Central sequence algebras via nilpotent elements

Joint work with Dominic Enders

July 17, 2022

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Example Take $z_n \in Z(A)$, $b_n \to 0$. Then

$$(z_n + b_n)$$

is a central sequence. Such central sequences are called *trivial*.

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Definition *Central sequence algebra* $F(A)$ of a unital C*-algebra A is

$$F(A) := A' \bigcap A_\omega.$$

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Definition For a non-unital C*-algebra A, its *central sequence algebra* $F(A)$ is

$$F(A) := \left(A' \bigcap A_\omega\right) / Ann(A, A_\omega).$$

von Neumann algebra

Let A be a II_1-factor.

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J. Phillips 1988: If A is unital and either simple or $A \supset K(H)$, then $F(A)$ is not abelian.

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E.g. a separable nuclear A is simple purely infinite $\Leftrightarrow$ $F(A)$ is simple and $F(A) \ncong \mathbb{C}$.

E.g. a unital separable A is $\mathcal{Z}$-absorbing ($A \cong A \otimes \mathcal{Z}$) $\Leftrightarrow$ $\mathcal{Z} \hookrightarrow F(A)$.

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If A is not type I, then $F(A)$ is not abelian.

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If A is not type I, then $F(A)$ is not abelian (not subhomogeneous).

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Toeplitz algebra is type I but not CCR.

Extensions by compact operators

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One needs some other technique for the non-unital case and to show non-subhomogeneity

Nilpotents

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Nilpotents are liftable: suppose $x \in A/I$ with $x^n = 0$, then x lifts to $a \in A$ with $a^n = 0$.

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Corollary

Given $n \in \mathbb{N}$ and $\epsilon > 0$, there exists δ such that the following holds: for any C*-algebra A and any $x \in A$ satisfying $\|x^n\| \leq \delta$ and $\|x\| \leq 1$ there is $y \in A$ such that $y^n = 0$, $\|y\| \leq 1$ and $\|y - x\| \leq \epsilon$.

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Theorem (Hadwin 1997)

A C*-algebra A is n-subhomogeneous if and only if each nilpotent element in A has order not larger than n.

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Proposition (Sh. 2019)

If the closure of nilpotents in a C*-algebra A contains a normal element, then A is not residually type I.

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Lemma

Let $A \subset B(H)$ be a separable C*-algebra, then $q(A)'$ contains a copy of $B(H)$.

Step 1: to prove that A is type I but not CCR $\Rightarrow F(A)$ is not subhomogeneous.

Sufficient: to prove that $A \supset K(H) \Rightarrow F(A)$ is not subhomogeneous.

Theorem

Let $A \subset B(H)$ be a separable C*-algebra such that $A \supset K(H)$. Then $F(A)$ is not residually type I.

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Theorem

Let $A \subset B(H)$ be a separable C*-algebra such that $A \supset K(H)$. Then $F(A)$ is not residually type I. In particular $F(A)$ is not type I and not RFD.

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Fell's condition

Definition $\pi_0 \in \hat{A}$ satisfies *Fell's condition* if there exist $b \in A^+$ and an open neighbourhood U of π_0 in $\hat{A}$ such that $\pi_0(b) \neq 0$ and

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whenever $\pi \in U$.

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$$A = \{f \in C([0,1], M_2) \mid f(1) \text{ is diagonal }\},\ \pi_0(f) = (f(1))_{11}$$

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Take $b :=$ any function s.t. $b(t) = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}$ in a nbhd of 1.

Examples

Example 2

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Indeed if $\pi_0(b) \neq 0$, then *rank* $b(1) = 2$. Since rank is lower semicontinuous, *rank* $\pi(b) = 2$ in a nbhd of π_0.

Fell's condition

Definition A C*-algebra A is said to satisfy *Fell's condition* (also is called *Fell algebra*) if every irreducible representation of A satisfies Fell's condition.

Fell's condition of higher order

Definition An irreducible representation π_0 of A satisfies *Fell's condition of order n* if there exist $b \in A^+$ and an open neighbourhood U of π_0 in $\hat{A}$ such that $\pi_0(b) \neq 0$ and

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$$A = \{ f \in C([0,1], M_3) \mid f(1) = \begin{pmatrix} \lambda & & \\ & \lambda & \\ & & \mu \end{pmatrix}, \lambda, \mu \in \mathbb{C} \}$$

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Then A satisfies Fell's condition of order 2 but not of order 1.

Examples

Example 4 Consider the UHF algebra

$$\mathbb{C} \subset M_2 \subset M_4 \subset \ldots \subset M_{2^\infty},$$

and its telescopic algebra

$$T\left(M_{2^\infty}\right) = \{f \in C([0,\infty], M_{2^\infty}) \mid t \leq i \Rightarrow f(t) \in M_{2^i}\}.$$

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$$A = \{f \in T(M_{2^\infty}) \mid f(\infty) \in \mathbb{C}1\}.$$

Then A is CCR but does not satisfy Fell's condition of order n, for any $n \in \mathbb{N}$.

Reformulation of Fell's condition of order n

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Case $n = 1$: an Huef, Kumjian, Sims 2011

Step 2: For CCR-algebras, $F(A)$ is n-subhomogeneous $\Leftrightarrow$ Fell's condition of order n

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Will show strategy for $\Leftarrow$.

Fell's condition of order $n \Rightarrow F(A)$ is n-subhomogeneous

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Observation: If in $B(H)$ we have $x^2{=}0$, $e \geq 0$ is of rank 1, then

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Lemma

For any $\epsilon > 0$ there is a $\delta > 0$ such that whenever $e \in (B(H))_{+,1}$ with *rank* $e = 1$ and $x \in (B(H))_1$ with $x^2 = 0$, then

$$\|[x, e]\| \leq \delta \quad \Rightarrow \quad \|ex\| \leq \epsilon \text{ and } \|xe\| \leq \epsilon.$$

Fell's condition of order $n \Rightarrow F(A)$ is n-subhomogeneous

Observation: If in $B(H)$ we have $x^2=0$, $e \geq 0$ is of rank 1, then

$$[x, e] = 0 \quad \Rightarrow \quad ex = 0, \; xe = 0.$$

Lemma

For any $\epsilon > 0$ there is a $\delta > 0$ such that whenever $e \in (B(H))_{+,1}$ with *rank* $e \leq N$ and $x \in (B(H))_1$ with $x^{N+1} = 0$, then

$$\|[x, e]\| \leq \delta \quad \Rightarrow \quad \|ex^N\| \leq \epsilon \text{ and } \|x^N e\| \leq \epsilon.$$

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Lift it to a central sequence $(x_1, x_2, \ldots)$ with $x_i^{N+1} = 0$.

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$$x_i^N e \to 0, \;\; e x_i^N \to 0$$

$$x^N = 0 \text{ in } \; F(A)$$

Theorem

Let A be a separable CCR C*-algebra. Then $F(A)$ is n-subhomogeneous if and only if A satisfies Fell's condition of order n.

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One application

Theorem

If a C*-algebra A satisfies Fell's condition but does not have continuous trace, then A has an outer derivation.

Thank you!