On local isometries between algebras of C(Y)-valued differentiable maps

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Notations

Let E and F be Banach spaces. Denote:

$$F^E = \{\, T \text{ is a mapping from } E \text{ to } F \,\},$$
$$\mathcal{B}(E,F) = \{\, T \text{ is a continuous linear operator from } E \text{ to } F \,\},$$
$$\mathrm{Iso}(E,F) = \{\, T \text{ is a surjective linear isometry from } E \text{ to } F \,\}.$$

K denotes either

$$[0,1] = \{t \in \mathbb{R} \colon 0 \le t \le 1\}$$

or

$$\mathbb{T} = \{z \in \mathbb{C} \colon \ |z| = 1\}.$$

Algebraic and topological reflexivity (Molnár '02)

Let E and F be Banach spaces and $\emptyset \neq \mathcal{S} \subseteq \mathcal{B}(E, F)$.

Define the algebraic reflexive closure of $\mathcal{S}$ by

$$\mathrm{ref}_{\mathrm{alg}}(\mathcal{S}) = \{T \in \mathcal{B}(E, F) \colon \forall e \in E,\ \exists S_e \in \mathcal{S} \mid S_e(e) = T(e)\},$$

and the topological reflexive closure of $\mathcal{S}$ by

$$\mathrm{ref}_{\mathrm{top}}(\mathcal{S}) = \left\{T \in \mathcal{B}(E, F) \colon \forall e \in E,\ \exists \{S_{e,n}\}_{n\in\mathbb{N}} \subset \mathcal{S} \mid \lim_{n\to\infty} S_{e,n}(e) = T(e)\right\}$$

Clearly, $\mathcal{S} \subseteq \mathrm{ref}_{\mathrm{alg}}(\mathcal{S}) \subseteq \mathrm{ref}_{\mathrm{top}}(\mathcal{S})$.

The set $\mathcal{S}$ is said to be algebraically reflexive if $\mathrm{ref}_{\mathrm{alg}}(\mathcal{S}) \subseteq \mathcal{S}$.

The set $\mathcal{S}$ is said to be topologically reflexive if $\mathrm{ref}_{\mathrm{top}}(\mathcal{S}) \subseteq \mathcal{S}$.

2-Algebraic and 2-topological reflexivity (Šemrl '97)

Let E and F be Banach spaces and $\emptyset \neq \mathcal{S} \subseteq \mathcal{B}(E, F)$.

Define the 2-algebraic reflexive closure of $\mathcal{S}$, $2\text{-ref}_{\text{alg}}(\mathcal{S})$, by

$$\left\{\Delta \in F^E \colon \forall e, u \in E,\ \exists S_{e,u} \in \mathcal{S} \mid S_{e,u}(e) = \Delta(e),\ S_{e,u}(u) = \Delta(u)\right\}$$

and the 2-topological reflexive closure of $\mathcal{S}$, $2\text{-ref}_{\text{top}}(\mathcal{S})$, by

$$\{\Delta \in F^E \colon \forall e, u \in E,\ \exists \{S_{e,u,n}\}_{n\in\mathbb{N}} \subset \mathcal{S} \\ \mid \lim_{n\to\infty} S_{e,u,n}(e) = \Delta(e),\ \lim_{n\to\infty} S_{e,u,n}(u) = \Delta(u)\}.$$

Clearly, $\mathcal{S} \subseteq 2\text{-ref}_{\text{alg}}(\mathcal{S}) \subseteq 2\text{-ref}_{\text{top}}(\mathcal{S})$.

The set $\mathcal{S}$ is called 2-algebraically reflexive if $2\text{-ref}_{\text{alg}}(\mathcal{S}) \subseteq \mathcal{S}$.

The set $\mathcal{S}$ is called 2-topologically reflexive if $2\text{-ref}_{\text{top}}(\mathcal{S}) \subseteq \mathcal{S}$.

