r-Fredholm theory in general ordered Banach algebras

Ronalda Benjamin

Stellenbosch University

Banach algebras and Applications Conference
Granada
18 - 23 July 2022

(This is joint work with Sonja Mouton)

Terminology and fixed notation

- Banach algebra : complex unital Banach algebra

Terminology and fixed notation

- Banach algebra : complex unital Banach algebra
- A^{-1} : the set of invertible elements of a Banach algebra A

Terminology and fixed notation

- Banach algebra : complex unital Banach algebra
- A^{-1} : the set of invertible elements of a Banach algebra A

$$\boxed{a \in A}$$

Terminology and fixed notation

- Banach algebra : complex unital Banach algebra
- A^{-1} : the set of invertible elements of a Banach algebra A

$$\boxed{a \in A}$$

- $\sigma(a, A) = \{\lambda \in \mathbb{C} : \lambda\mathbf{1} - a \notin A^{-1}\}$: the spectrum of a in A

Terminology and fixed notation

- Banach algebra : complex unital Banach algebra
- A^{-1} : the set of invertible elements of a Banach algebra A

$$\boxed{a \in A}$$

- $\sigma(a, A) = \{\lambda \in \mathbb{C} : \lambda\mathbf{1} - a \notin A^{-1}\}$: the spectrum of a in A
- $r(a, A) := \sup\limits_{\lambda \in \sigma(a,A)} |\lambda|$: spectral radius of a in A

Terminology and fixed notation

- Banach algebra : complex unital Banach algebra
- A^{-1} : the set of invertible elements of a Banach algebra A

$$\boxed{a \in A}$$

- $\sigma(a, A) = \{\lambda \in \mathbb{C} : \lambda \mathbf{1} - a \notin A^{-1}\}$: the spectrum of a in A
- $r(a, A) := \sup\limits_{\lambda \in \sigma(a,A)} |\lambda|$: spectral radius of a in A
- $p(a, \lambda)$: spectral idempotent of $a \in A$ corresponding to $\lambda \in \text{iso}\, \sigma(a)$

(Algebra) homomorphism: $T : A \to B$
preserves algebraic structure and satisfies $T\mathbf{1}_A = \mathbf{1}_B$

(Algebra) homomorphism: $T : A \to B$
preserves algebraic structure and satisfies $T\mathbf{1}_A = \mathbf{1}_B$

- $\mathrm{N}(T) = \{a \in A : Ta = 0\}$: the null space of T

(Algebra) homomorphism: $T : A \to B$
preserves algebraic structure and satisfies $T\mathbf{1}_A = \mathbf{1}_B$

- $\mathrm{N}(T) = \{a \in A : Ta = 0\}$: the null space of T
- T has the Riesz property whenever the spectrum of each element of $\mathrm{N}(T)$ is either finite or a sequence converging to zero

(Algebra) homomorphism: $T : A \to B$
preserves algebraic structure and satisfies $T\mathbf{1}_A = \mathbf{1}_B$

- $\mathrm{N}(T) = \{a \in A : Ta = 0\}$: the null space of T
- T has the Riesz property whenever the spectrum of each element of $\mathrm{N}(T)$ is either finite or a sequence converging to zero

$$\Updownarrow$$

(Algebra) homomorphism: $T : A \to B$
preserves algebraic structure and satisfies $T\mathbf{1}_A = \mathbf{1}_B$

- $\mathrm{N}(T) = \{a \in A : Ta = 0\}$: the null space of T
- T has the Riesz property whenever the spectrum of each element of $\mathrm{N}(T)$ is either finite or a sequence converging to zero

$$\Updownarrow$$

$$\operatorname{acc} \sigma(a, A) \subseteq \{0\} \text{ for all } a \in \mathrm{N}(T)$$

(Algebra) homomorphism: $T : A \to B$
preserves algebraic structure and satisfies $T\mathbf{1}_A = \mathbf{1}_B$

- $\mathrm{N}(T) = \{a \in A : Ta = 0\}$: the null space of T
- T has the Riesz property whenever the spectrum of each element of $\mathrm{N}(T)$ is either finite or a sequence converging to zero

$$\Updownarrow$$

$$\text{acc}\, \sigma(a, A) \subseteq \{0\} \text{ for all } a \in \mathrm{N}(T)$$

- T has the strong Riesz property whenever $\sigma(a) \subseteq \eta\sigma(Ta) \cup \text{iso}\, \sigma(a)$ for all $a \in A$.

(Algebra) homomorphism: $T : A \to B$
preserves algebraic structure and satisfies $T\mathbf{1}_A = \mathbf{1}_B$

- $N(T) = \{a \in A : Ta = 0\}$: the null space of T
- T has the Riesz property whenever the spectrum of each element of $N(T)$ is either finite or a sequence converging to zero

$$\Updownarrow$$

$$\operatorname{acc} \sigma(a, A) \subseteq \{0\} \text{ for all } a \in N(T)$$

- T has the strong Riesz property whenever $\sigma(a) \subseteq \eta\sigma(Ta) \cup \operatorname{iso} \sigma(a)$ for all $a \in A$.

- ηK: Connected hull of a compact set $K \subseteq \mathbb{C}$ (the complement of the unique unbounded component of $\mathbb{C} \backslash K$)

(Algebra) homomorphism: $T : A \to B$
preserves algebraic structure and satisfies $T\mathbf{1}_A = \mathbf{1}_B$

- $N(T) = \{a \in A : Ta = 0\}$: the null space of T
- T has the Riesz property whenever the spectrum of each element of $N(T)$ is either finite or a sequence converging to zero

$$\Updownarrow$$

$$\text{acc}\, \sigma(a, A) \subseteq \{0\} \text{ for all } a \in N(T)$$

- T has the strong Riesz property whenever $\sigma(a) \subseteq \eta\sigma(Ta) \cup \text{iso}\, \sigma(a)$ for all $a \in A$.

$$\Updownarrow$$

- ηK: Connected hull of a compact set $K \subseteq \mathbb{C}$ (the complement of the unique unbounded component of $\mathbb{C} \backslash K$)

(Algebra) homomorphism: $T : A \to B$
preserves algebraic structure and satisfies $T\mathbf{1}_A = \mathbf{1}_B$

- $\mathrm{N}(T) = \{a \in A : Ta = 0\}$: the null space of T
- T has the Riesz property whenever the spectrum of each element of $\mathrm{N}(T)$ is either finite or a sequence converging to zero

$$\Updownarrow$$

$$\operatorname{acc} \sigma(a, A) \subseteq \{0\} \text{ for all } a \in \mathrm{N}(T)$$

- T has the strong Riesz property whenever $\sigma(a) \subseteq \eta\sigma(Ta) \cup \operatorname{iso} \sigma(a)$ for all $a \in A$.

$$\Updownarrow$$

$$\operatorname{acc} \sigma(a, A) \subseteq \eta\sigma(Ta, B)$$

- ηK: Connected hull of a compact set $K \subseteq \mathbb{C}$
(the complement of the unique unbounded component of $\mathbb{C} \backslash K$)

(Algebra) homomorphism: $T : A \to B$
preserves algebraic structure and satisfies $T\mathbf{1}_A = \mathbf{1}_B$

- $\mathrm{N}(T) = \{a \in A : Ta = 0\}$: the null space of T
- T has the Riesz property whenever the spectrum of each element of $\mathrm{N}(T)$ is either finite or a sequence converging to zero

$$\Updownarrow$$

$$\operatorname{acc} \sigma(a, A) \subseteq \{0\} \text{ for all } a \in \mathrm{N}(T)$$

- T has the strong Riesz property whenever $\sigma(a) \subseteq \eta\sigma(Ta) \cup \operatorname{iso} \sigma(a)$ for all $a \in A$.

$$\Updownarrow$$

$$\operatorname{acc} \sigma(a, A) \subseteq \eta\sigma(Ta, B)$$

$$\text{strong Riesz} \Longrightarrow \text{Riesz}$$

- ηK: Connected hull of a compact set $K \subseteq \mathbb{C}$ (the complement of the unique unbounded component of $\mathbb{C} \backslash K$)

(Algebra) homomorphism: $T : A \to B$
preserves algebraic structure and satisfies $T\mathbf{1}_A = \mathbf{1}_B$

- $\mathrm{N}(T) = \{a \in A : Ta = 0\}$: the null space of T
- T has the Riesz property whenever the spectrum of each element of $\mathrm{N}(T)$ is either finite or a sequence converging to zero

$$\Updownarrow$$

$$\text{acc } \sigma(a, A) \subseteq \{0\} \text{ for all } a \in \mathrm{N}(T)$$

- T has the strong Riesz property whenever $\sigma(a) \subseteq \eta\sigma(Ta) \cup \text{iso } \sigma(a)$ for all $a \in A$.

$$\Updownarrow$$

$$\text{acc } \sigma(a, A) \subseteq \eta\sigma(Ta, B)$$

$$\text{strong Riesz} \Longrightarrow \text{Riesz}$$

$$\text{strong Riesz} \Longleftarrow \text{Riesz and closed range}$$

- ηK: Connected hull of a compact set $K \subseteq \mathbb{C}$
(the complement of the unique unbounded component of $\mathbb{C} \backslash K$)

(Algebra) homomorphism: $T : A \to B$
preserves algebraic structure and satisfies $T\mathbf{1}_A = \mathbf{1}_B$

- $\mathrm{N}(T) = \{a \in A : Ta = 0\}$: the null space of T
- T has the Riesz property whenever the spectrum of each element of $\mathrm{N}(T)$ is either finite or a sequence converging to zero

$$\Updownarrow$$

$$\operatorname{acc} \sigma(a, A) \subseteq \{0\} \text{ for all } a \in \mathrm{N}(T)$$

- T has the strong Riesz property whenever $\sigma(a) \subseteq \eta\sigma(Ta) \cup \operatorname{iso} \sigma(a)$ for all $a \in A$.

$$\Updownarrow$$

$$\operatorname{acc} \sigma(a, A) \subseteq \eta\sigma(Ta, B)$$

$$\text{strong Riesz} \Longrightarrow \text{Riesz}$$

$$\text{strong Riesz} \Longleftarrow \text{Riesz and closed range}$$

$$\boxed{\pi : \mathcal{L}(X) \to \mathcal{L}(X)/\mathcal{K}(X)}$$

- ηK: Connected hull of a compact set $K \subseteq \mathbb{C}$ (the complement of the unique unbounded component of $\mathbb{C} \backslash K$)

Fredholm theory in Banach algebras and some motivation

Definition (Harte, 1982)

Definition (Harte, 1982)

Let $T : A \to B$ be a homomorphism. W.r.t. T, an element $a \in A$ is called:

Definition (Harte, 1982)

Let $T : A \to B$ be a homomorphism. W.r.t. T, an element $a \in A$ is called:

- Fredholm if $Ta \in B^{-1}$,

Definition (Harte, 1982)

Let $T : A \to B$ be a homomorphism. W.r.t. T, an element $a \in A$ is called:

- Fredholm if $Ta \in B^{-1}$,
- Weyl if $a \in A^{-1} + \mathrm{N}(T)$,

Definition (Harte, 1982)

Let $T : A \to B$ be a homomorphism. W.r.t. T, an element $a \in A$ is called:

- Fredholm if $Ta \in B^{-1}$,
- Weyl if $a \in A^{-1} + \mathrm{N}(T)$,
- Browder if $a \in A^{-1} \mathbin{\underline{+}} \mathrm{N}(T)$.

