Newton-Girard-Vieta and Waring-Lagrange theorems for two non-commuting variables

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Power sums

François Viète (1593) and Albert Girard (1629) gave formulae for the sums of powers of the zeros of a polynomial in terms of the coefficients of the polynomial. The formulae are often attributed to Newton (*Algebra Universalis*, 1707).

For any integer n and commuting variables x, y let

$$\begin{aligned} p_n(x,y) &= x^n + y^n \\ \alpha &= x + y \\ \beta &= xy. \end{aligned}$$

x, y are the zeros of the polynomial $\lambda^2 - \alpha\lambda + \beta$.

François Viète

Albert Girard

The first few Girard-Viète formulae

$$\begin{aligned} p_1 &= x + y &=& \quad \alpha \\ p_2 &= x^2 + y^2 &=& \quad \alpha^2 - 2\beta \\ p_3 &= x^3 + y^3 &=& \quad \alpha^3 - 3\alpha\beta \\ p_4 &= x^4 + y^4 &=& \quad \alpha^4 - 4\alpha^2\beta + 2\beta^2 \end{aligned}$$

where

$$\alpha = x + y, \quad \beta = xy.$$

Symmetric polynomials

A polynomial p is *symmetric* if it is unchanged by a permutation of the variables.

The Waring-Lagrange theorem

Every symmetric polynomial is expressible as a polynomial in the elementary symmetric polynomials.

Thus any symmetric polynomial in x and y can be written as a polynomial in $x + y$ and xy.

1762: *Meditationes Algebraicae*, by Edward Waring, Lucasian Professor at Cambridge.
1798: *Traité de la Résolution des Équations Numériques de tous les Degrés*, by Joseph Louis Lagrange.

Edward Waring

What if x and y do not commute?

The symmetric free polynomial

$$xyx + yxy$$

cannot be written as a free polynomial in $x+y$ and $xy+yx$.

Show this by substituting 2×2 matrices for x, y in such a way that $xy + yx = 0$.

Theorem (Margarete Wolf, 1936)

There is no finite basis for the algebra of symmetric free polynomials in d indeterminates over $\mathbb{C}$ when $d > 1$.

Thus there is no reason to expect that the free polynomials $p_n = x^n + y^n$, for integer n, can be written as free polynomials in some finite collection of ‘elementary symmetric nc-functions’ of x and y.

Nevertheless, we do find three symmetric free polynomials α, β, γ in x and y such that every p_n can be written as a free polynomial in α, β, γ and β^{-1}.

A free Newton-Girard-Vieta formula

Let

$$u = \tfrac{1}{2}(x + y), \qquad v = \tfrac{1}{2}(x - y)$$

and let

$$\alpha = u, \qquad \beta = v^2, \qquad \gamma = vuv.$$

Then α, β, γ are symmetric free polynomials in x, y, and, for every positive integer n, there exists a free rational function P_n in three variables such that

$$p_n(x, y) = P_n(\alpha, \beta, \gamma).$$

Moreover P_n can be written as a free polynomial in α, β, γ and β^{-1}.

Proof

Show by induction that p_n is the sum of all monomials in u, v of total degree n and of even degree in v.

Any monomial in u and v, in which v occurs with even degree, can be written as a monomial in α, β, γ and β^{-1}.

For example

$$\begin{aligned} u^2vu^2v &= u^2(vuv)(v^2)^{-1}(vuv) \\ &= \alpha^2\gamma\beta^{-1}\gamma. \end{aligned}$$

Hence p_n is a sum of monomials in α, β, γ and β^{-1}.

The first few P_n

$$x^n + y^n = P_n(\alpha, \beta, \gamma)$$

where $\alpha = \frac{1}{2}(x+y), \beta = \frac{1}{4}(x-y)^2, \gamma = \frac{1}{8}(x-y)(x+y)(x-y)$.

$$\begin{aligned}
P_1 &= 2\alpha \\
P_2 &= 2(\alpha^2 + \beta) \\
P_3 &= 2(\alpha^3 + \alpha\beta + \gamma + \beta\alpha) \\
P_4 &= 2(\alpha^4 + \alpha^2\beta + \alpha\gamma + \gamma\beta^{-1}\gamma + \alpha\beta\alpha + \gamma\alpha + \beta\alpha^2 + \beta^2) \\
P_{-1} &= 2(\alpha - \beta\gamma^{-1}\beta)^{-1} \\
P_{-2} &= 2\left(\alpha^2 + \beta - (\alpha\beta + \gamma)(\gamma\beta^{-1}\gamma + \beta^2)^{-1}(\beta\alpha + \gamma)\right)^{-1} \\
P_{-3} &= 2\Big(\alpha^3 + \alpha\beta + \beta\alpha + \gamma - (\alpha^2\beta + \alpha\gamma + \gamma\beta^{-1}\gamma + \beta^2)\times \\
&\qquad (\gamma\beta^{-1}\gamma\beta^{-1}\gamma + \gamma\beta + \beta\gamma + \beta\alpha\beta)^{-1}(\beta\alpha^2 + \gamma\alpha + \gamma\beta^{-1}\gamma + \beta^2)\Big)^{-1}.
\end{aligned}$$

A free Waring-Lagrange theorem

Let

$$u = \tfrac{1}{2}(x + y), \qquad v = \tfrac{1}{2}(x - y)$$

and let

$$\alpha = u, \qquad \beta = v^2, \qquad \gamma = vuv.$$

Every symmetric free polynomial in x and y can be written as a free polynomial in α, β, γ and β^{-1}.

Proof is by dimension-counting.

