Almost contractive maps

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Motivation

Based on joint works with Jean Roydor

- Extension of “contractive results” in harmonic analysis
- Quantitive perturbation theories
- Gap phenomena

Basic facts on contractive maps on C^*-algebras

A, B unital C^*-algebra and let $T : A \to B$ be contractive

$$T(1) = 1 \quad \Leftrightarrow \quad T \geqslant 0$$

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In particular $T(x^*) = T(x)^*$.
We say that T is completely contractive if $T \otimes id : M_n(A) \to M_n(B)$ is contractive.
We say that T is completely positive if $T \otimes id : M_n(A) \to M_n(B)$ is positive for all $n \geqslant 1$.

Actually if $T(1) = 1$, then

$$T \text{ is (completely) contractive} \quad \Leftrightarrow \quad T \text{ is (completely) positive}$$

Basic facts on contractive maps on C^*-algebras

Recall that Jordan product is defined by

$$x \circ y = \frac{xy + yx}{2}$$

$$x \circ x^* = Re(x)^2 + Im(x)^2$$

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Kadison's inequality

Assume $T(1) = 1$ and T is contractive (or positive), then

$$\forall x \in A, \qquad T(x \circ x^*) \geqslant T(x) \circ T(x)^*$$

If T is 2-positive then

$$\forall x \in A, \qquad T(xx^*) \geqslant T(x)T(x)^*$$

Basic facts on contractive maps on C^*-algebras

If T is unital cp then $T(x) = a^*\pi(x)a$ with π a unital $*$-representation and $a^*a = 1$:

$$T(xx^*) - T(x)T(x)^* = a^*\pi(x)(1 - aa^*)\pi(x)^*a \geqslant 0$$

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Considering, the sesquilinear map $(x, y) \mapsto T(xy^*) - T(x)T(y)^*$
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Multiplicative domain

Assume $T(1) = 1$ and T is completely contractive, then

$$L = \{x \mid T(xx^*) = T(x)T(x)^*\} = \{x \mid \forall y, T(xy) = T(x)T(y)^*\}$$

L is called the left multiplicative domain.
$R \cap L$ is a C^*-algebra.

Basic facts on contractive maps on C^*-algebras

Similarly,

Jordan Multiplicative domain

Assume $T(1) = 1$ and T is contractive, then

$$M = \{x \mid T(x \circ x^*) = T(x) \circ T(x)^*\} = \{x \mid \forall y, T(x \circ y) = T(x) \circ T(y)^*\}$$

M is a Jordan algebra

Beware that the Jordan product is not associative.

An application

Korovkin's Theorem

Let $T_n : C([0,1]) \to C([0,1]$ be a sequence of unital positive maps such that $\|T_n(x) - x\| \to 0$ and $\|T_n(x^2) - x^2\|$, then for any $f \in C([0,1])$, $\|T_n(f) - f\| \to 0$

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$A = B = \prod_{\mathfrak{U}} C[0,1]$, $T = \prod_{\mathfrak{U}} T_n$, then by assumption $T(x) = x$ and x is in the multiplicative domain, hence $T(f) = f$ for all $f \in C([0,1])$ by Weierstrass' theorem.

What can be said if one drop the assumption to $\|T\| < 1 + \varepsilon$?
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Since we have algebraic characterization, one can use

The ultraproduct argument

For every $\delta > 0$, there exists $\varepsilon > 0$ such that for any $T : A \to B$ with $T(1) = 1$ and $\|T\| < 1 + \varepsilon$ and any $x \in A^+$ with $\|x\| = 1$,

$$dist(T(x), B^+) < \delta$$

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Pf : assume it is false, then there is $\delta > 0$ and $\varepsilon_n \to 0$ and $T_n : A_n \to B_n$, $x_n \in A_n^+$ such that $dist(T_n(x_n), B^+) \geqslant \delta$.
Let $A = \prod_{\mathfrak{U}} A_n$, $B = \prod_{\mathfrak{U}} B_n$, $T = \prod_{\mathfrak{U}} T_n$ and $x = (x_n)$.
Then T is positive thus $T(x) \geqslant 0$, but then $0 = \lim dist(T_n(x_n), B^+)$.

It can be applied in many situations but it has a major drawback it is not explicit.
With Jean Roydor, we made them explicit

Examples of quantitative estimates

Let $T : A \to B$ with $T(1) = 1$ and $\|T\| < 1 + \varepsilon$, then for any $x \in A$:

$$\|T(x)^* - T(x^*)\| \leqslant 4\sqrt{2\varepsilon + \varepsilon^2}\|x\|$$

$$\|K_T(x, y)\| \leqslant 2(\|K_T(x, x)\| + \delta_\varepsilon \|x\|^2)^{1/2}(\|K_T(y, y)\| + \delta_\varepsilon \|x\|^2)^{1/2} + v_\varepsilon \|x\|\|y\|,$$

where $K_T(x, y) = T(x^* \circ y) - T(x^*) \circ T(y)$ and $\delta_\varepsilon = 2(4 + 3\varepsilon)\sqrt{2\varepsilon + \varepsilon^2}$ and $v_\varepsilon = 2(5 + 4\varepsilon)\sqrt{2\varepsilon + \varepsilon^2}$.