Definition (Harte, 1982)

Let $T : A \to B$ be a homomorphism. W.r.t. T, an element $a \in A$ is called:

- Fredholm if $Ta \in B^{-1}$,
- Weyl if $a \in A^{-1} + \mathrm{N}(T)$,
- Browder if $a \in A^{-1} \mathbin{\in\!\!\!\!+} \mathrm{N}(T)$.

Evidently,

$$\text{Browder} \Rightarrow \text{Weyl}$$

Definition (Harte, 1982)

Let $T : A \to B$ be a homomorphism. W.r.t. T, an element $a \in A$ is called:

- Fredholm if $Ta \in B^{-1}$,
- Weyl if $a \in A^{-1} + \mathrm{N}(T)$,
- Browder if $a \in A^{-1} \mathbin{\in\!\!\!\!+} \mathrm{N}(T)$.

Evidently,

$$\text{invertible} \Rightarrow \text{Browder} \Rightarrow \text{Weyl} \Rightarrow \text{Fredholm}$$

Definition (Harte, 1982)

Let $T : A \to B$ be a homomorphism. W.r.t. T, an element $a \in A$ is called:

- Fredholm if $Ta \in B^{-1}$,
- Weyl if $a \in A^{-1} + \mathrm{N}(T)$,
- Browder if $a \in A^{-1} \mathbin{\in\!\!\!\!+} \mathrm{N}(T)$.

Evidently,

$$\text{invertible} \Rightarrow \text{Browder} \Rightarrow \text{Weyl} \Rightarrow \text{Fredholm}$$

$a \in A$ almost invertible: $0 \notin \operatorname{acc} \sigma(a)$

Definition (Harte, 1982)

Let $T : A \to B$ be a homomorphism. W.r.t. T, an element $a \in A$ is called:

- Fredholm if $Ta \in B^{-1}$,
- Weyl if $a \in A^{-1} + \mathrm{N}(T)$,
- Browder if $a \in A^{-1} \mathbin{\in\!\!\!\!\!+} \mathrm{N}(T)$.

Evidently,

$$\text{invertible} \Rightarrow \text{Browder} \Rightarrow \text{Weyl} \Rightarrow \text{Fredholm}$$

$a \in A$ almost invertible: $0 \notin \operatorname{acc} \sigma(a)$

Clearly, invertible $\Rightarrow$ almost invertible

Definition (Harte, 1982)

Let $T : A \to B$ be a homomorphism. W.r.t. T, an element $a \in A$ is called:

- Fredholm if $Ta \in B^{-1}$,
- Weyl if $a \in A^{-1} + \mathrm{N}(T)$,
- Browder if $a \in A^{-1} \mathbin{\in\!\!\!\!+} \mathrm{N}(T)$.

Evidently,

$$\text{invertible} \Rightarrow \text{Browder} \Rightarrow \text{Weyl} \Rightarrow \text{Fredholm}$$

$a \in A$ almost invertible: $0 \notin \operatorname{acc} \sigma(a)$

Clearly, invertible $\Rightarrow$ almost invertible Fredholm

Definition (Harte, 1982)

Let $T : A \to B$ be a homomorphism. W.r.t. T, an element $a \in A$ is called:

- Fredholm if $Ta \in B^{-1}$,
- Weyl if $a \in A^{-1} + \mathrm{N}(T)$,
- Browder if $a \in A^{-1} \mathbin{\in\!\!\!\!+} \mathrm{N}(T)$.

Evidently,

$$\text{invertible} \Rightarrow \text{Browder} \Rightarrow \text{Weyl} \Rightarrow \text{Fredholm}$$

$a \in A$ almost invertible: $0 \notin \operatorname{acc} \sigma(a)$

Clearly, invertible $\Rightarrow$ almost invertible Fredholm $\Rightarrow$ Browder

Definition (Harte, 1982)

Let $T : A \to B$ be a homomorphism. W.r.t. T, an element $a \in A$ is called:

- Fredholm if $Ta \in B^{-1}$,
- Weyl if $a \in A^{-1} + \mathrm{N}(T)$,
- Browder if $a \in A^{-1} \mathbin{\in\!\!\!\!+} \mathrm{N}(T)$.

Evidently,

$$\text{invertible} \Rightarrow \text{Browder} \Rightarrow \text{Weyl} \Rightarrow \text{Fredholm}$$

$a \in A$ almost invertible: $0 \notin \operatorname{acc} \sigma(a)$

Clearly, invertible $\Rightarrow$ almost invertible Fredholm $\Rightarrow$ Browder

Theorem (Harte ,1982)

$$\text{almost invertble Fredholm} = \text{Browder}$$

if and only if

$T : A \to B$ has the Riesz property

Definition (Harte, 1982)

Definition (Harte, 1982)

Let $T : A \to B$ be a homomorphism and $a \in A$. W.r.t T, the

Definition (Harte, 1982)

Let $T : A \to B$ be a homomorphism and $a \in A$. W.r.t T, the

- Fredholm spectrum of a is given by $\tau_T(a) := \{\lambda \in \mathbb{C} : \lambda \mathbf{1} - a \notin \mathcal{F}_T\}$,

Definition (Harte, 1982)

Let $T : A \to B$ be a homomorphism and $a \in A$. W.r.t T, the

- Fredholm spectrum of a is given by $\tau_T(a) := \{\lambda \in \mathbb{C} : \lambda \mathbf{1} - a \notin \mathcal{F}_T\}$,
- Weyl spectrum of a is given by $\omega_T(a) = \{\lambda \in \mathbb{C} : \lambda \mathbf{1} - a \notin \mathcal{W}_T\}$,

Definition (Harte, 1982)

Let $T : A \to B$ be a homomorphism and $a \in A$. W.r.t T, the

- Fredholm spectrum of a is given by $\tau_T(a) := \{\lambda \in \mathbb{C} : \lambda\mathbf{1} - a \notin \mathcal{F}_T\}$,
- Weyl spectrum of a is given by $\omega_T(a) = \{\lambda \in \mathbb{C} : \lambda\mathbf{1} - a \notin \mathcal{W}_T\}$,
- Browder spectrum of a is given by $\beta_T(a) = \{\lambda \in \mathbb{C} : \lambda\mathbf{1} - a \notin \mathcal{B}_T\}$.

Definition (Harte, 1982)

Let $T : A \to B$ be a homomorphism and $a \in A$. W.r.t T, the

- Fredholm spectrum of a is given by $\tau_T(a) := \{\lambda \in \mathbb{C} : \lambda \mathbf{1} - a \notin \mathcal{F}_T\}$,
- Weyl spectrum of a is given by $\omega_T(a) = \{\lambda \in \mathbb{C} : \lambda \mathbf{1} - a \notin \mathcal{W}_T\}$,
- Browder spectrum of a is given by $\beta_T(a) = \{\lambda \in \mathbb{C} : \lambda \mathbf{1} - a \notin \mathcal{B}_T\}$.

Recall: invertible $\Rightarrow$ Browder $\Rightarrow$ Weyl $\Rightarrow$ Fredholm

Definition (Harte, 1982)

Let $T : A \to B$ be a homomorphism and $a \in A$. W.r.t T, the

- Fredholm spectrum of a is given by $\tau_T(a) := \{\lambda \in \mathbb{C} : \lambda \mathbf{1} - a \notin \mathcal{F}_T\}$,
- Weyl spectrum of a is given by $\omega_T(a) = \{\lambda \in \mathbb{C} : \lambda \mathbf{1} - a \notin \mathcal{W}_T\}$,
- Browder spectrum of a is given by $\beta_T(a) = \{\lambda \in \mathbb{C} : \lambda \mathbf{1} - a \notin \mathcal{B}_T\}$.

Recall: invertible $\Rightarrow$ Browder $\Rightarrow$ Weyl $\Rightarrow$ Fredholm

Now:

Definition (Harte, 1982)

Let $T : A \to B$ be a homomorphism and $a \in A$. W.r.t T, the

- Fredholm spectrum of a is given by $\tau_T(a) := \{\lambda \in \mathbb{C} : \lambda \mathbf{1} - a \notin \mathcal{F}_T\}$,
- Weyl spectrum of a is given by $\omega_T(a) = \{\lambda \in \mathbb{C} : \lambda \mathbf{1} - a \notin \mathcal{W}_T\}$,
- Browder spectrum of a is given by $\beta_T(a) = \{\lambda \in \mathbb{C} : \lambda \mathbf{1} - a \notin \mathcal{B}_T\}$.

Recall: invertible $\Rightarrow$ Browder $\Rightarrow$ Weyl $\Rightarrow$ Fredholm
Now:

$$\tau_T(a) \quad \subseteq \quad \omega_T(a) \quad \subseteq \quad \beta_T(a) \quad \subseteq \quad \sigma(a).$$

Definition (Harte, 1982)

Let $T: A \to B$ be a homomorphism and $a \in A$. W.r.t T, the

- Fredholm spectrum of a is given by $\tau_T(a) := \{\lambda \in \mathbb{C} : \lambda \mathbf{1} - a \notin \mathcal{F}_T\}$,
- Weyl spectrum of a is given by $\omega_T(a) = \{\lambda \in \mathbb{C} : \lambda \mathbf{1} - a \notin \mathcal{W}_T\}$,
- Browder spectrum of a is given by $\beta_T(a) = \{\lambda \in \mathbb{C} : \lambda \mathbf{1} - a \notin \mathcal{B}_T\}$.

