On Aupetit's Scarcity Theorem

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- Define the set D_f as follows:

$$D_f := \{\lambda \in D\colon \#\sigma(f(\lambda)) < \infty\}.$$

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Bernard Aupetit's original Scarcity Theorem [1, Theorem 3.4.25, p. 63] is as follows:

Theorem (Scarcity of Elements with Finite Spectrum)

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2. the n points of $\sigma(f(\lambda))$ are **locally** holomorphic functions on $D \setminus E$.

Mechanics of the Scarcity Theorem

Assuming that D_f has nonzero capacity, we have that

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What one has to be aware of—amongst the many boons that this theorem grants us—is that if we have two **distinct** points γ, κ in $D \setminus E$, the following observation can be made:

The First Observation

There are disks $B_\gamma = B(\gamma, r_\gamma)$ and $B_\kappa = B(\kappa, r_\kappa)$ in $D \setminus E$ and $2n$ functions, $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ such that each α_i is holomorphic on B_γ and each β_i is holomorphic on B_κ.

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The question then arises, what happens when $B_\gamma \cap B_\kappa \neq \emptyset$?

Surely, for all λ in this intersection we must have that

$$\sigma(f(\lambda)) = \{\alpha_1(\lambda), \ldots, \alpha_n(\lambda)\} = \{\beta_1(\lambda), \ldots, \beta_n(\lambda)\}. \qquad (\star)$$

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Notice that this must happen for uncountably many λ's, and then by the Pigeonhole Principle we conclude without loss of generality that $\alpha_i \equiv \beta_i$ on $B_\gamma \cap B_\kappa$ for each $i \in \{1, \ldots, n\}$.

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So taking M to be $B_\gamma \cup B_\kappa$ and define new functions, say, $\theta_1, \ldots, \theta_n$, as below:

$$\theta_i(\lambda) = \begin{cases} \alpha_i(\lambda) & \text{if } \lambda \in B_\gamma, \\ \beta_i(\lambda) & \text{if } \lambda \in B_\kappa. \end{cases}$$

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We have obtained some 'larger' set M in $D \setminus E$ on which $\sigma(f(\lambda))$ is given by a single set of n holomorphic functions.

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If we do not assume anything other than A being a complex Banach algebra with unit—and of course D_f having nonzero capacity—the best we do (so far) is that M is a dense open subset of $D \setminus E$.

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The next example illustrates how a simple construction can be impede our aim of achieving $M = D \setminus E$.

A Simple Example Which Foiled Us...

Example

Set $D = \mathbb{C}$ and take $f\colon \mathbb{C} \to M_2(\mathbb{C})$ to be the map

$$\lambda \mapsto \begin{bmatrix} 0 & \lambda \\ 1 & 0 \end{bmatrix}.$$

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Now notice that if our largest set M is all of $D \setminus E$ then we would obtain a square root function which is holomorphic on all of $\mathbb{C} \setminus \{0\}$.

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In the second part of the talk, we show a case where $M = D \setminus E$ can be obtained.

Some Applications using the "Vanilla" Scarcity Theorem

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- Brits et al. in [2] (2021) use the theorem to show that a Banach algebra A is commutative if and only if

  $$\#\sigma(e^{x+y} - e^x e^y) = 1 \quad \text{for all } x, y \in A.$$

- Jacobson's Lemma: $\sigma(xy) \cup \{0\} = \sigma(yx) \cup \{0\}$ for all $x, y \in A$. But this does not necessarily hold for all permutations of three elements if A is not commutative.

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  Now if $\#\sigma(xa)$ is finite for all $a \in A$, then it is possible to prove using the Scarcity Theorem that $rank(x)$ is finite.

The Scarcity Theorem (new and improved)

Theorem (Scarcity of Elements with Finite Spectrum)

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4. M is dense in D.

Proof Outline

(1) The original theorem gives us existence of a closed discrete subset E of D and an integer $n \geq 1$ such that $\#\sigma(f(\lambda)) = n$ on $D \setminus E$ and $\#\sigma(f(\lambda)) < n$ otherwise.

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Define another set $\mathcal{C}$ as follows:

$$\mathcal{C} := \{H_F \neq \emptyset \colon F \text{ is a nonempty open subset of } D \setminus E\}.$$

Proof Outline

(3) Define the following partial order on $\mathcal{C}$: Given $H_F = (\alpha_i)_{i=1}^n$ and $H_G = (\beta_i)_{i=1}^n$, we will say $H_F \leq H_G$ whenever

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M is an open subset of $D \setminus E$ with $H_M = (\omega_1, \ldots, \omega_n)$ which are functions holomorphic on M such that $\sigma(f(\lambda)) = \{\omega_i(\lambda)\}_{i=1}^n$ for all $\lambda \in M$.

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(4) To show that M is dense in D, we need only show $\overline{M} = D \setminus E$, which is really obvious using a proof by contradiction. □

The Most Optimal Conclusion

Our issue previously was that, at best, the spectrum of $f(\lambda)$ was given in terms of functions holomorphic on a dense open subset M of $D \setminus E$—but $M \neq D \setminus E$ necessarily.

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If, however, we assume that $f(\lambda)f(\kappa) = f(\kappa)f(\lambda)$ for all $\lambda, \kappa \in D$, then it is possible to obtain this conclusion.

In other words, we need to have that the set $\{f(\lambda) \colon \lambda \in D\}$ is a commutative subset of A.

The *commutant* of a set S in A is defined to be

$$C_S := \{x \in A \colon xa = ax \text{ for all } a \in S\},$$

with the *bicommutant* (denoted B_S) being just the commutant of the commutant of S.

B_S is a commutative Banach subalgebra of A provided that S is commutative itself. Furthermore, bicommutants preserve the spectrum of its elements in A.

A *character* ϕ on a Banach algebra is a nonzero bounded linear functional such that $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in A$. The character space of A will be denoted $\mathfrak{M}$.

The spectrum of an element x in a commutative Banach algebra is precisely

$$\{\phi(x) \colon \phi \in \mathfrak{M}\}.$$

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3. the functions $\alpha_1, \ldots, \alpha_n$ are holomorphic on D.

Proof Outline

(1) If λ_1 is a fixed point in $D \setminus E$, there is a disk B_1 contained in $D \setminus E$ centered at λ_1 and functions $\alpha_1, \ldots, \alpha_n$ such that

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With this, we argue and show that each α_i is assigned to at least one $\phi_i \circ f$ on B_S such that $\alpha_i \equiv \phi_i \circ f$ on B_1.

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It remains to show that the above statement holds for all $\lambda \in D$.

Notice that $\{(\phi_i \circ f)(\lambda)\}_{i=1}^n \subseteq \sigma(f(\lambda))$, for any $\lambda \in D \setminus E$.

And we have that $\#\sigma(f(\lambda)) = n$ outside E, so to show equality here, we need only prove that

$$\#\{(\phi_i \circ f)(\lambda)\}_{i=1}^n = n \text{ for all } \lambda \in D \setminus E.$$

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(3) The Identity Theorem for holomorphic functions easily establishes this fact.

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So we are almost there, as the following is now true:

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(4) Finally, using a short argument, we can extend these functions holomorphically to points of E, and that $(\star)$ holds for all $\lambda \in E$ as well.

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