

# Uniqueness of Algebra Norms of Quotient Algebras of Bounded Operators

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*Based on joint work with Niels Laustsen.*

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## Previous Results

- ▶ Meyer showed that the Calkin algebras ( $= \mathcal{B}(X)/\mathcal{K}(X)$ ) of  $X = c_0$  and  $X = \ell_p$  ( $1 \leq p < \infty$ ) have unique algebra norms.



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- ▶ Johnson, Phillips, and Schechtman proved that for ( $1 < p < \infty$ ),  $L_p([0, 1])$  has a Calkin algebra with a unique norm, but there is a closed ideal  $\mathcal{I}$  of  $\mathcal{B}(L_p([0, 1]))$  for which  $\mathcal{B}(L_p([0, 1]))/\mathcal{I}$  does not have a unique algebra norm.



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For each of these spaces, we know the entire lattice of closed ideals of  $\mathcal{B}(X)$ .

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- ▶ We in fact prove the stronger condition of *uniform incompressibility*:  $\mathcal{A}$  is uniformly incompressible if there exists  $f : (0, \infty) \rightarrow (0, \infty)$  such that for any Banach algebra  $\mathcal{B}$ , for all  $a \in \mathcal{A}$  with  $\|a\| = 1$ , and for every continuous, injective algebra homomorphism  $\phi : \mathcal{A} \rightarrow \mathcal{B}$ , we have that  $\|\phi(a)\| \geq f(\|\phi\|)$ .

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This notion is helpful to us because we have access to the following method, courtesy of Johnson, Phillips, and Schechtman:

## A Uniform Incompressibility Method

- ▶ Let  $(\mathcal{A}, \|\cdot\|)$  be a Banach algebra. Suppose that there is a nonzero idempotent  $a \in \mathcal{A}$  and a constant  $M \in (0, \infty)$  such that for each norm-one  $x \in \mathcal{A}$ , there exist  $b, c \in \mathcal{A}$  for which  $\|b\|\|c\| \leq M$  and such that  $a = bxc$ . Then  $\mathcal{A}$  is uniformly incompressible.

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Then, for each unit vector  $x \in \mathcal{A}$ , we have

$$1 \leq \|\phi(a)\| \leq \|\phi(b)\|\|\phi(x)\|\|\phi(c)\| \leq \|\phi\|^2 \|b\|\|c\|\|\phi(x)\| \leq M\|\phi\|^2 \|\phi(x)\| .$$



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So, a lower bound of  $\phi$  is  $(M\|\phi\|^2)^{-1}$ .



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One step in proving this ideal classification was to show that  $\forall T \in \mathcal{B}(X) \setminus \overline{\mathcal{G}_{c_0}}(X)$ , there are  $A, B$  such that  $I_X = ATB$ .

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$\forall T \in \mathcal{B}(X) \setminus \overline{\mathcal{G}_{c_0}}(X)$ , there are  $A, B$  such that  $I_X = ATB$ .

Our job is then for  $\|T\|_q = 1$ , to calculate an upper bound on the norms of  $A$  and  $B$ .

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- ▶ It remains to be proved that  $\mathcal{B}(X)/\overline{\mathcal{G}}_{c_0}(X)$  has a minimal algebra norm. We prove this via uniform incompressibility.

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For  $n \in \mathbb{N}$ ,  $H_1, \dots, H_n$  Hilbert spaces,  $E$  a Banach space, and  $\epsilon > 0$ , let  $T \in \mathcal{B}(H_1 \oplus \dots \oplus H_n; E)$ . Define the index  $m_\epsilon(T)$  as:

$$m_\epsilon(T) := \sup\{m \in \mathbb{N}_0 : \|T(P_{G_1^\perp}^{H_1} \oplus P_{G_2^\perp}^{H_2} \cdots \oplus P_{G_n^\perp}^{H_n})\| > \epsilon\}$$

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In words,  $m_\epsilon(T)$  is the largest number  $m$  of dimensions, such that if you remove  $m$  dimensions from each of the Hilbert spaces in the domain of  $T$ , you are still left with an operator with norm bigger than  $\epsilon$ .

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- ▶ Expressing each  $T \in \mathcal{B}(X)$  as a matrix  $(T_{i,j})_{i,j \in \mathbb{N}}$ , where  $T_{i,j} : \ell_2^j \rightarrow \ell_2^i$ , we can approximate  $T$  via a compact perturbation with an operator which has a matrix of finitely supported rows and columns.



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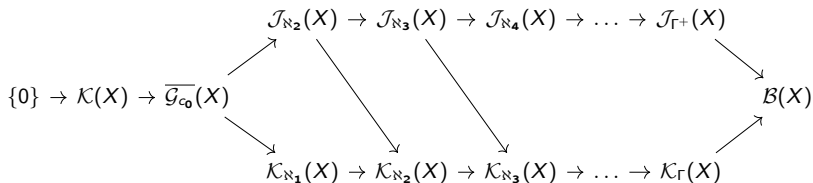
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A similar method also works to prove uniqueness of norm of  $\mathcal{B}(X^*)/\overline{\mathcal{G}_{\ell_1}}(X^*)$ .

Quotients of  $\mathcal{B}(X)$  for  $X = E \oplus c_0(\Gamma)$ , for  $E = \left(\bigoplus_{n \in \mathbb{N}} \ell_2^n\right)_{c_0}$

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- ▶ *Question: Does the Calkin algebra of  $C[0, \omega^\omega)$  have a unique algebra norm?*

*Thank you for listening!*