Harmonic analysis

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Example : $L_1(G)$
At Banach space level, isometrically $L_1(G)$ only remembers the measure space (G, μ). If ones want the group structure one has to take the convolution $*$ product into account:

Wendel's thms (1951, 1952)

If there is an isometric algebra isomorphism $(L_1(G), *) \to (L_1(H), *)$ then G and H are homeomorphic as topological groups.

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Johnson and Rigelhof also did it for measure algebras.

The assumption in Wendel's theorem can be relaxed a little bit

Kalton and Wood (1976)

Assume that $T : (L_1(G), *) \to (L_1(H), *)$ is an algebra isomorphism with norm less than 1.247 then G and H are homeomorphic as topological groups.

If G and H are abelian, one can take $\sqrt{2}$, this is the best possible.

The Fourier-Stieltjes algebra

G s.c. locally countable group
$\omega : G \to B(K)$ the universal unitary representation of G

$$B(G) = \{g \mapsto \langle \omega(g)\xi, \eta \rangle \,;\, \xi, \eta \in K\}$$

$$\|f\|_{B(G)} = \inf\{\|\xi\|.\|\eta\| \;;\; f(g) = \langle \omega(g)\xi, \eta \rangle\}$$

With the pointwise product, the Fourier-Stieltjes algebra $(B(G), .)$ is a Banach algebra.

$$B(G) = C^*(G)^*, \qquad B(G)^* = W(G) = \{\omega(g) \;;\; g \in G\}''$$

The Fourier algebra

$\lambda : G \to B(L_2(G,\mu))$ the left regular representation of G

$$A(G) = \{g \mapsto \langle \lambda(g)\xi, \eta\rangle \,;\, \xi, \eta \in L_2(G)\}$$

This is the same as considering

$$A(G) = \{g \mapsto \langle \sum_{i=1}^{\infty} \lambda(g)\xi_i, \eta_i\rangle \,;\, \xi_i, \eta_i \in L_2(G), \sum \|\xi_i\|.\|\eta_i\| < \infty\}$$

$$\|f\|_{A(G)} = \inf\{\|\xi\|.\|\eta\| \;;\; f(g) = \langle \lambda(g)\xi, \eta\rangle\}$$

With the pointwise product, the Fourier algebra $(A(G), .)$ is a Banach algebra.

$$A(G) \subset C_c^*(G)^*, \qquad A(G)^* = L(G) = \{\lambda(g) \;;\; g \in G\}''$$

$A(G) \subset B(G)$ is isometrically a subspace and is an ideal

General problem : To understand algebra homomorphisms
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They are automatically continuous (Silov).

We have $\sigma(A(G)) \simeq G$, it follows that

There exist an open subset $\Omega \subset H$ and a continuous map $\alpha : \Omega \to G$ such that

$$\Phi(f)(h) = \begin{cases} f(\alpha(h)) & \text{if } h \in \Omega \\ 0 & \text{if } h \notin \Omega \end{cases}$$

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Idea : to look at Φ^* !

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Extensions by Host, Ilie

Ilie and Spronk (2005)

If G is amenable and H arbitrary, this also works for completely bounded homomorphisms iff α is a continuous piecewise affine map. This is fall if $G \supset F_2$.

By analogy with Wendel :

Walter (1972)

If there is an isometric algebra isomorphism $\Phi : A(G) \to A(H)$, then $G \simeq H$.
Moreover, we have $\Phi(f)(h) = f(g_0\tau(h_0^{-1}h))$ for τ a group isomorphism or anti-isomorphism for some $g_0 \in G$, $h_0 \in H$.

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By analogy with Wendel 2 :

Le Pham (2010)

If there is an conctractive algebra homorphism $\Phi : A(G) \to B(H)$,
There is an open subgroup Ω and τ a continuous group homomorphism or anti-homomorphism and $g_0 \in G$, $h_0 \in H$ so that

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Pb : to relax the contractive assumption !

A first cb version, analogous to Kalton and Wood

Kuznetsova and Roydor (2015)

If there is an algebra homomorphism $\Phi : A(G) \to B(H)$ with $\|Id_{M_2} \otimes \Phi\| \leqslant \sqrt{5}/2$. Then, there is an open subgroup Ω and τ a group isomorphism and $g_0 \in G$, $h_0 \in H$ so that

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In particular $\|\Phi\|_{cb} \leqslant 1$.

The proof relies on some estimates on “almost multiplicative maps” between von Neumann algebras in a previous work of Roydor and myself.

Using the ultraproduct argument :

Kuznetsova and Roydor (2015)

There is some $\varepsilon > 0$ so that if $T : A(G) \to A(H)$ is an algebra homomorphism with $\|T\|.\|T^{-1}\| < 1 + \varepsilon$ then we have $\Phi(f)(h) = f(g_0\tau(h_0^{-1}h))$ for τ a group isomorphism or anti-isomorphism for some $g_0 \in G$, $h_0 \in H$.
In particular $\|T\| = \|T^{-1}\| = 1$.