More suggestive terminologies

Let E and F be Banach spaces.

$$\mathrm{ref}_{\mathrm{alg}}(\mathrm{Iso}(E,F)) = \{\text{local isometries from E to F}\},$$
$$\mathrm{ref}_{\mathrm{top}}(\mathrm{Iso}(E,F)) = \{\text{approximate local isometries from E to F}\}.$$

$$2\text{-}\mathrm{ref}_{\mathrm{alg}}(\mathrm{Iso}(E,F)) = \{\text{2-local isometries from E to F}\},$$
$$2\text{-}\mathrm{ref}_{\mathrm{top}}(\mathrm{Iso}(E,F)) = \{\text{approximate 2-local isometries from E to F}\}.$$

Banach algebras of C(Y)-valued differentiable maps

Definition

Let K be either $[0,1]$ or $\mathbb{T}$, and let Y be a Hausdorff compact space. A mapping $F \in C(K, C(Y))$ is said to be continuously differentiable if there exists a map $G \in C(K, C(Y))$ such that

$$\lim_{x \to x_0} \left\| \frac{F(x) - F(x_0)}{x - x_0} - G(x_0) \right\|_\infty = 0$$

for every $x_0 \in K$. We denote $F' = G$.
The linear space

$$C^1(K, C(Y)) = \{F \in C(K, C(Y)) \colon F \text{ is continuously differentiable}\},$$

equipped with the Σ-norm:

$$\|F\|_\Sigma = \|F\|_\infty + \|F'\|_\infty \qquad (F \in C^1(K, C(Y))),$$

is a unital semisimple commutative complex Banach algebra.

The unity is the mapping $1_K \otimes 1_Y \colon K \to C(Y)$ given by

$$(1_K \otimes 1_Y)(x) = 1_K(x)1_Y = 1_Y \quad (x \in K).$$

Given $f \in C^1(K)$ and $g \in C(Y)$, the map $f \otimes g \colon K \to C(Y)$, given by

$$(f \otimes g)(x) = f(x)g \qquad (x \in K),$$

belongs to $C^1(K, C(Y))$ with

$$\begin{aligned} \|f \otimes g\|_\infty &= \|f\|_\infty \|g\|_\infty , \\ \|(f \otimes g)'\|_\infty &= \|f'\|_\infty \|g\|_\infty , \\ \|f \otimes g\|_\Sigma &= \|f\|_\Sigma \|g\|_\infty . \end{aligned}$$

If $\#(Y) = 1$, then $C(Y) \cong \mathbb{C}$, and we write $C^1(X) = C^1(X, C(Y))$.

Background

The reflexivity of the isometry group of spaces of differentiable mappings has been studied for:

- $C^1([0,1])$, the Banach algebra of all continuously differentiable complex-valued functions on $[0,1] \subseteq \mathbb{R}$ (Hatori & Oi, '19).
- $C^{(n)}(X)$, the Banach algebra of all n-times continuously differentiable complex-valued functions on an open subset $X \subseteq \mathbb{R}$. (Hosseini & JV, '21).
- $C^{(n)}(X, E)$, the Banach space of all n-times continuously differentiable Banach-valued functions on an open subset $X \subseteq \mathbb{R}^n$ (Miao & X. Wang & Li & L. Wang, '20).

Objective

Our main goal is to prove:

$\mathrm{Iso}(C^1(K, C(Y)))$ is topologically reflexive and 2-topologically reflexive whenever $\mathrm{Iso}(C(Y))$ is topologically reflexive.

Is our result applicable? Yes, there are Hausdorff compact spaces Y for which $\mathrm{Iso}(C(Y))$ is topologically reflexive.