Recall: invertible $\Rightarrow$ Browder $\Rightarrow$ Weyl $\Rightarrow$ Fredholm

Now:

$$\tau_T(a) \subseteq \omega_T(a) \subseteq \beta_T(a) \subseteq \sigma(a).$$

$$\tau_T(a) = \sigma(Ta, B)$$

Definition (Harte, 1982)

Let $T : A \to B$ be a homomorphism and $a \in A$. W.r.t T, the

- Fredholm spectrum of a is given by $\tau_T(a) := \{\lambda \in \mathbb{C} : \lambda \mathbf{1} - a \notin \mathcal{F}_T\}$,
- Weyl spectrum of a is given by $\omega_T(a) = \{\lambda \in \mathbb{C} : \lambda \mathbf{1} - a \notin \mathcal{W}_T\}$,
- Browder spectrum of a is given by $\beta_T(a) = \{\lambda \in \mathbb{C} : \lambda \mathbf{1} - a \notin \mathcal{B}_T\}$.

Recall: invertible $\Rightarrow$ Browder $\Rightarrow$ Weyl $\Rightarrow$ Fredholm

Now:

$$\bigcap_{c \in \mathrm{N}(T)} \sigma(a+c)$$

$$||$$

$$\tau_T(a) \quad \subseteq \quad \omega_T(a) \quad \subseteq \quad \beta_T(a) \quad \subseteq \quad \sigma(a).$$

$$||$$

$$\sigma(Ta, B)$$

Definition (Harte, 1982)

Let $T : A \to B$ be a homomorphism and $a \in A$. W.r.t T, the

- Fredholm spectrum of a is given by $\tau_T(a) := \{\lambda \in \mathbb{C} : \lambda \mathbf{1} - a \notin \mathcal{F}_T\}$,
- Weyl spectrum of a is given by $\omega_T(a) = \{\lambda \in \mathbb{C} : \lambda \mathbf{1} - a \notin \mathcal{W}_T\}$,
- Browder spectrum of a is given by $\beta_T(a) = \{\lambda \in \mathbb{C} : \lambda \mathbf{1} - a \notin \mathcal{B}_T\}$.

Recall: invertible $\Rightarrow$ Browder $\Rightarrow$ Weyl $\Rightarrow$ Fredholm

Now:

$$\omega_T(a) = \bigcap_{c \in \mathrm{N}(T)} \sigma(a+c)$$

$$\tau_T(a) \subseteq \omega_T(a) \subseteq \beta_T(a) \subseteq \sigma(a).$$

$$\tau_T(a) = \sigma(Ta, B), \qquad \beta_T(a) = \bigcap_{\substack{c \in \mathrm{N}(T) \\ ac=ca}} \sigma(a+c)$$

Definition (Harte, 1982)

Let $T : A \to B$ be a homomorphism and $a \in A$. W.r.t T, the

- Fredholm spectrum of a is given by $\tau_T(a) := \{\lambda \in \mathbb{C} : \lambda \mathbf{1} - a \notin \mathcal{F}_T\}$,
- Weyl spectrum of a is given by $\omega_T(a) = \{\lambda \in \mathbb{C} : \lambda \mathbf{1} - a \notin \mathcal{W}_T\}$,
- Browder spectrum of a is given by $\beta_T(a) = \{\lambda \in \mathbb{C} : \lambda \mathbf{1} - a \notin \mathcal{B}_T\}$.

Recall: invertible $\Rightarrow$ Browder $\Rightarrow$ Weyl $\Rightarrow$ Fredholm

Now:

$$\sigma(Ta, B) = \tau_T(a) \subseteq \omega_T(a) \subseteq \beta_T(a) \subseteq \sigma(a).$$

$$\omega_T(a) = \bigcap_{c \in \mathrm{N}(T)} \sigma(a+c), \qquad \beta_T(a) = \bigcap_{\substack{c \in \mathrm{N}(T) \\ ac = ca}} \sigma(a+c)$$

Theorem (R. Harte, S. Živković-Zlatanović, 2014)

If $T : A \to B$ has the strong Riesz property, then

$$\eta\sigma(Ta) = \eta\omega_T(a) = \eta\beta_T(a) \text{ for all } a \in A.$$

Interplay between Fredholm theory and ordering

Theorem (R. Harte, S. Živković-Zlatanović, 2014)

If $T : A \to B$ has the strong Riesz property, then

$$\eta\sigma(Ta) = \eta\omega_T(a) = \eta\beta_T(a) \text{ for all } a \in A.$$

In particular, $r(a) \notin \sigma(Ta) \iff r(a) \notin \omega_T(a) \iff r(a) \notin \beta_T(a)$.

Interplay between Fredholm theory and ordering

Theorem (R. Harte, S. Živković-Zlatanović, 2014)

If $T : A \to B$ has the strong Riesz property, then

$$\eta\sigma(Ta) = \eta\omega_T(a) = \eta\beta_T(a) \text{ for all } a \in A.$$

In particular, $r(a) \notin \sigma(Ta) \iff r(a) \notin \omega_T(a) \iff r(a) \notin \beta_T(a)$.

Applied to $\pi : \mathcal{L}(X) \to \mathcal{L}(X)/\mathcal{K}(X)$

Interplay between Fredholm theory and ordering

Theorem (R. Harte, S. Živković-Zlatanović, 2014)

If $T : A \to B$ has the strong Riesz property, then

$$\eta\sigma(Ta) = \eta\omega_T(a) = \eta\beta_T(a) \text{ for all } a \in A.$$

In particular, $r(a) \notin \sigma(Ta) \iff r(a) \notin \omega_T(a) \iff r(a) \notin \beta_T(a)$.

Applied to $\pi : \mathcal{L}(X) \to \mathcal{L}(X)/\mathcal{K}(X)$

$$\eta\sigma(\pi T) = \eta\omega_\pi(T) = \eta\beta_\pi(T) \text{ for all } T \in \mathcal{L}(X).$$

In particular, $r(T) \notin \sigma(\pi T) \iff r(T) \notin \omega_\pi(T) \iff r(T) \notin \beta_\pi(T)$.

Interplay between Fredholm theory and ordering

Theorem (R. Harte, S. Živković-Zlatanović, 2014)

If $T : A \to B$ has the strong Riesz property, then

$$\eta\sigma(Ta) = \eta\omega_T(a) = \eta\beta_T(a) \text{ for all } a \in A.$$

In particular, $r(a) \notin \sigma(Ta) \iff r(a) \notin \omega_T(a) \iff r(a) \notin \beta_T(a)$.

Applied to $\pi : \mathcal{L}(X) \to \mathcal{L}(X)/\mathcal{K}(X)$

$$\eta\sigma(\pi T) = \eta\omega_\pi(T) = \eta\beta_\pi(T) \text{ for all } T \in \mathcal{L}(X).$$

In particular, $r(T) \notin \sigma(\pi T) \iff r(T) \notin \omega_\pi(T) \iff r(T) \notin \beta_\pi(T)$.

Recall for $T \in \mathcal{L}(X)$: $\bigcap_{S \in \mathcal{K}(X)} \sigma(T+S)$

Interplay between Fredholm theory and ordering

Theorem (R. Harte, S. Živković-Zlatanović, 2014)

If $T : A \to B$ has the strong Riesz property, then

$$\eta\sigma(Ta) = \eta\omega_T(a) = \eta\beta_T(a) \text{ for all } a \in A.$$

In particular, $r(a) \notin \sigma(Ta) \iff r(a) \notin \omega_T(a) \iff r(a) \notin \beta_T(a)$.

Applied to $\pi : \mathcal{L}(X) \to \mathcal{L}(X)/\mathcal{K}(X)$

$$\eta\sigma(\pi T) = \eta\omega_\pi(T) = \eta\beta_\pi(T) \text{ for all } T \in \mathcal{L}(X).$$

In particular, $r(T) \notin \sigma(\pi T) \iff r(T) \notin \omega_\pi(T) \iff r(T) \notin \beta_\pi(T)$.

Recall for $T \in \mathcal{L}(X)$: $\bigcap_{S \in \mathcal{K}(X)} \sigma(T+S)$

Now for $T \in \mathcal{L}(E)$: $\bigcap_{S \in K \cap \mathcal{K}(E)} \sigma(T+S)$, K: positive operators on E

Problem [Alekhno, 2007]

For $T \in K$, does $r(T) \notin \sigma(\pi T)$ imply $r(T) \notin \bigcap_{S \in K \cap \mathcal{K}(E)} \sigma(T + S)$?

Problem [Alekhno, 2007]

For $T \in K$, does $r(T) \notin \sigma(\pi T)$ imply $r(T) \notin \bigcap_{S \in K \cap \mathcal{K}(E)} \sigma(T+S)$?

YES!

Problem [Alekhno, 2007]

For $T \in K$, does $r(T) \notin \sigma(\pi T)$ imply $r(T) \notin \bigcap_{S \in K \cap \mathcal{K}(E)} \sigma(T+S)$?

YES!

Theorem (Alekhno, 2009)

For $T \in \mathcal{L}(E)$,

$$\omega_\pi(T) = \bigcap_{S \in K \cap \mathcal{K}(E)} \sigma(T+S).$$

Problem [Alekhno, 2007]

For $T \in K$, does $r(T) \notin \sigma(\pi T)$ imply $r(T) \notin \bigcap_{S \in K \cap \mathcal{K}(E)} \sigma(T+S)$?

YES!

Theorem (Alekhno, 2009)

For $T \in \mathcal{L}(E)$,

$$\omega_\pi(T) = \bigcap_{S \in K \cap \mathcal{K}(E)} \sigma(T+S).$$

- $\mathcal{L}(E)^{-1} + \mathcal{K}(E) = \mathcal{L}(E)^{-1} + \mathcal{F}(E)$

Problem [Alekhno, 2007]

For $T \in K$, does $r(T) \notin \sigma(\pi T)$ imply $r(T) \notin \bigcap_{S \in K \cap \mathcal{K}(E)} \sigma(T+S)$?

