UNIVERSITY OF GRANADA.
Ordinary Exam: Mathematical Methods II.
Degree in Physics
January 21th 2019.

- **Name and Surname:**
- *Duration: 2.5 hours*

1. [*3 points*] Given the Riccati differential equation

$$y' + \frac{y}{x} - 2y^2 = -\frac{2}{x^2},$$

   a) Make the change of variable $y \to z = (y + \frac{1}{x})^{-1}$ and check that you arrive to a linear equation.

   b) Solve the linear equation and determine the general solution $y(x)$ (and its domain) for the original Riccati equation.

2. [*3 points*] Find the general solution to the Cauchy-Euler differential equation

$$x^2y'' - 3xy' + 5y = x^2\ln(x) \qquad (x > 0)$$

3. [*4 points*] Find the general solution, by power series about $x = 0$, to the differential equation

$$9x^2y'' + 9xy' + (x^2 - 1)y = 0\,.$$

   a) Are there analytic solutions about $x = 0$? Justify the answer.

   b) What is the radius of convergence?

SOLUTIONS ORDINARY EXAM "MATHEMATICAL METHODS II"

1] Riccati equation $y' + \frac{y}{x} - 2y^2 = \frac{-2}{x^2}$

$$z = \left(y+\frac{1}{x}\right)^{-1} \Rightarrow y = \frac{1}{z} - \frac{1}{x}, \quad y' = -\frac{z'}{z^2} + \frac{1}{x^2} \Rightarrow \overbrace{-\frac{1}{z^2}}^{0}\left(\overbrace{z' - \frac{5}{x}z + 2}^{0}\right) = 0$$

Case 1| $\frac{1}{z^2} = 0 \Rightarrow \left(y+\frac{1}{x}\right)^2 = 0 \Rightarrow \tilde{y} = -\frac{1}{x}$ particular solution

Case 2| $z' - \frac{5}{x}z + 2 = 0$ (linear) $\Rightarrow z = Cx^5 + \frac{1}{2}x \Rightarrow$

$$\Rightarrow y = \frac{1}{Cx^5 + \frac{1}{2}x} - \frac{1}{x} \quad \text{(general solution)}$$

Domain$(\tilde{y}) = \mathbb{R} - \{0\}$, Domain$(y) = \mathbb{R} - \{0, (-2C)^{-1/4}\}$

2] Cauchy-Euler $x^2y'' - 3xy' + 5y = x^2\ln(x)$

Homogeneous part: Try $y(x) = x^m \Rightarrow m(m-1) - 3m + 5 = 0$

$$m = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i$$

General solution of the homog. eq. $y_h(x) = x^2\left(C_1\cos(\ln x) + C_2\sin(\ln x)\right)$

Particular solution: Try $y_p(x) = x^2(A\ln x + B)$

$$y_p'(x) = 2x(A\ln x + B) + Ax, \quad y_p''(x) = 2(A\ln x + B) + 3A$$

$$x^2y_p'' - 3xy_p' + 5y_p = 2x^2(A\ln x + B) + \cancel{3Ax^2} - 6x^2(A\ln x + B) - \cancel{3Ax^2} + 5x^2(A\ln x + B)$$
$$\| \quad x^2\ln x \qquad = x^2(A\ln x + B) \Rightarrow A = 1, \; B = 0$$

General Solution: $y(x) = y_h(x) + y_p(x) = x^2\left(C_1\cos(\ln x) + C_2\sin(\ln x)\right) + x^2\ln x$

3] $9x^2y'' + 9xy' + (x^2-1)y = 0$ $\quad x=0$ singular regular

Try $y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$, $y'(x) = \sum_{n=0}^{\infty} c_n (n+r)x^{n+r-1}$, $y''(x) = \sum_{n=0}^{\infty} c_n (n+r)(n+r-1)x^{n+r-2}$

$$\sum_{n=0}^{\infty} 9c_n\Big(\underbrace{(n+r)(n+r-1) + (n+r)}_{(n+r)^2}\Big)x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r+2} - \sum_{n=0}^{\infty} c_n x^{n+r} = 0 \qquad n+2=m$$

$$c_0x^r(\underbrace{9r^2-1}_{r=\pm 1/3}) + c_1x^{r+1}\left(9(1+r)^2 - 1\right) + \sum_{m=2}^{\infty} x^{m+r}\Big(\underbrace{c_m\left(9(m+r)^2-1\right) + c_{m-2}}_{=0 \text{ recurrence}}\Big) = 0$$

$c_1 = 0$

$$c_m = \frac{-c_{m-2}}{\underbrace{9(m+r)^2-1}_{(3(m+r)+1)(3(m+r)-1)}} \xRightarrow[c_{2k}=a_k]{m=2k} a_k = \frac{-a_{k-1}}{(3(2k+r)+1)(3(2k+r)-1)} =$$

$$a_k = \frac{-a_{k-1}}{3^2\cdot 2^2\underbrace{\left(k+\frac{r}{2}+\frac{1}{6}\right)}_{\alpha+k-1}\underbrace{\left(k+\frac{r}{2}-\frac{1}{6}\right)}_{\beta+k-1}} = \cdots = \frac{(-1)^k a_0}{3^{2k}\cdot 2^{2k}\Big(\underbrace{\frac{r}{2}+\frac{1}{6}+1}_{\alpha}\Big)_k\Big(\underbrace{\frac{r}{2}-\frac{1}{6}+1}_{\beta}\Big)_k}$$

$\boxed{\text{For } r=\frac{1}{3}}$
$$a_k = \frac{(-1)^k a_0}{3^{2k}\cdot 2^{2k}\left(\frac{4}{3}\right)_k k!} \Rightarrow \boxed{y_1(x) = x^{1/3}\sum_{k=0}^{\infty} a_k x^{2k} = x^{1/3} a_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\left(\frac{4}{3}\right)_k}\left(\frac{x}{6}\right)^{2k}}$$

$\boxed{\text{For } r=-\frac{1}{3}}$
$$a_k = \frac{(-1)^k a_0}{6^{2k} k!\left(\frac{2}{3}\right)_k} \Rightarrow \boxed{y_2(x) = x^{-1/3} a_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\left(\frac{2}{3}\right)_k}\left(\frac{x}{6}\right)^{2k}}$$

$y_1(x)$ is finite at $x=0$ with radius of convergence $R=\infty$

$y_2(x)$ is not finite at $x=0$.

The solutions $y_1(x)$ and $y_2(x)$ can be written in terms of Bessel functions

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(1+\nu+k)}\left(\frac{x}{2}\right)^{2k+\nu} \quad \text{where} \quad (1+\nu)_k = \Gamma(1+\nu+k)/\Gamma(1+\nu)$$

Actually $y_1(x) \propto J_{1/3}\left(\frac{x}{3}\right)$ and $y_2(x) \propto J_{-1/3}\left(\frac{x}{3}\right)$