FIRST PART

THE TOOLS

Spherical variants of Gleason–Kahane-Żelazko and Kowalski–Słodkowski theorems

Theorem (Li, Peralta, L. Wang & Y.-S. Wang, '19)

Let $\mathcal{A}$ be a unital complex Banach algebra, and let $F\colon \mathcal{A} \to \mathbb{C}$ be a continuous linear functional such that $F(x) \in \mathbb{T}\sigma(x)$ for every $x \in \mathcal{A}$. Then $\overline{F(\mathbf{1})}F$ is multiplicative.

Theorem (Li, Peralta, L. Wang & Y.-S. Wang, '19)

Let $\mathcal{A}$ be a unital complex Banach algebra, and let $\Delta\colon \mathcal{A} \to \mathbb{C}$ be a function such that

1. Δ is homogeneous: $\Delta(\lambda x) = \lambda\Delta(x)$ for every $\lambda \in \mathbb{C}$ and $x \in \mathcal{A}$.
2. Δ satisfies the spectral condition: $\Delta(x) - \Delta(y) \in \mathbb{T}\sigma(x - y)$ for every $x, y \in \mathcal{A}$.

Then Δ is linear, and there exists $\lambda_0 \in \mathbb{T}$ such that $\lambda_0\Delta$ is multiplicative.

Onto linear isometries of $C^1(K, C(Y))$-spaces

Theorem (Hatori & Oi, '18)

Let K be either $[0,1]$ or $\mathbb{T}$ and let Y_1, Y_2 be Hausdorff compact spaces. A map $T\colon C^1(K, C(Y_1)) \to C^1(K, C(Y_2))$ is a surjective linear isometry with respect to the Σ-norms if and only if it has a representation of BJ type, that is, there exist:

- a function $h \in C(Y_2, \mathbb{T})$,
- a function $\phi \in C(K \times Y_2, K)$ with $\phi^y \in \mathrm{Iso}(K)$ for each $y \in Y_2$,
- a mapping $\tau \in \mathrm{Homeo}(Y_2, Y_1)$,

such that

$$T(F)(x,y) = h(y)F(\phi(x,y), \tau(y)) \qquad ((x,y) \in K \times Y_2),$$

for all $F \in C^1(K, C(Y_1))$.

-For each $y \in Y$, $\varphi^y\colon K \to K$ is defined by $\varphi^y(x) = \varphi(x,y)$ for all $x \in K$.
-τ depends only on the second variable.

Unital algebra homom. of $C^1(K, C(Y))$-algebras

Theorem (Hosseini & JV & Ramírez, '22)

Let K be either $[0,1]$ or $\mathbb{T}$ and let Y_1, Y_2 be Hausdorff compact spaces. If T is a unital algebra homomorphism of $C^1(K, C(Y_1))$ to $C^1(K, C(Y_2))$, then there exist:

- a function $\phi \in C(K \times Y_2, K)$ so that $\phi^y \in C^1(K)$ for each $y \in Y_2$,
- a map $\tau \in C(K \times Y_2, Y_1)$,

such that

$$T(F)(x,y) = F(\phi(x,y), \tau(x,y)) \qquad ((x,y) \in K \times Y_2)$$

for all $F \in C^1(K, C(Y_1))$.

SECOND PART

THE RESULTS

Topological reflexivity in $C^1(K)$-algebras

Using the descriptions of onto linear isometries and unital algebra homomorphisms of $C^1(K)$, the spherical variant of Gleason–Kahane–Żelazko theorem and the Arzelá–Ascoli theorem:

Theorem (Hosseini & JV & Ramírez, '22)

For $K = [0, 1]$ or $\mathbb{T}$, $\mathrm{Iso}(C^1(K))$ is topologically reflexive.

Topological reflexivity in $C^1(K, C(Y))$-algebras

Theorem (Hosseini & JV & Ramírez, '22)

Let K be either $[0,1]$ or $\mathbb{T}$ and let Y_1, Y_2 be Hausdorff compact spaces. Suppose that $\mathrm{Iso}(C(Y_1), C(Y_2))$ is topologically reflexive. Then

$$\mathrm{Iso}(C^1(K, C(Y_1)), C^1(K, C(Y_2)))$$

is topologically reflexive.

