BALANCED FOURIER TRUNCATIONS ON THE FREE GROUP

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BANACH ALGEBRAS AND APPLICATIONS
GRANADA

Outline

1. Motivation
2. An invariant
3. Harmonic analysis elements
4. Metric consequences

1- MOTIVATION

INCLUSIONS BETWEEN BANACH SPACES

- Old question: given $\mathbb{X}, \mathbb{Y}$ Banach spaces, when is $\mathbb{X}$ isomorphic to a linear subspace of $\mathbb{Y}$?
- $1 < q, p < \infty$: when is $L_q \underset{\text{linear}}{\subset} L_p$?

- Some facts:
  * $L_2 \subset L_p$: always.
  * $q < \min\{2, p\}$ or $q > \max\{2, p\}$: never.
  * Banach's conjecture: rest of cases work.
  * Kadeć (1958): $L_q \subset L_p$ if $p < q < 2$.
  * $\boxed{\text{Paley (1936): } p > q > 2 \text{ does not work}}$

Bi-Lipschitz inclusions

- From now on, $p > q > 2$.
- Still possible: is there $T : L_q \to L_p$ bi-Lipschitz?
- When such a T exists between $\mathcal{X}$ and $\mathcal{Y}$,

$$\boxed{c_{\mathcal{Y}}(\mathcal{X}) := \inf_{T : \mathcal{X} \to \mathcal{Y}} \|T\|_{\text{bi-Lip}}}$$

$c_{\mathcal{Y}}(\mathcal{X})$ is called <u>distortion</u>.

- Mankiewicz (1972): There is no such $T : L_q \to L_p$
  * Proof uses differentiability of Lip maps $\Longrightarrow$ full power of the linear structure of L_p.
  * Question (related to Ribe's program): Non-linear proof?

2- An Invariant

A Bi-Lipschitz invariant

- A proof of the non-bi-Lipschitz embeddability of L_q into L_p can be given without differentiability
- Naor-Schechtman (2016): bi-Lipschitz invariant called X_p, in terms <u>only</u> of metric.
- L_p is an X_p space, and not an X_q space if $p \neq q$

$$\Longrightarrow$$

metric + quantitative proof of Mankiewicz theorem.

- Consequences (Naor, 2016):

  * $C_p([m]_q^n) \sim_{p,q} \min\left\{ m^{1-2/q}, n^{\frac{(p-q)(q-2)}{q^2(p-2)}} \right\}$

(phase transition at $m \sim n^{(p-q)/q(p-2)}$).

  * $(L_q, \|x-y\|_{L_q}^{\theta}) \hookrightarrow L_p$ only if $\theta \leq \frac{q}{p}$

THE KEY STEP IN THE METRIC X_p INEQUALITY

<u>Theorem</u> (Naor '16): $p \geq 2$, $k \in [n]$,

$$\underbrace{\frac{1}{\binom{n}{k}} \sum_{\substack{S \subseteq [n] \\ |S| = k}}}_{\substack{\text{Avg. over subsets} \\ \text{of size } k.}} \Big\| \sum_{A \subseteq S} \hat{f}(A) W_A \Big\|^p_{L_p(\Omega_n)} \lesssim_p \underbrace{\frac{k}{n}}_{\substack{\text{Gain} \\ \text{over Poincaré-type} \\ \text{inequality}}} \sum_{j=1}^{n} \| \partial_j f \|^p_{L_p(\Omega_n)} + \underbrace{\left(\frac{k}{n} \right)^{p/2}}_{\substack{\text{Gain} \\ \text{over trivial} \\ \text{inequality}}} \| f \|^p_{L_p(\Omega_n)} \qquad (N_p)$$

OUR GOAL

- Main goal: extend (N_p) to derive metric consequences within $L_p(\mathcal{M})$, $\mathcal{M}$ $\underbrace{\text{von Neumann algebra}}_{\text{A BANACH ALGEBRA! :)}}$.

- Precedent: if $f = \sum_{j \in [n]} \hat{f}(\{j\}) \varepsilon_j$,

$$\frac{1}{\binom{n}{k}} \sum_{S \subseteq [n]} \Big\| \sum_{j \in S} \hat{f}(\{j\}) \varepsilon_j \Big\|_p^p \approx_p \frac{k}{n} \sum_{j=1}^{n} \| \hat{f}(\{j\}) \varepsilon_j \|_p^p + \left(\frac{k}{n}\right)^{p/2} \|f\|_p^p$$

(Johnson, Maurey, Schechtman, Tzafriri 1979)

- Junge, Xu (2008): true also if $\hat{f}(\{j\}) \in S_p$, or even $\hat{f}(\{j\}) \in L_p(\mathcal{M})$.