YES!

Theorem (Alekhno, 2009)

For $T \in \mathcal{L}(E)$,

$$\omega_\pi(T) = \bigcap_{S \in K \cap \mathcal{K}(E)} \sigma(T+S).$$

- $\mathcal{L}(E)^{-1} + \mathcal{K}(E) = \mathcal{L}(E)^{-1} + \mathcal{F}(E)$
- $\mathcal{F}(E) \subseteq \operatorname{span}(K \cap \mathcal{K}(E))$

Problem [Alekhno, 2007]

For $T \in K$, does $r(T) \notin \sigma(\pi T)$ imply $r(T) \notin \bigcap_{S \in K \cap \mathcal{K}(E)} \sigma(T+S)$?

YES!

Theorem (Alekhno, 2009)

For $T \in \mathcal{L}(E)$,

$$\omega_\pi(T) = \bigcap_{S \in K \cap \mathcal{K}(E)} \sigma(T+S).$$

- $\mathcal{L}(E)^{-1} + \mathcal{K}(E) = \mathcal{L}(E)^{-1} + \mathcal{F}(E)$
- $\mathcal{F}(E) \subseteq \mathrm{span}(K \cap \mathcal{K}(E))$

Natural generalization

Ordered Banach algebra (A, C), $a \in A$: $\bigcap_{c \in C \cap \mathrm{N}(T)} \sigma(a+c)$

In a general ordered Banach algebra (A, C),

$$\omega_T(a) := \bigcap_{c \in \mathrm{N}(T)} \sigma(a+c) \subsetneq \bigcap_{c \in C \cap \mathrm{N}(T)} \sigma(a+c)$$

In a general ordered Banach algebra (A, C),

$$\omega_T(a) := \bigcap_{c \in \mathrm{N}(T)} \sigma(a+c) \subsetneq \bigcap_{c \in C \cap \mathrm{N}(T)} \sigma(a+c)$$

Example ($(C(K,\mathbb{C}), C)$; $C := \{f \in C(K) : f(x) \in \mathbb{R}^+ \text{ for all } x \in K\}$.)

In a general ordered Banach algebra (A, C),

$$\omega_T(a) := \bigcap_{c \in \mathrm{N}(T)} \sigma(a+c) \subsetneq \bigcap_{c \in C \cap \mathrm{N}(T)} \sigma(a+c)$$

Example $(C(K,\mathbb{C}), C)$; $C := \{f \in C(K) : f(x) \in \mathbb{R}^+ \text{ for all } x \in K\}$.)

Let $K := [0,1]$ and $T : C(K,\mathbb{C}) \to C(K,\mathbb{C})$ be the homomorphism induced by composition with the unit function $\mathbf{1}$; i.e.

$$Tf = f \circ \mathbf{1} \text{ for all } f \in C(K).$$

Consider $g \in C(K)$ defined by $g(z) = -z$ for all $z \in K$. Then $\omega_T(g) = \{-1\}$ and $\bigcap_{c \in C \cap \mathrm{N}(T)} \sigma(a+c) = [-1, 0]$.

In a general ordered Banach algebra (A, C),

$$\omega_T(a) := \bigcap_{c \in \mathrm{N}(T)} \sigma(a+c) \subsetneq \bigcap_{c \in C \cap \mathrm{N}(T)} \sigma(a+c) = \omega_T^+(a)$$

Example $(C(K,\mathbb{C}), C)$; $C := \{f \in C(K) : f(x) \in \mathbb{R}^+ \text{ for all } x \in K\}$.)

Let $K := [0,1]$ and $T : C(K,\mathbb{C}) \to C(K,\mathbb{C})$ be the homomorphism induced by composition with the unit function $\mathbf{1}$; i.e.

$$Tf = f \circ \mathbf{1} \text{ for all } f \in C(K).$$

Consider $g \in C(K)$ defined by $g(z) = -z$ for all $z \in K$. Then $\omega_T(g) = \{-1\}$ and $\bigcap_{c \in C \cap \mathrm{N}(T)} \sigma(a+c) = [-1, 0]$.

In a general ordered Banach algebra (A, C),

$$\omega_T(a) := \bigcap_{c \in \mathrm{N}(T)} \sigma(a+c) \subsetneq \bigcap_{c \in C \cap \mathrm{N}(T)} \sigma(a+c) = \omega_T^+(a)$$

Example $(C(K, \mathbb{C}), C)$; $C := \{f \in C(K) : f(x) \in \mathbb{R}^+ \text{ for all } x \in K\}$.)

Let $K := [0, 1]$ and $T : C(K, \mathbb{C}) \to C(K, \mathbb{C})$ be the homomorphism induced by composition with the unit function $\mathbf{1}$; i.e.

$$Tf = f \circ \mathbf{1} \text{ for all } f \in C(K).$$

Consider $g \in C(K)$ defined by $g(z) = -z$ for all $z \in K$. Then $\omega_T(g) = \{-1\}$ and $\bigcap_{c \in C \cap \mathrm{N}(T)} \sigma(a+c) = [-1, 0]$.

Question: Let (A, C) be an ordered Banach algebra and $T : A \to B$ a homomorphism with the strong Riesz property. If $a \in C$, does $r(a) \notin \sigma(Ta, B)$ imply that $r(a) \notin \omega_T^+(a)$?

In a general ordered Banach algebra (A, C),

$$\omega_T(a) := \bigcap_{c \in \mathrm{N}(T)} \sigma(a+c) \subsetneq \bigcap_{c \in C \cap \mathrm{N}(T)} \sigma(a+c) = \omega_T^+(a)$$

Example ($(C(K, \mathbb{C}), C)$; $C := \{f \in C(K) : f(x) \in \mathbb{R}^+ \text{ for all } x \in K\}$.)

Let $K := [0, 1]$ and $T : C(K, \mathbb{C}) \to C(K, \mathbb{C})$ be the homomorphism induced by composition with the unit function $\mathbf{1}$; i.e.

$$Tf = f \circ \mathbf{1} \text{ for all } f \in C(K).$$

Consider $g \in C(K)$ defined by $g(z) = -z$ for all $z \in K$. Then $\omega_T(g) = \{-1\}$ and $\bigcap_{c \in C \cap \mathrm{N}(T)} \sigma(a+c) = [-1, 0]$.

Question: Let (A, C) be an ordered Banach algebra and $T : A \to B$ a homomorphism with the strong Riesz property. If $a \in C$, does $r(a) \notin \sigma(Ta, B)$ imply that $r(a) \notin \omega_T^+(a)$? YES!

The upper Browder spectrum property

Recall for $T \in \mathcal{L}(X)$: $\bigcap_{\substack{S \in \mathcal{K}(X) \\ TS = ST}} \sigma(T+S)$

The upper Browder spectrum property

Recall for $T \in \mathcal{L}(X)$: $\bigcap_{\substack{S \in \mathcal{K}(X) \\ TS = ST}} \sigma(T+S)$

Now for $T \in \mathcal{L}(E)$: $\bigcap_{\substack{S \in K \cap \mathcal{K}(E) \\ TS = ST}} \sigma(T+S)$, K: positive operators on E

The upper Browder spectrum property

Recall for $T \in \mathcal{L}(X)$: $\bigcap_{\substack{S \in \mathcal{K}(X) \\ TS=ST}} \sigma(T+S)$

Now for $T \in \mathcal{L}(E)$: $\bigcap_{\substack{S \in K \cap \mathcal{K}(E) \\ TS=ST}} \sigma(T+S)$, $\quad K$: positive operators on E

For $T \in K$, does $r(T) \notin \sigma(\pi T)$ imply $r(T) \notin \bigcap_{\substack{S \in K \cap \mathcal{K}(E) \\ TS=ST}} \sigma(T+S)$?

The upper Browder spectrum property

Recall for $T \in \mathcal{L}(X)$: $\bigcap_{\substack{S \in \mathcal{K}(X) \\ TS=ST}} \sigma(T+S)$

Now for $T \in \mathcal{L}(E)$: $\bigcap_{\substack{S \in K \cap \mathcal{K}(E) \\ TS=ST}} \sigma(T+S)$, K: positive operators on E

For $T \in K$, does $r(T) \notin \sigma(\pi T)$ imply $r(T) \notin \bigcap_{\substack{S \in K \cap \mathcal{K}(E) \\ TS=ST}} \sigma(T+S)$?

Definition (R. Benjamin and S. Mouton, 2016)

Let (A, C) be an ordered Banach algebra and $T : A \to B$ be a homomorphism. For $a \in A$, the upper Browder spectrum of a (relative to T) is given by

$$\bigcap_{\substack{c \in C \cap \mathrm{N}(T) \\ ac=ca}} \sigma(a+c)$$

The upper Browder spectrum property

Recall for $T \in \mathcal{L}(X)$: $\displaystyle\bigcap_{\substack{S \in \mathcal{K}(X) \\ TS=ST}} \sigma(T+S)$

Now for $T \in \mathcal{L}(E)$: $\displaystyle\bigcap_{\substack{S \in K \cap \mathcal{K}(E) \\ TS=ST}} \sigma(T+S)$, K: positive operators on E

For $T \in K$, does $r(T) \notin \sigma(\pi T)$ imply $r(T) \notin \displaystyle\bigcap_{\substack{S \in K \cap \mathcal{K}(E) \\ TS=ST}} \sigma(T+S)$?

Definition (R. Benjamin and S. Mouton, 2016)

Let (A, C) be an ordered Banach algebra and $T : A \to B$ be a homomorphism. For $a \in A$, the upper Browder spectrum of a (relative to T) is given by

$$\beta_T^+(a) = \bigcap_{\substack{c \in C \cap \mathrm{N}(T) \\ ac=ca}} \sigma(a+c)$$

The upper Browder spectrum property

Recall for $T \in \mathcal{L}(X)$: $\bigcap_{\substack{S \in \mathcal{K}(X) \\ TS=ST}} \sigma(T+S)$

Now for $T \in \mathcal{L}(E)$: $\bigcap_{\substack{S \in K \cap \mathcal{K}(E) \\ TS=ST}} \sigma(T+S)$, K: positive operators on E

For $T \in K$, does $r(T) \notin \sigma(\pi T)$ imply $r(T) \notin \bigcap_{\substack{S \in K \cap \mathcal{K}(E) \\ TS=ST}} \sigma(T+S)$?