A sketch of the proof (steps 1–3)

Let $T \in \mathrm{ref}_{\mathrm{top}}(\mathrm{Iso}(C^1(K, C(Y_1)), C^1(K, C(Y_2))))$. We prove that T has a representation of BJ type:

$$T(F)(x,y) = h(y)F(\phi(x,y), \tau(y)) \qquad ((x,y) \in K \times Y_2),$$

for all $F \in C^1(K, C(Y_1))$, and so

$$T \in \mathrm{Iso}(C^1(K, C(Y_1)), C^1(K, C(Y_2))).$$

Steps:

(1) Using (Hat-Oi'18), for every $F \in C^1(K, C(Y_1))$, we have $\|T(F)\|_\infty = \|F\|_\infty$, $\|T(F)'\|_\infty = \|F'\|_\infty$ and $\|T(F)\|_\Sigma = \|F\|_\Sigma$.

(2) Using (Hat-Oi'18), for every $F \in C^1(K, C(Y_1))$, there exist three sequences: $\{h_{F,n}\}_{n\in\mathbb{N}}$ in $C(Y_2, \mathbb{T})$, $\{\phi_{F,n}\}_{n\in\mathbb{N}}$ in $C(K \times Y_2, K)$ such that, for each $y \in Y_2$, $\phi_{F,n}^y \in \mathrm{Iso}(K)$ for all $n \in \mathbb{N}$, and $\{\tau_{F,n}\}_{n\in\mathbb{N}}$ in $\mathrm{Homeo}(Y_2, Y_1)$ satisfying that

$$\lim_{n\to\infty} \|h_{F,n}F(\phi_{F,n}, \tau_{F,n}) - T(F)\|_\Sigma = 0.$$

(3) There exists a $h \in C(Y_2, \mathbb{T})$ such that $T(1_K \otimes 1_{Y_1}) = 1_K \otimes h$.

A sketch of the proof (steps 4–7)

(4) Using (Li-Per-Wan-Wan'19), for each $(x, y) \in K \times Y_2$, the functional $S_{(x,y)}\colon C^1(K, C(Y_1)) \to \mathbb{C}$ defined by

$$S_{(x,y)}(F) = \overline{h(y)}\,T(F)(x, y) \qquad (F \in C^1(K, C(Y_1))),$$

is linear, unital and multiplicative.

(5) Using (Hos-JV-Ram'22), there exist two maps $\phi \in C(K \times Y_2, K)$ with $\phi^y \in C^1(K)$ for each $y \in Y_2$, and $\tau \in C(K \times Y_2, Y_1)$ such that

$$T(F)(x, y) = h(y)F(\phi(x, y), \tau(x, y)) \qquad ((x, y) \in K \times Y_2),$$

for all $F \in C^1(K, C(Y_1))$.

(6) Using (Hat-Oi'18 & Hos-JV-Ram'22), for each $y \in Y_2$, $\phi^y \in \mathrm{Iso}(K)$.

(7) Using Banach–Stone theorem, there exists a map $\beta \in \mathrm{Homeo}(Y_2, Y_1)$ such that

$$\beta(y) = \tau(x, y) \qquad (y \in Y_2),$$

where x is any point of K.

2-Topological reflexivity in $C^1(K, C(Y))$-algebras

Applying also the spherical variant of Kowalski–Słodkowski theorem:

Corollary (Hosseini & JV & Ramírez, '22)

Let K be either $[0,1]$ or $\mathbb{T}$ and let Y_1, Y_2 be Hausdorff compact spaces. Suppose that $\mathrm{Iso}(C(Y_1), C(Y_2))$ is topologically reflexive. Then

$$\mathrm{Iso}(C^1(K, C(Y_1)), C^1(K, C(Y_2)))$$

is 2-topologically reflexive.

MANY THANKS
FOR YOUR TIME!