Corollary: symmetric nc-functions

Let

$$\mathcal{M}^d \stackrel{\text{def}}{=} \bigcup_{n=1}^{\infty} (\mathcal{M}_n)^d.$$

Let $\pi : \mathcal{M}^2 \to \mathcal{M}^3$ be given by

$$\pi(x,y) = (\alpha, \beta, \gamma) = \left(\tfrac{1}{2}(x+y), \tfrac{1}{4}(x-y)^2, \tfrac{1}{8}(x-y)(x+y)(x-y)\right).$$

For every symmetric free polynomial map $f : \mathcal{M}^2 \to \mathcal{M}$ there exists a rational nc-map $F : \mathcal{M}^3 \to \mathcal{M}$ such that $f = F \circ \pi$.

We could try to prove: for every symmetric *freely holomorphic map* $f : \mathcal{M}^2 \to \mathcal{M}$ there exists a freely holomorphic map $F : \mathcal{M}^3 \to \mathcal{M}$ such that $f = F \circ \pi$. Unfortunately it's false.

A holomorphic free Waring-Lagrange theorem 1

There exists a two-dimensional topological nc-manifold $\mathcal{G}$ and a holomorphic map $\pi : \mathbb{M}^2 \to \mathcal{G}$ which induces a canonical isomorphism between

- the algebra of symmetric freely holomorphic functions on $\mathcal{M}^2$ and
- the algebra of holomorphic functions on $\mathcal{G}$ that have a certain local boundedness property.

$\mathcal{G}$ is embedded in $\mathcal{M}^3$ and π is the map (u, v^2, vuv) as before.

The free topology on $\mathcal{M}^d$

For any $I \times J$ matrix $\delta = [\delta_{ij}]$ of free polynomials in d non-commuting variables define

$$B_\delta = \{x \in \mathcal{M}^d : \|\delta(x)\| < 1\}.$$

The *free topology* on $\mathcal{M}^d$ is the topology for which a base consists of the sets B_δ.

Conditionally nc-functions

Let D be a subset of $\mathbb{M}^d$ and f be a mapping from D to $\mathbb{M}$. We say that f is *conditionally nc* if f is a graded function and

(i) if $x, s^{-1}xs \in D$, then $f(s^{-1}xs) = s^{-1}f(x)s$;

(ii) there exists a graded function $\hat{f}$ defined on the set

$$\hat{D} \stackrel{\text{def}}{=} \{y \in \mathbb{M}^d : \text{ there exists } x \in D \text{ such that } x \oplus y \in D\}$$

such that, for all $x \in D$ such that $x \oplus y \in D$,

$$f\left(\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}\right) = \begin{bmatrix} f(x) & 0 \\ 0 & \hat{f}(y) \end{bmatrix}.$$

A holomorphic free Waring-Lagrange theorem 2

There exists a two-dimensional Zariski-free manifold $\mathcal{G}$ and a holomorphic map $\pi : \mathbb{M}^2 \to \mathcal{G}$ with the following property. There is a canonical bijection between the classes of

(i) freely holomorphic symmetric nc functions f on $\mathbb{M}^2$, and

(ii) holomorphic functions F defined on the nc-manifold $\mathcal{G}$ that are conditionally nc and are locally bounded, meaning that, for every $w \in \mathbb{M}^2$, there is a free neighbourhood U of w such that F is bounded on $\pi(U) \cap \mathcal{G}$.

Reference

Jim Agler, John E. McCarthy and N. J. Young,

Non-commutative manifolds, the free square root and symmetric functions in two non-commuting variables,

Trans. London Math. Soc. (2018) 5(1) 132 – 183

A free Newton-Girard-Vieta formula for three noncommuting variables x, y, z

Let ω be a primitive cube root of 1 and let

$$\begin{aligned} u &= x + y + z, \\ v &= x + \omega y + \omega^2 z, \\ w &= x + \omega^2 y + \omega z. \end{aligned}$$

Let also

$$\begin{aligned} p_n &= x^n + y^n + z^n, \\ q_n &= x^n + \omega y^n + \omega^2 z^n, \\ r_n &= x + \omega^2 y^n + \omega z^n. \end{aligned}$$

A recursion

It is easy to show that, for $n \geq 1$,

$$\begin{bmatrix} p_n \\ q_n \\ r_n \end{bmatrix} = \frac{1}{3} \begin{bmatrix} u & w & v \\ v & u & w \\ w & v & u \end{bmatrix} \begin{bmatrix} p_{n-1} \\ q_{n-1} \\ r_{n-1} \end{bmatrix}.$$

Let

$$T = \frac{1}{3} \begin{bmatrix} u & w & v \\ v & u & w \\ w & v & u \end{bmatrix} = \frac{1}{3} \left(uI + w \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} + v \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^2 \right).$$

Then

$$\begin{bmatrix} p_n \\ q_n \\ r_n \end{bmatrix} = T \begin{bmatrix} p_{n-1} \\ q_{n-1} \\ r_{n-1} \end{bmatrix} = T^n \begin{bmatrix} p_0 \\ q_0 \\ r_0 \end{bmatrix} = T^n \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}.$$

A formula for $x^n + y^n + z^n$ in terms of u, v, w

Thus

$$\begin{aligned} x^n + y^n + z^n &= p_n \\ &= 3 \times \text{ the } (1,1) \text{ entry of } T^n \\ &= \frac{1}{3^{n-1}} \sum_{0 \le j,k,\ell \le n,\, j+k+\ell=n,\, 3|2k+\ell} m_{jk\ell}(u,v,w) \end{aligned}$$

where $m_{jk\ell}(u,v,w)$ is the sum of all monomials in u, v, w of degree j, k, ℓ in u, v, w respectively.