Definition (R. Benjamin and S. Mouton, 2016)

Let (A, C) be an ordered Banach algebra and $T : A \to B$ be a homomorphism. For $a \in A$, the upper Browder spectrum of a (relative to T) is given by

$$\beta_T^+(a) = \bigcap_{\substack{c \in C \cap \mathrm{N}(T) \\ ac=ca}} \sigma(a+c) = \{\lambda \in \mathbb{C} : \lambda \mathbf{1}_A - a \notin A^{-1} \Subset (C \cap \mathrm{N}(T))\}$$

Example

Consider the homomorphism $T : M_3^u(\mathbb{C}) \to \mathbb{C}$ defined by

$$T \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ 0 & x_{22} & x_{23} \\ 0 & 0 & x_{33} \end{bmatrix} = x_{11}$$

and $M := \left(\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right) \in M_3^u(\mathbb{C})$. Then

$$\eta\sigma(TM) = \eta\beta_T(M) = \eta\omega_T(M) = \eta\omega_T^+(M) = \{1\} \neq \{0,1\} = \eta\beta_T^+(M).$$

Example

Consider the homomorphism $T : M_3^u(\mathbb{C}) \to \mathbb{C}$ defined by

$$T \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ 0 & x_{22} & x_{23} \\ 0 & 0 & x_{33} \end{bmatrix} = x_{11}$$

and $M := \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in M_3^u(\mathbb{C})$. Then

$$\eta\sigma(TM) = \eta\beta_T(M) = \eta\omega_T(M) = \eta\omega_T^+(M) = \{1\} \neq \{0,1\} = \eta\beta_T^+(M).$$

Question: Let (A, C) be an ordered Banach algebra and $T : A \to B$ a homomorphism with the strong Riesz property. If $a \in C$, does $r(a) \notin \sigma(Ta, B)$ imply that $r(a) \notin \beta_T^+(a)$?

Example

Consider the homomorphism $T : M_3^u(\mathbb{C}) \to \mathbb{C}$ defined by

$$T \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ 0 & x_{22} & x_{23} \\ 0 & 0 & x_{33} \end{bmatrix} = x_{11}$$

and $M := \left(\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right) \in M_3^u(\mathbb{C})$. Then

$$\eta\sigma(TM) = \eta\beta_T(M) = \eta\omega_T(M) = \eta\omega_T^+(M) = \{1\} \neq \{0,1\} = \eta\beta_T^+(M).$$

Question: Let (A, C) be an ordered Banach algebra and $T : A \to B$ a homomorphism with the strong Riesz property. If $a \in C$, does $r(a) \notin \sigma(Ta, B)$ imply that $r(a) \notin \beta_T^+(a)$?

- $r(a) \in \text{iso}\, \sigma(a, A)$

Example

Consider the homomorphism $T : M_3^u(\mathbb{C}) \to \mathbb{C}$ defined by

$$T \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ 0 & x_{22} & x_{23} \\ 0 & 0 & x_{33} \end{bmatrix} = x_{11}$$

and $M := \left(\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right) \in M_3^u(\mathbb{C})$. Then

$$\eta\sigma(TM) = \eta\beta_T(M) = \eta\omega_T(M) = \eta\omega_T^+(M) = \{1\} \neq \{0,1\} = \eta\beta_T^+(M).$$

Question: Let (A, C) be an ordered Banach algebra and $T : A \to B$ a homomorphism with the strong Riesz property. If $a \in C$, does $r(a) \notin \sigma(Ta, B)$ imply that $r(a) \notin \beta_T^+(a)$?

- $r(a) \in \operatorname{iso} \sigma(a, A)$
- In ordered Banach algebras, we have information about $p(a, r(a))$

r-Fredholm theory in (ordered) Banach algebras

r-Fredholm theory relative to T

r-Fredholm theory relative to T

Let $T : A \to B$ be a homomorphism. W.r.t. T an element $a \in A$ is called:

r-Fredholm theory relative to T

Let $T : A \to B$ be a homomorphism. W.r.t. T an element $a \in A$ is called:

- Fredholm if $0 \notin \sigma(Ta, B)$,

r-Fredholm theory relative to T

Let $T : A \to B$ be a homomorphism. W.r.t. T an element $a \in A$ is called:

- Fredholm if $0 \notin \sigma(Ta, B)$,
- Browder if there exist commuting elements $b \in A^{-1}$ $(0 \notin \sigma(b, A))$ and $c \in \mathrm{N}(T)$ such that $a = b + c$.

r-Fredholm theory relative to T

Let $T : A \to B$ be a homomorphism. W.r.t. T an element $a \in A$ is called:

- Fredholm if $0 \notin \sigma(Ta, B)$,
- Browder if there exist commuting elements $b \in A^{-1}$ ($0 \notin \sigma(b, A)$) and $c \in \mathrm{N}(T)$ such that $a = b + c$.

a almost invertible: $0 \notin \operatorname{acc} \sigma(a)$

r-Fredholm theory relative to T

Let $T : A \to B$ be a homomorphism. W.r.t. T an element $a \in A$ is called:

- Fredholm if $0 \notin \sigma(Ta, B)$,
- Browder if there exist commuting elements $b \in A^{-1}$ ($0 \notin \sigma(b, A)$) and $c \in \mathrm{N}(T)$ such that $a = b + c$.

a invertible: $0 \notin \sigma(a)$
a almost invertible: $0 \notin \operatorname{acc} \sigma(a)$

r-Fredholm theory relative to T

Let $T : A \to B$ be a homomorphism. W.r.t. T an element $a \in A$ is called:

- Fredholm if $0 \notin \sigma(Ta, B)$,
- Browder if there exist commuting elements $b \in A^{-1}$ $(0 \notin \sigma(b, A))$ and $c \in \mathrm{N}(T)$ such that $a = b + c$.

a invertible: $0 \notin \sigma(a)$
a almost invertible: $0 \notin \operatorname{acc} \sigma(a)$

- invertible $\Rightarrow$ almost invertible

r-Fredholm theory relative to T

Let $T : A \to B$ be a homomorphism. W.r.t. T an element $a \in A$ is called:

- Fredholm if $0 \notin \sigma(Ta, B)$,
- Browder if there exist commuting elements $b \in A^{-1}$ $(0 \notin \sigma(b, A))$ and $c \in \mathrm{N}(T)$ such that $a = b + c$.

a invertible: $0 \notin \sigma(a)$
a almost invertible: $0 \notin \operatorname{acc} \sigma(a)$

- invertible $\Rightarrow$ almost invertible
  $\Downarrow$
  Browder

r-Fredholm theory relative to T

Let $T : A \to B$ be a homomorphism. W.r.t. T an element $a \in A$ is called:

- Fredholm if $0 \notin \sigma(Ta, B)$,
- Browder if there exist commuting elements $b \in A^{-1}$ ($0 \notin \sigma(b, A)$) and $c \in \mathrm{N}(T)$ such that $a = b + c$.

a invertible: $0 \notin \sigma(a)$
a almost invertible: $0 \notin \operatorname{acc} \sigma(a)$

- invertible $\Rightarrow$ almost invertible
  $\Downarrow$
  Browder $\Rightarrow$ Fredholm

r-Fredholm theory relative to T

Let $T : A \to B$ be a homomorphism. W.r.t. T an element $a \in A$ is called:

- Fredholm if $0 \notin \sigma(Ta, B)$,
- Browder if there exist commuting elements $b \in A^{-1}$ $(0 \notin \sigma(b, A))$ and $c \in \mathrm{N}(T)$ such that $a = b + c$.

a invertible: $0 \notin \sigma(a)$

a almost invertible: $0 \notin \operatorname{acc} \sigma(a)$

a r-invertible: $r(a) \notin \sigma(a)$

- invertible $\Rightarrow$ almost invertible
  $\Downarrow$
  Browder $\Rightarrow$ Fredholm

r-Fredholm theory relative to T

Let $T : A \to B$ be a homomorphism. W.r.t. T an element $a \in A$ is called:

- Fredholm if $0 \notin \sigma(Ta, B)$,
- Browder if there exist commuting elements $b \in A^{-1}$ $(0 \notin \sigma(b, A))$ and $c \in \mathrm{N}(T)$ such that $a = b + c$.

a invertible: $0 \notin \sigma(a)$

a almost invertible: $0 \notin \operatorname{acc} \sigma(a)$

a r-invertible: $r(a) \notin \sigma(a)$

a almost r-invertible: $r(a) \notin \operatorname{acc} \sigma(a)$

- invertible $\Rightarrow$ almost invertible
  $\Downarrow$
  Browder $\Rightarrow$ Fredholm

r-Fredholm theory relative to T

Let $T : A \to B$ be a homomorphism. W.r.t. T an element $a \in A$ is called:

- Fredholm if $0 \notin \sigma(Ta, B)$,
- r-Fredholm if $r(a) \notin \sigma(Ta, B)$,
- Browder if there exist commuting elements $b \in A^{-1}$ $(0 \notin \sigma(b, A))$ and $c \in \mathrm{N}(T)$ such that $a = b + c$.

a invertible: $0 \notin \sigma(a)$

a almost invertible: $0 \notin \operatorname{acc} \sigma(a)$

a r-invertible: $r(a) \notin \sigma(a)$

a almost r-invertible: $r(a) \notin \operatorname{acc} \sigma(a)$

- invertible $\Rightarrow$ almost invertible

  $\Downarrow$

  Browder $\Rightarrow$ Fredholm

r-Fredholm theory relative to T

Let $T : A \to B$ be a homomorphism. W.r.t. T an element $a \in A$ is called:

- Fredholm if $0 \notin \sigma(Ta, B)$,
- r-Fredholm if $r(a) \notin \sigma(Ta, B)$,
- Browder if there exist commuting elements $b \in A^{-1}$ ($0 \notin \sigma(b, A)$) and $c \in \mathrm{N}(T)$ such that $a = b + c$.
- r-Browder if there exist commuting elements b ($r(b) \notin \sigma(b, A)$) and $c \in \mathrm{N}(T)$ such that $a = b + c$.

