Uniqueness of Algebra Norms of Quotient Algebras of Bounded Operators

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Based on joint work with Niels Laustsen.

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We say that $\|\cdot\|$ is *maximal* if for any alternative algebra norm $|\!|\!|\cdot|\!|\!|$ on $\mathcal{A}$, there is some $D > 0$ for which $D|\!|\!|a|\!|\!| \leq \|a\|$ for all $a \in \mathcal{A}$.

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- $\|\cdot\|$ is maximal on $\mathcal{A}$ if and only if all algebra homomorphisms from $(\mathcal{A}, \|\cdot\|)$ into a normed algebra are continuous.
- $\|\cdot\|$ is minimal on $\mathcal{A}$ if and only every injective homomorphism from $(\mathcal{A}, \|\cdot\|)$ into a normed algebra is bounded below.

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- Johnson, Phillips, and Schechtman proved that for ($1 < p < \infty$), $L_p([0,1])$ has a Calkin algebra with a unique norm, but there is a closed ideal $\mathcal{J}$ of $\mathcal{B}(L_p([0,1]))$ for which $\mathcal{B}(L_p([0,1]))/\mathcal{J}$ does not have a unique algebra norm.

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For each of these spaces, we know the entire lattice of closed ideals of $\mathcal{B}(X)$.

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- We in fact prove the stronger condition of *uniform incompressibility*: $\mathcal{A}$ is uniformly incompressible if there exists $f : (0, \infty) \to (0, \infty)$ such that for any Banach algebra $\mathcal{B}$, for all $a \in \mathcal{A}$ with $\|a\| = 1$, and for every continuous, injective algebra homomorphism $\phi : \mathcal{A} \to \mathcal{B}$, we have that $\|\phi(a)\| \geq f(\|\phi\|)$.

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This notion is helpful to us because we have access to the following method, courtesy of Johnson, Phillips, and Schechtman:

A Uniform Incompressibility Method

- Let $(\mathcal{A}, \|\cdot\|)$ be a Banach algebra. Suppose that there is a nonzero idempotent $a \in \mathcal{A}$ and a constant $M \in (0, \infty)$ such that for each norm-one $x \in \mathcal{A}$, there exist $b, c \in \mathcal{A}$ for which $\|b\|\|c\| \leq M$ and such that $a = bxc$. Then $\mathcal{A}$ is uniformly incompressible.

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Then, for each unit vector $x \in \mathcal{A}$, we have

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So, a lower bound of ϕ is $(M\|\phi\|^2)^{-1}$.

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Our job is then for $\|T\|_q = 1$, to calculate an upper bound on the norms of A and B.

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- It remains to be proved that $\mathcal{B}(X)/\overline{\mathcal{G}_{c_0}}(X)$ has a minimal algebra norm. We prove this via uniform incompressibility.

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For $n \in \mathbb{N}$, $H_1, \ldots, H_n$ Hilbert spaces, E a Banach space, and $\epsilon > 0$, let $T \in \mathcal{B}(H_1 \oplus \cdots \oplus H_n; E)$. Define the index $m_\epsilon(T)$ as:

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In words, $m_\epsilon(T)$ is the largest number m of dimensions, such that if you remove m dimensions from each of the Hilbert spaces in the domain of T, you are still left with an operator with norm bigger than ϵ.

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- Expressing each $T \in \mathcal{B}(X)$ as a matrix $(T_{i,j})_{i,j\in\mathbb{N}}$, where $T_{i,j} : \ell_2^j \to \ell_2^i$, we can approximate T via a compact perturbation with an operator which has a matrix of finitely supported rows and columns.

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- Letting $Q_n : X \to \ell_2^n$ denote the standard projection, and for $T \in \mathcal{B}(X)$ an operator of finite rows and columns, we can naturally define $m_\epsilon(Q_n T)$ by simply ignoring the Hilbert spaces in the domain of T on which $Q_n T$ acts trivially.

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A similar method also works to prove uniqueness of norm of $\mathcal{B}(X^*)/\overline{\mathcal{G}_{\ell_1}}(X^*)$.

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Found by Niels Laustsen and myself, the lattice of closed ideals of

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- *Question: Does the Calkin algebra of $C[0, \omega^\omega)$ have a unique algebra norm?*

Thank you for listening!