This is not totally satisfactory because it is in term of the distortion of T.

R. and Roydor

If $T : A(G) \to A(H)$ is an algebra isomorphism with $\|T\| < 1.0005$ then we have $\Phi(f)(h) = f(g_0\tau(h_0^{-1}h))$ for τ a group isomorphism or anti-isomorphism for some $g_0 \in G$, $h_0 \in H$.
In particular $\|T\| = \|T^{-1}\| = 1$.

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If there is an algebra homomorphism $\Phi : A(G) \to B(H)$ with $\|\Phi\| \leqslant 1.00018$. Then, there is an open subgroup Ω and τ a group morphism and $g_0 \in G$, $h_0 \in H$ so that

$$\Phi(f)(h) = \begin{cases} f(g_0\tau(h_0^{-1}h)) & \text{if } h \in h_0\Omega \\ 0 & \text{if } h \notin h_0\Omega \end{cases}$$

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Ideas of the proof :
Look at $\Phi^* : W^*(H) \to L(G)$, assume $\Phi\ \emptyset$, then there is some $h_0 \in H$ such that $\Phi(\omega_{h_0}) = \lambda_{g_0}$.

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Do appropriate translations to define a unital $T : W^*(H) \to L(G)$ with $\|T\| = \|\Phi\|$ (as well as Ω; assume $\Omega = H$).
Hence T is almost positive !

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We must have $T(\omega_h) = \lambda_{f(h)}$.
Thus $T(\omega_h \circ \omega_h^*) \approx T(\omega_h) \circ T(\omega_h)^*$.

Hence T is almost Jordan multiplicative on generators.

Two easy lemmas to conclude

Lemma 1

Unless there are trivial simplifications to make it 0

$$\|\lambda_{g_1} + \lambda_{g_2} - \lambda_{g_3} - \lambda_{g_4}\| \geqslant \sqrt{3}$$

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i.e. $\lambda_{f(h_1 h_2)} + \lambda_{f(h_2 h_1)} - \lambda_{f(h_2)f(h_1)} - \lambda_{f(h_2)f(h_1)} = 0$

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Lemma 2

Let $f : G \to H$ be a map such that $f(\{xy, yx\}) = \{f(x)f(y), f(y)f(x)\}$.
Then f is a group morphism or a group anti-ismomorphism.

Another application of almost contractive maps

A non commutative Amir-Cambern theorem

$$d_{cb}(A,B) = \inf\{\|T\|_{cb}.\|T^{-1}\|_{cb}\ ;\ T : A \to B \text{ cb-isomorphism}\}$$

R. and Roydor

Let A be a separable nuclear C^*-algebra or a von Neumann algebra, then there exists an explicit $\varepsilon_0 > 0$ such that for any C^*-algebra B, the inequality $d_{cb}(A,B) < 1 + \varepsilon_0$ implies that A and B $*$-isomorphic as C^*-algebras or von Neumann algebras.

When A and B are $C(K)$-spaces $\varepsilon_0 = 1$!
This is false for non separable C^*-algebras or without cb (Connes) !
It relies on a deep result by Christensen, Sinclair, Smith and White for nuclear C^*-algebras.
It relies on *cb*-cohomology stuff for vN algebras.

Another application of almost multiplicativity

Assume A is a von Neumann with a nsf trace τ.
One can define for $1 \leqslant p < \infty$:

$$L_p(A, \tau) = \text{closure of } \{x \in A \mid \tau(|x|^p) < \infty\}$$

with $\|x\|_p^p = \tau(|x|^p)$.

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with $\|x\|_p^p = \tau(|x|^p)$.
If $T : A \to A$ is unital and trace preserving then T extends to a contraction on all L_p-spaces.

Caspers, Parcet, Perrin and R.

For any $x \in L_{2p}(A)^+$

$$\|T(x) - T(\sqrt{x})^2\|_{2p} \leqslant \frac{1}{2}\|T(x^2) - T(x)^2\|_p^{1/2}.$$

Caspers, Parcet, Perrin and R.

There is some $C > 0$ such that for any $x \in L_2^+$ and any $0 < \theta \leqslant 1$

$$\|T(x^\theta) - x^\theta\|_{2/\theta} \leqslant C\|T(x) - x\|_2^{\theta/2}\|x\|_2^{\theta/2}.$$

It gives an answer to :
If f has Fourier support in $[-\varepsilon, \varepsilon]$, then f^2 has Fourier support in $[-2\varepsilon, 2\varepsilon]$. What about the opposite ?

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If f has Fourier support in $[-\varepsilon, \varepsilon]$, then f^2 has Fourier support in $[-2\varepsilon, 2\varepsilon]$. What about the opposite ?

If f is positive in L_2 with Fourier support in $[-\varepsilon, \varepsilon]$, then $\sqrt{f}$ must be close in L_4 to a function whose Fourier transform has support in $[-\varepsilon^\alpha, \varepsilon^\alpha]$ $(0 < \alpha < 1)$.

THANK YOU !