a invertible: $0 \notin \sigma(a)$

a almost invertible: $0 \notin \operatorname{acc} \sigma(a)$

a r-invertible: $r(a) \notin \sigma(a)$

a almost r-invertible: $r(a) \notin \operatorname{acc} \sigma(a)$

- invertible $\Rightarrow$ almost invertible

  $\Downarrow$

  Browder $\Rightarrow$ Fredholm

r-Fredholm theory relative to T

Let $T : A \to B$ be a homomorphism. W.r.t. T an element $a \in A$ is called:

- Fredholm if $0 \notin \sigma(Ta, B)$,
- r-Fredholm if $r(a) \notin \sigma(Ta, B)$,
- Browder if there exist commuting elements $b \in A^{-1}$ $(0 \notin \sigma(b, A))$ and $c \in \mathrm{N}(T)$ such that $a = b + c$.
- r-Browder if there exist commuting elements b $(r(b) \notin \sigma(b, A))$ and $c \in \mathrm{N}(T)$ such that $a = b + c$.

a invertible: $0 \notin \sigma(a)$

a almost invertible: $0 \notin \operatorname{acc} \sigma(a)$

- invertible $\Rightarrow$ almost invertible

  $\Downarrow$

  Browder $\Rightarrow$ Fredholm

a r-invertible: $r(a) \notin \sigma(a)$

a almost r-invertible: $r(a) \notin \operatorname{acc} \sigma(a)$

- r-invertible $\Rightarrow$ almost r-invertible

r-Fredholm theory relative to T

Let $T : A \to B$ be a homomorphism. W.r.t. T an element $a \in A$ is called:

- Fredholm if $0 \notin \sigma(Ta, B)$,
- r-Fredholm if $r(a) \notin \sigma(Ta, B)$,
- Browder if there exist commuting elements $b \in A^{-1}$ $(0 \notin \sigma(b, A))$ and $c \in \mathrm{N}(T)$ such that $a = b + c$.
- r-Browder if there exist commuting elements b $(r(b) \notin \sigma(b, A))$ and $c \in \mathrm{N}(T)$ such that $a = b + c$.

a invertible: $0 \notin \sigma(a)$
a almost invertible: $0 \notin \operatorname{acc} \sigma(a)$

a r-invertible: $r(a) \notin \sigma(a)$
a almost r-invertible: $r(a) \notin \operatorname{acc} \sigma(a)$

- invertible $\Rightarrow$ almost invertible
  $\Downarrow$
  Browder $\Rightarrow$ Fredholm

- r-invertible $\Rightarrow$ almost r-invertible
  $\Downarrow$
  r-Browder

r-Fredholm theory relative to T

Let $T : A \to B$ be a homomorphism. W.r.t. T an element $a \in A$ is called:

- Fredholm if $0 \notin \sigma(Ta, B)$,
- r-Fredholm if $r(a) \notin \sigma(Ta, B)$,
- Browder if there exist commuting elements $b \in A^{-1}$ $(0 \notin \sigma(b, A))$ and $c \in \mathrm{N}(T)$ such that $a = b + c$.
- r-Browder if there exist commuting elements b $(r(b) \notin \sigma(b, A))$ and $c \in \mathrm{N}(T)$ such that $a = b + c$.

a invertible: $0 \notin \sigma(a)$

a almost invertible: $0 \notin \text{acc } \sigma(a)$

a r-invertible: $r(a) \notin \sigma(a)$

a almost r-invertible: $r(a) \notin \text{acc } \sigma(a)$

- invertible $\Rightarrow$ almost invertible
  $\Downarrow$
  Browder $\Rightarrow$ Fredholm
- r-invertible $\Rightarrow$ almost r-invertible
  $\Downarrow$
  r-Browder $\not\Rightarrow$ r-Fredholm

r-Fredholm theory relative to T

Let $T : A \to B$ be a homomorphism. W.r.t. T an element $a \in A$ is called:

- Fredholm if $0 \notin \sigma(Ta, B)$,
- r-Fredholm if $r(a) \notin \sigma(Ta, B)$,
- Browder if there exist commuting elements $b \in A^{-1}$ ($0 \notin \sigma(b, A)$) and $c \in \mathrm{N}(T)$ such that $a = b + c$.
- r-Browder if there exist commuting elements b ($r(b) \notin \sigma(b, A)$) and $c \in \mathrm{N}(T)$ such that $a = b + c$.

a invertible: $0 \notin \sigma(a)$
a almost invertible: $0 \notin \operatorname{acc} \sigma(a)$

- invertible $\Rightarrow$ almost invertible
  $\Downarrow$
  Browder $\Rightarrow$ Fredholm
  $\Uparrow$
  almost invertible Fredholm

a r-invertible: $r(a) \notin \sigma(a)$
a almost r-invertible: $r(a) \notin \operatorname{acc} \sigma(a)$

- r-invertible $\Rightarrow$ almost r-invertible
  $\Downarrow$
  r-Browder $\not\Rightarrow$ r-Fredholm

r-Fredholm theory relative to T

Let $T : A \to B$ be a homomorphism. W.r.t. T an element $a \in A$ is called:

- Fredholm if $0 \notin \sigma(Ta, B)$,
- r-Fredholm if $r(a) \notin \sigma(Ta, B)$,
- Browder if there exist commuting elements $b \in A^{-1}$ ($0 \notin \sigma(b, A)$) and $c \in \mathrm{N}(T)$ such that $a = b + c$.
- r-Browder if there exist commuting elements b ($r(b) \notin \sigma(b, A)$) and $c \in \mathrm{N}(T)$ such that $a = b + c$.

a invertible: $0 \notin \sigma(a)$
a almost invertible: $0 \notin \operatorname{acc} \sigma(a)$

a r-invertible: $r(a) \notin \sigma(a)$
a almost r-invertible: $r(a) \notin \operatorname{acc} \sigma(a)$

- invertible $\Rightarrow$ almost invertible
  $\Downarrow$
  Browder $\Rightarrow$ Fredholm
  $\Uparrow$
  almost invertible Fredholm

- r-invertible $\Rightarrow$ almost r-invertible
  $\Downarrow$
  r-Browder $\not\Rightarrow$ r-Fredholm
  $\Uparrow$
  almost r-invertible r-Fredholm

r-Fredholm theory relative to T

Let $T : A \to B$ be a homomorphism. W.r.t. T an element $a \in A$ is called:

- Fredholm if $0 \notin \sigma(Ta, B)$,
- r-Fredholm if $r(a) \notin \sigma(Ta, B)$,
- Browder if there exist commuting elements $b \in A^{-1}$ $(0 \notin \sigma(b, A))$ and $c \in \mathrm{N}(T)$ such that $a = b + c$.
- r-Browder if there exist commuting elements b $(r(b) \notin \sigma(b, A))$ and $c \in \mathrm{N}(T)$ such that $a = b + c$.

a invertible: $0 \notin \sigma(a)$
a almost invertible: $0 \notin \operatorname{acc} \sigma(a)$

a r-invertible: $r(a) \notin \sigma(a)$
a almost r-invertible: $r(a) \notin \operatorname{acc} \sigma(a)$

- invertible $\Rightarrow$ almost invertible
  $\Downarrow$
  Browder $\Rightarrow$ Fredholm
  $\|$
  almost invertible Fredholm

- r-invertible $\Rightarrow$ almost r-invertible
  $\Downarrow$
  r-Browder $\not\Rightarrow$ r-Fredholm
  $\Uparrow$
  almost r-invertible r-Fredholm

r-Fredholm theory relative to T

Let $T : A \to B$ be a homomorphism. W.r.t. T an element $a \in A$ is called:

- Fredholm if $0 \notin \sigma(Ta, B)$,
- r-Fredholm if $r(a) \notin \sigma(Ta, B)$,
- Browder if there exist commuting elements $b \in A^{-1}$ ($0 \notin \sigma(b, A)$) and $c \in \mathrm{N}(T)$ such that $a = b + c$.
- r-Browder if there exist commuting elements b ($r(b) \notin \sigma(b, A)$) and $c \in \mathrm{N}(T)$ such that $a = b + c$.

a invertible: $0 \notin \sigma(a)$
a almost invertible: $0 \notin \operatorname{acc} \sigma(a)$

a r-invertible: $r(a) \notin \sigma(a)$
a almost r-invertible: $r(a) \notin \operatorname{acc} \sigma(a)$

- invertible $\Rightarrow$ almost invertible
  $\Downarrow$
  Browder $\Rightarrow$ Fredholm
  $\|$
  almost invertible Fredholm

- r-invertible $\Rightarrow$ almost r-invertible
  $\Downarrow$
  r-Browder $\not\Rightarrow$ r-Fredholm
  $\nparallel$
  almost r-invertible r-Fredholm

Example

Let K and L be compact Hausdorff spaces and $T : C(K) \to C(L)$ be the homomorphism induced by composition with the continuous map $\theta : L \to K$; i.e.

$$Tf = f \circ \theta \text{ for all } f \in C(K).$$

Then

- r-Fredholm elements: $\{f \in C(K) : r(f) \notin f(\theta(L))\}$
- r-Browder elements $= \begin{cases} C(K)^r & \text{if } \theta(L) = K \\ C(K) & \text{otherwise} \end{cases}$

Example

Let K and L be compact Hausdorff spaces and $T : C(K) \to C(L)$ be the homomorphism induced by composition with the continuous map $\theta : L \to K$; i.e.

$$Tf = f \circ \theta \text{ for all } f \in C(K).$$

Then

- r-Fredholm elements: $\{f \in C(K) : r(f) \notin f(\theta(L))\}$
- r-Browder elements $= \begin{cases} C(K)^r & \text{if } \theta(L) = K \\ C(K) & \text{otherwise} \end{cases}$

Example

Let $A = M_2^u(\mathbb{C})$ and $T : A \to \mathbb{C}$ be the homomorphism defined by $Ta = a_1$, where $a := \begin{pmatrix} a_1 & a_2 \\ 0 & a_4 \end{pmatrix}$. Then

- a is r-Fredholm if and only $r(a) \neq a_1$
- r-Browder elements: $A \backslash \left\{ \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} : x \geq 0 \text{ and } y \neq 0 \right\}$