3- Harmonic Analysis

(on group von Neumann algebras)

Group von Neumann algebras

- Let G be discrete
- Left regular representation:

$$\lambda : G \longrightarrow B(\ell_2(G))$$
$$g \longmapsto \lambda(g) : \delta_h \mapsto \delta_{g^{-1}h}$$

- $\lambda(g)$ plays the role of the characters on $\mathbb{Z}$.
- $\mathcal{L}(G) = \overline{\mathrm{span}\{\lambda(g)\}}^{w*}$ is a von Neumann algebra.
- $f \in \mathcal{L}(G) \Rightarrow f = \sum_{g \in G} \hat{f}(g)\lambda(g)$
- Natural trace on $\mathcal{L}(G)$: $\tau(f) = \hat{f}(e)$.
- We want an analogue of (N_p) on $L_p(\mathcal{L}(\mathbb{F}_n))$.

$$\mathbb{F}_n = \langle g_1, g_2, \cdots, g_n \rangle .$$

The Boolean cube Ω_n vs	the free group $\mathbb{F}_n$
$\Omega_n = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$	$\mathbb{F}_n = \mathbb{Z} * \mathbb{Z} * \cdots \mathbb{Z}$
$\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$	$w = g_{j_1}^{\ell_1} g_{j_2}^{\ell_2} \cdots g_{j_m}^{\ell_m}$
$f : \Omega_n \longrightarrow \mathbb{C}$	$f \in \mathcal{L}(\mathbb{F}_n)$
W_A	$\lambda(w)$
$\hat{f}(A) = \mathbb{E}_\varepsilon(f W_A)$	$\hat{f}(w) = \tau(\lambda(w)^* f)$
$\partial_j f(\varepsilon) = f(\varepsilon) - f(\varepsilon - 2\varepsilon_j e_j)$	$\partial_j f = \sum_{\substack{w \geq g_j \\ \text{or} \\ w \geq g_j}} \hat{f}(w)\lambda(w)$
$E_{[n]\setminus S} f = \sum_{A \subset S} \hat{f}(A) W_A$	$E_{[n]\setminus S} f = \begin{cases} \sum_{w \in \mathbb{F}_S} \hat{f}(w)\lambda(w) \\ \sum_{w \in \mathbb{A}_S} \hat{f}(w)\lambda(w) \end{cases}$

Main result

Theorem (Cuevas-Méndez, C-A, Parcet 2022). Let $k \in [n]$. For any $f \in L_p(\mathcal{L}(\mathbb{F}_n))$ such that $\tau(f) = 0$,

$$\frac{1}{\binom{n}{k}} \sum_{\substack{S \subset [n] \\ |S| = k}} \| E_{[n] \setminus S} f \|_p^p \lesssim_p \frac{k}{n} \sum_{j=1}^{n} \| \partial_j f \|_p^p + \left(\frac{k}{n} \right)^{p/2} \| f \|_p^p$$

- Two options for $E_{[n] \setminus S}$:

$$\mathbb{F}_S = \{ w \in \mathbb{F}_n : w = g_{j_1}^{\ell_1} g_{j_2}^{\ell_2} \cdots g_{j_k}^{\ell_k}, \ j_i \in S \text{ for all } i \in [k] \}$$

$$\mathbb{A}_S = \{ w \in \mathbb{F}_n : w = g_{j_1}^{\ell_1} g_{j_2}^{\ell_2} \cdots g_{j_k}^{\ell_k}, \ j_1 \in S \}$$

- $\mathbb{F}_S \subset \mathbb{A}_S$, so inequality for $\mathbb{A}_S$ is stronger.

Elements of the proof + Other scenarios/extensions

- Key elements of the proof: n-independent estimates for Riesz transforms on groups (Junge, Mei, Parcet 2018) (true for many groups).
- Boundedness of free Hilbert transforms (Mei, Ricard 2017).
- Result valid for groups G with:
  * Product structure: $\mathbb{Z}^n$, $\mathbb{Z}_2^{*n}$, $\mathbb{T}^n$ ...
  * A suitable differential structure (i.e. cocycles)
  * Good truncations of Fourier series.
- Result holds for $f \in L_p(\mathcal{L}(G) \bar{\otimes} M)$, M any other semifinite von Neumann algebra.

4- METRIC CONSEQUENCES

Corollary (Naor 2016 + Conde-Alonso, C-A, Parcet 2022):

Let $\mathcal{M}$ be semifinite. Then, $L_p(\mathcal{M})$ is a metric X_p space. Therefore, no commutative L_q space can be embedded into $L_p(\mathcal{M})$ (with a bi-Lipschitz embedding).

- $C_{L_p(\mathcal{M})}([m]_q^n) \sim C_{L_p}([m]_q^n)$

- $(L_q, \|x-y\|_q^\theta) \hookrightarrow L_p(\mathcal{M})$ only if $\theta \leq \frac{q}{p}$

THANK YOU

VERY MUCH!!