie on a **vertical** axis L; see **Figure 1**.

We will find our free boundary annuli within $\mathcal{F}$. To do so, we consider slices of immersions $\Sigma_\tau := \psi([-\tau, \tau] \times \mathbb{R})$ satisfying several properties:

1. **Horizontal symmetry:** we want the immersion to have a horizontal symmetry plane P; see **Figure 1**.

2. **Annular** condition: we ask the v-lines to be closed, so that Σ_τ is an annulus whose boundary components lie on the spheres $\mathbb{S}(\tau), \mathbb{S}(-\tau)$.

3. **Orthogonality** condition: we choose $\tau > 0$ so that Σ_τ meets the boundary spheres $\mathbb{S}(\tau), \mathbb{S}(-\tau)$ orthogonally; see **Figure 1**.

4. **Spherical boundary:** we ask the two boundary spheres to coincide. Due to the symmetry in P, this is achieved when the center of $\mathbb{S}(\tau)$ lies in $L \cap P$. In such a case, both boundary components of Σ_τ lie on $\mathbb{S}(\tau) \equiv \mathbb{S}(-\tau)$.

Let Σ_0 be an annulus satisfying properties 1-4. If, moreover, Σ_0 is contained in the ball $\mathbb{B}^3$ whose boundary is $\mathbb{S}(\tau)$, then the annulus is **free boundary** in $\mathbb{B}^3$.

It can be shown that the family of annuli $\Upsilon \subset \mathcal{F}$ satisfying the aforementioned properties is analytic and 1-parametric. Moreover, in Υ we detect an **embedded** piece of nodoid contained in its corresponding ball, so it is free boundary in it; see **Figure 2**. By analyticity, any annulus in Υ near this nodoid will also be embedded and free boundary in a certain ball.

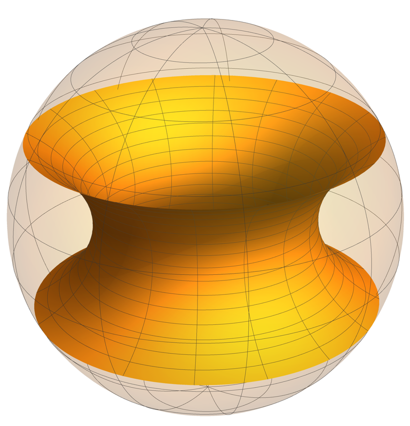

Figure 2. Free boundary nodoid in Υ.

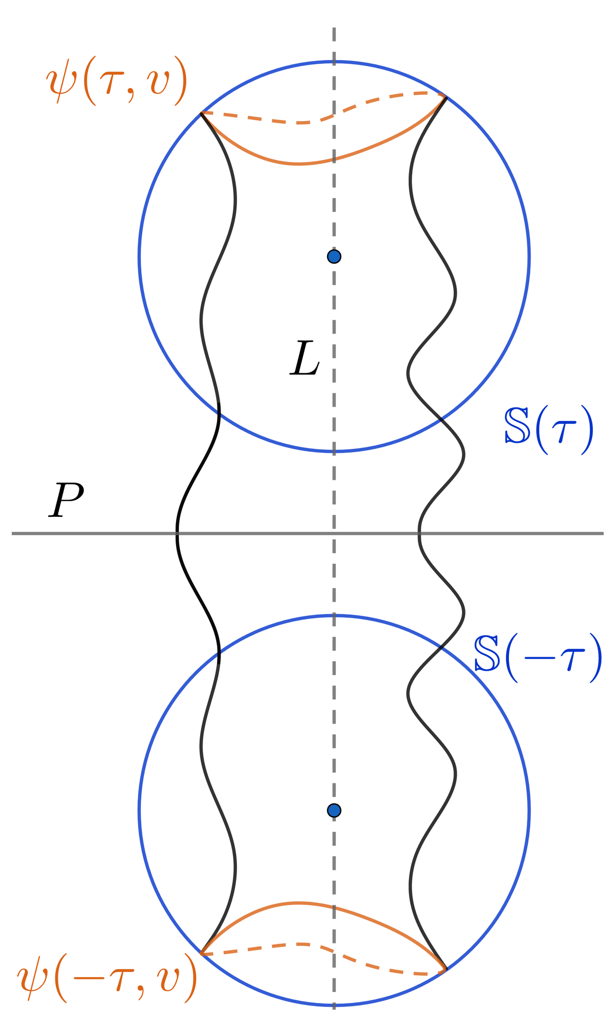

Figure 1. Annulus Σ_τ.

References

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[N] J.C.C. Nitsche: Stationary partitioning of convex bodies (1985).

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Acknowledgements & contact

Research financially supported by Project PID2020-118137GB-I00 funded by MCIN/AEI /10.13039/501100011033 and by ESF+.

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