Theorem (Benjamin, Laustsen, Mouton [2019])

If $T : A \to B$ has the Riesz property, then

$A \ni a$ is almost r-invertble r-Fredholm $\Longleftrightarrow$ a is contractive r-Browder

Theorem (Benjamin, Laustsen, Mouton [2019])

If $T : A \to B$ has the Riesz property, then

$A \ni a$ is almost r-invertble r-Fredholm $\iff$ a is contractive r-Browder

Recall:

a r-Browder:

Theorem (Benjamin, Laustsen, Mouton [2019])

If $T : A \to B$ has the Riesz property, then

$$A \ni a \text{ is almost } r\text{-invertble } r\text{-Fredholm} \iff a \text{ is contractive } r\text{-Browder}$$

Recall:

a r-Browder: $a = b + c, bc = cb,$

Theorem (Benjamin, Laustsen, Mouton [2019])

If $T : A \to B$ has the Riesz property, then

$$A \ni a \text{ is almost } r\text{-invertble } r\text{-Fredholm} \iff a \text{ is contractive } r\text{-Browder}$$

Recall:

a r-Browder: $a = b + c, bc = cb,$ where b is r-invertible and $c \in \mathrm{N}(T)$

Theorem (Benjamin, Laustsen, Mouton [2019])

If $T : A \to B$ has the Riesz property, then

$$A \ni a \text{ is almost } r\text{-invertble } r\text{-Fredholm} \iff a \text{ is contractive } r\text{-Browder}$$

Recall:

a r-Browder: $a = b + c, bc = cb,$ where b is r-invertible and $c \in \mathrm{N}(T)$

(Now) a contractive r-Browder:

Theorem (Benjamin, Laustsen, Mouton [2019])

If $T : A \to B$ has the Riesz property, then

$A \ni a$ is almost r-invertble r-Fredholm $\iff$ a is contractive r-Browder

Recall:

a r-Browder: $a = b + c, bc = cb,$ where b is r-invertible and $c \in \mathrm{N}(T)$

(Now) a contractive r-Browder:
$a = b + c, bc = cb,$

Theorem (Benjamin, Laustsen, Mouton [2019])

If $T : A \to B$ has the Riesz property, then

$A \ni a$ is almost r-invertble r-Fredholm $\iff$ a is contractive r-Browder

Recall:

a r-Browder: $a = b + c, bc = cb,$ where b is r-invertible and $c \in \mathrm{N}(T)$

(Now) a contractive r-Browder:
$a = b + c, bc = cb,$ where b is r-invertible, $c \in \mathrm{N}(T)$

Theorem (Benjamin, Laustsen, Mouton [2019])

If $T : A \to B$ has the Riesz property, then

$A \ni a$ is almost r-invertble r-Fredholm $\iff a$ is contractive r-Browder

Recall:

a r-Browder: $a = b + c, bc = cb,$ where b is r-invertible and $c \in \mathrm{N}(T)$

(Now) a contractive r-Browder:
$a = b + c, bc = cb,$ where b is r-invertible, $c \in \mathrm{N}(T)$ and $r(b) \leq r(a)$

Theorem (Benjamin, Laustsen, Mouton [2019])

If $T : A \to B$ has the Riesz property, then

$A \ni a$ is almost r-invertble r-Fredholm $\Longleftrightarrow$ a is contractive r-Browder

Recall:

a r-Browder: $a = b + c, bc = cb,$ where b is r-invertible and $c \in \mathrm{N}(T)$

(Now) a contractive r-Browder:
$a = b + c, bc = cb,$ where b is r-invertible, $c \in \mathrm{N}(T)$ and $r(b) \leq r(a)$

a almost r-invertible r-Fredholm

Theorem (Benjamin, Laustsen, Mouton [2019])

If $T : A \to B$ has the Riesz property, then

$A \ni a$ is almost r-invertble r-Fredholm $\Longleftrightarrow$ a is contractive r-Browder

Recall:

a r-Browder: $a = b + c, bc = cb,$ where b is r-invertible and $c \in \mathrm{N}(T)$

(Now) a contractive r-Browder:
$a = b + c, bc = cb,$ where b is r-invertible, $c \in \mathrm{N}(T)$ and $r(b) \leq r(a)$

a almost r-invertible r-Fredholm $\Leftrightarrow$

Theorem (Benjamin, Laustsen, Mouton [2019])

If $T : A \to B$ has the Riesz property, then

$$A \ni a \text{ is almost } r\text{-invertble } r\text{-Fredholm} \iff a \text{ is contractive } r\text{-Browder}$$

Recall:

a r-Browder: $a = b + c, bc = cb,$ where b is r-invertible and $c \in \mathrm{N}(T)$

(Now) a contractive r-Browder:
$a = b + c, bc = cb,$ where b is r-invertible, $c \in \mathrm{N}(T)$ and $r(b) \leq r(a)$

a almost r-invertible r-Fredholm $\Leftrightarrow r(a) \notin \operatorname{acc} \sigma(a, A)$

Theorem (Benjamin, Laustsen, Mouton [2019])

If $T : A \to B$ has the Riesz property, then

$A \ni a$ is almost r-invertble r-Fredholm $\iff$ a is contractive r-Browder

Recall:

a r-Browder: $a = b + c, bc = cb,$ where b is r-invertible and $c \in \mathrm{N}(T)$

(Now) a contractive r-Browder:
$a = b + c, bc = cb,$ where b is r-invertible, $c \in \mathrm{N}(T)$ and $r(b) \leq r(a)$

a almost r-invertible r-Fredholm $\Leftrightarrow$ $r(a) \notin$ acc $\sigma(a, A)$ and $r(a) \notin \sigma(Ta, B)$

Theorem (Benjamin, Laustsen, Mouton [2019])

If $T : A \to B$ has the Riesz property, then

$A \ni a$ is almost r-invertble r-Fredholm $\iff$ a is contractive r-Browder

Recall:

a r-Browder: $a = b + c, bc = cb,$ where b is r-invertible and $c \in \mathrm{N}(T)$

(Now) a contractive r-Browder:
$a = b + c, bc = cb,$ where b is r-invertible, $c \in \mathrm{N}(T)$ and $r(b) \leq r(a)$

a almost r-invertible r-Fredholm $\Leftrightarrow$ $r(a) \notin \operatorname{acc} \sigma(a, A)$ and $r(a) \notin \sigma(Ta, B)$

- $r(a) \notin \sigma(a, A)$
- $r(a) \in \operatorname{iso} \sigma(a, A)$

Theorem (Benjamin, Laustsen, Mouton [2019])

If $T : A \to B$ has the Riesz property, then

$A \ni a$ is almost r-invertble r-Fredholm $\iff$ a is contractive r-Browder

Recall:

a r-Browder: $a = b + c, bc = cb,$ where b is r-invertible and $c \in \mathrm{N}(T)$

(Now) a contractive r-Browder:
$a = b + c, bc = cb,$ where b is r-invertible, $c \in \mathrm{N}(T)$ and $r(b) \leq r(a)$

a almost r-invertible r-Fredholm $\Leftrightarrow$ $r(a) \notin$ acc $\sigma(a, A)$ and $r(a) \notin \sigma(Ta, B)$

- $r(a) \notin \sigma(a, A)$
  $\Updownarrow$
  a is r-invertible
- $r(a) \in$ iso $\sigma(a, A)$

Theorem (Benjamin, Laustsen, Mouton [2019])

If $T : A \to B$ has the Riesz property, then

$$A \ni a \text{ is almost } r\text{-invertble } r\text{-Fredholm} \iff a \text{ is contractive } r\text{-Browder}$$

Recall:

a r-Browder: $a = b + c, bc = cb,$ where b is r-invertible and $c \in \mathrm{N}(T)$

(Now) a contractive r-Browder:
$a = b + c, bc = cb,$ where b is r-invertible, $c \in \mathrm{N}(T)$ and $r(b) \leq r(a)$

a almost r-invertible r-Fredholm $\Leftrightarrow r(a) \notin \operatorname{acc} \sigma(a, A)$ and $r(a) \notin \sigma(Ta, B)$

- $r(a) \notin \sigma(a, A)$
  $\Updownarrow$
  a is r-invertible
  $\Downarrow$
  a is contractive r-Browder
- $r(a) \in \operatorname{iso} \sigma(a, A)$

Theorem (Benjamin, Laustsen, Mouton [2019])

If $T : A \to B$ has the Riesz property, then

$$A \ni a \text{ is almost } r\text{-invertble } r\text{-Fredholm} \iff a \text{ is contractive } r\text{-Browder}$$

Recall:

a r-Browder: $a = b + c, bc = cb,$ where b is r-invertible and $c \in \mathrm{N}(T)$

(Now) a contractive r-Browder:
$a = b + c, bc = cb,$ where b is r-invertible, $c \in \mathrm{N}(T)$ and $r(b) \leq r(a)$

a almost r-invertible r-Fredholm $\Leftrightarrow r(a) \notin \operatorname{acc} \sigma(a, A)$ and $r(a) \notin \sigma(Ta, B)$

- $r(a) \notin \sigma(a, A)$
  $\Updownarrow$
  a is r-invertible
  $\Downarrow$
  a is contractive r-Browder

- $r(a) \in \operatorname{iso} \sigma(a, A)$
- $a = b + c,$

Theorem (Benjamin, Laustsen, Mouton [2019])

If $T : A \to B$ has the Riesz property, then

$$A \ni a \text{ is almost } r\text{-invertble } r\text{-Fredholm} \iff a \text{ is contractive } r\text{-Browder}$$

Recall:

a r-Browder: $a = b + c, bc = cb,$ where b is r-invertible and $c \in \mathrm{N}(T)$

(Now) a contractive r-Browder:
$a = b + c, bc = cb,$ where b is r-invertible, $c \in \mathrm{N}(T)$ and $r(b) \leq r(a)$

a almost r-invertible r-Fredholm $\Leftrightarrow r(a) \notin \text{acc } \sigma(a, A)$ and $r(a) \notin \sigma(Ta, B)$

- $r(a) \notin \sigma(a, A)$
  $\Updownarrow$
  a is r-invertible
  $\Downarrow$
  a is contractive r-Browder

- $r(a) \in \text{iso } \sigma(a, A)$
- $a = b + c$, where
  $b := a(\mathbf{1} - p(a, r(a)) - r(a)p(a, r(a))$

Theorem (Benjamin, Laustsen, Mouton [2019])

If $T : A \to B$ has the Riesz property, then

$$A \ni a \text{ is almost } r\text{-invertble } r\text{-Fredholm} \iff a \text{ is contractive } r\text{-Browder}$$

Recall:

a r-Browder: $a = b + c, bc = cb,$ where b is r-invertible and $c \in \mathrm{N}(T)$

(Now) a contractive r-Browder:
$a = b + c, bc = cb,$ where b is r-invertible, $c \in \mathrm{N}(T)$ and $r(b) \le r(a)$

a almost r-invertible r-Fredholm $\Leftrightarrow r(a) \notin \operatorname{acc} \sigma(a, A)$ and $r(a) \notin \sigma(Ta, B)$

- $r(a) \notin \sigma(a, A)$
  $\Updownarrow$
  a is r-invertible
  $\Downarrow$
  a is contractive r-Browder

- $r(a) \in \operatorname{iso} \sigma(a, A)$
- $a = b + c$, where
  $b := a(\mathbf{1} - p(a, r(a)) - r(a)p(a, r(a))$
  is r-invertible,

Theorem (Benjamin, Laustsen, Mouton [2019])

If $T : A \to B$ has the Riesz property, then

$$A \ni a \text{ is almost } r\text{-invertble } r\text{-Fredholm} \iff a \text{ is contractive } r\text{-Browder}$$

Recall:

a r-Browder: $a = b + c, bc = cb,$ where b is r-invertible and $c \in \mathrm{N}(T)$

(Now) a contractive r-Browder:
$a = b + c, bc = cb,$ where b is r-invertible, $c \in \mathrm{N}(T)$ and $r(b) \leq r(a)$

a almost r-invertible r-Fredholm $\Leftrightarrow r(a) \notin \operatorname{acc} \sigma(a, A)$ and $r(a) \notin \sigma(Ta, B)$

- $r(a) \notin \sigma(a, A)$
  $\Updownarrow$
  a is r-invertible
  $\Downarrow$
  a is contractive r-Browder

- $r(a) \in \operatorname{iso} \sigma(a, A)$
- $a = b + c$, where
  $b := a(\mathbf{1} - p(a, r(a)) - r(a)p(a, r(a))$
  is r-invertible, $r(b) = r(a)$,

Theorem (Benjamin, Laustsen, Mouton [2019])

If $T : A \to B$ has the Riesz property, then

$$A \ni a \text{ is almost } r\text{-invertble } r\text{-Fredholm} \iff a \text{ is contractive } r\text{-Browder}$$

Recall:

a r-Browder: $a = b + c, bc = cb,$ where b is r-invertible and $c \in \mathrm{N}(T)$

(Now) a contractive r-Browder:
$a = b + c, bc = cb,$ where b is r-invertible, $c \in \mathrm{N}(T)$ and $r(b) \leq r(a)$

a almost r-invertible r-Fredholm $\Leftrightarrow r(a) \notin \operatorname{acc} \sigma(a, A)$ and $r(a) \notin \sigma(Ta, B)$

- $r(a) \notin \sigma(a, A)$
  $\Updownarrow$
  a is r-invertible
  $\Downarrow$
  a is contractive r-Browder

- $r(a) \in \operatorname{iso} \sigma(a, A)$
- $a = b + c$, where
  $b := a(\mathbf{1} - p(a, r(a)) - r(a)p(a, r(a))$
  is r-invertible, $r(b) = r(a)$, and
  $c := (a + r(a)\mathbf{1})p(a, r(a)) \in \mathrm{N}(T)$

Theorem (Benjamin, Laustsen, Mouton [2019])

If $T : A \to B$ has the Riesz property, then

$$A \ni a \text{ is almost } r\text{-invertble } r\text{-Fredholm} \iff a \text{ is contractive } r\text{-Browder}$$

Recall:

a r-Browder: $a = b + c$, $bc = cb$, where b is r-invertible and $c \in \mathrm{N}(T)$

(Now) a contractive r-Browder:
$a = b + c$, $bc = cb$, where b is r-invertible, $c \in \mathrm{N}(T)$ and $r(b) \leq r(a)$

Theorem (Benjamin, Laustsen, Mouton [2019])

If $T : A \to B$ has the Riesz property, then

$A \ni a$ is almost r-invertble r-Fredholm $\iff$ a is contractive r-Browder

Recall:

a r-Browder: $a = b + c, bc = cb$, where b is r-invertible and $c \in \mathrm{N}(T)$

(Now) a contractive r-Browder:
$a = b + c, bc = cb$, where b is r-invertible, $c \in \mathrm{N}(T)$ and $r(b) \leq r(a)$

Theorem [Benjamin, Mouton, 2020]

If $T : A \to B$ has the strong Riesz property, then

$A \ni a$ is almost r-invertble r-Fredholm $\iff$ a is contractive r-Browder

$\iff$ a is r-Fredholm

a is an almost r-invertible r-Fredholm element

$$\Downarrow$$

a is contractive r-Browder

$$\Updownarrow$$

$a = b + c, bc = cb$, where b is r-invertible, $c \in \mathrm{N}(T)$ and $r(b) \leq r(a)$

a is **a positive** almost r-invertible r-Fredholm element

$\Downarrow$

a is contractive **upper** r-Browder

$\Updownarrow$

$a = b + c$, $bc = cb$, where b is r-invertible, $c \in C \cap \mathrm{N}(T)$ and $r(b) \leq r(a)$

a is **a positive** almost r-invertible r-Fredholm element

$\Downarrow$

a is contractive **upper** r-Browder

$\Updownarrow$

$a = b + c$, $bc = cb$, where b is r-invertible, $c \in C \cap \mathrm{N}(T)$ and $r(b) \leq r(a)$

Theorem (Benjamin and Mouton)

Let $A = A_1 \oplus \cdots \oplus A_n$, where each A_j $(j = 1, \ldots, n)$ is a finite-dimensional simple OBA with algebra cone C_j, and $C = C_1 \oplus \cdots \oplus C_n$.
W.r.t. any $T : A \to B$,

positive almost r-invertible r-Fredholm $\Rightarrow$ contractive upper r-Browder

a is **a positive** almost r-invertible r-Fredholm element

$\Downarrow$

a is contractive **upper** r-Browder

$\Updownarrow$

$a = b + c$, $bc = cb$, where b is r-invertible, $c \in C \cap \mathrm{N}(T)$ and $r(b) \leq r(a)$

Theorem (Benjamin and Mouton)

Let $A = A_1 \oplus \cdots \oplus A_n$, where each A_j ($j = 1, \ldots, n$) is a finite-dimensional simple OBA with algebra cone C_j, and $C = C_1 \oplus \cdots \oplus C_n$.
W.r.t. any $T : A \to B$,

positive almost r-invertible r-Fredholm $\Rightarrow$ contractive upper r-Browder

Corollary

Any finite-dimensional semisimple OBA is algebraically isomorphic to an OBA (A, C) with the property that, w.r.t. any $T : A \to B$,

positive almost r-invertible r-Fredholm $\Rightarrow$ contractive upper r-Browder

Theorem (Benjamin and Mouton)

Let (A, C) be an OBA with closed algebra cone C. W.r.t. any $T : A \to B$,

Theorem (Benjamin and Mouton)

Let (A, C) be an OBA with closed algebra cone C. W.r.t. any $T : A \to B$,

a positive almost r-invertible r-Fredholm $\implies$ *a* contractive upper r-Browder

Theorem (Benjamin and Mouton)

Let (A, C) be an OBA with closed algebra cone C. W.r.t. any $T : A \to B$,

a positive almost r-invertible r-Fredholm		a contractive
	$\Longrightarrow$	upper
$r(a)$ simple pole of the resolvent of a		r-Browder

Theorem (Benjamin and Mouton)

Let (A, C) be an OBA with closed algebra cone C. W.r.t. any $T : A \to B$,

a positive almost r-invertible r-Fredholm		a contractive
	$\Longrightarrow$	upper
$r(a)$ simple pole of the resolvent of a		r-Browder

Corollary

Let (A, C) be a semisimple OBA with closed algebra cone. If A is either commutative or C is proper and inverse-closed, then, w.r.t. a homomorphism $T : A \to B$ with the strong Riesz property,

positive almost r-invertible r-Fredholm $\Rightarrow$ contractive upper r-Browder

Theorem (Benjamin and Mouton)

Let (A, C) be an OBA with closed algebra cone C. W.r.t. any $T : A \to B$,

a positive almost r-invertible r-Fredholm		a contractive
	$\Longrightarrow$	upper
$r(a)$ simple pole of the resolvent of a		r-Browder

Corollary

Let (A, C) be a semisimple OBA with closed algebra cone. If A is either commutative or C is proper and inverse-closed, then, w.r.t. a homomorphism $T : A \to B$ with the strong Riesz property,

positive almost r-invertible r-Fredholm $\Rightarrow$ contractive upper r-Browder

Theorem (Benjamin and Mouton)

Let (A, C) be an OBA with closed algebra cone C. W.r.t. any $T : A \to B$,

a positive almost r-invertible r-Fredholm		a contractive
	$\Longrightarrow$	upper
$r(a)$ simple pole of the resolvent of a		r-Browder

Corollary

Let (A, C) be a semisimple OBA with closed algebra cone. If A is either commutative or C is proper and inverse-closed, then, w.r.t. a homomorphism $T : A \to B$ with the strong Riesz property,

positive almost r-invertible r-Fredholm $\Rightarrow$ contractive upper r-Browder

- proper: $C \cap -C = \{0\}$

Theorem (Benjamin and Mouton)

Let (A, C) be an OBA with closed algebra cone C. W.r.t. any $T : A \to B$,

a positive almost r-invertible r-Fredholm		a contractive
	$\Longrightarrow$	upper
$r(a)$ simple pole of the resolvent of a		r-Browder

Corollary

Let (A, C) be a semisimple OBA with closed algebra cone. If A is either commutative or C is proper and inverse-closed, then, w.r.t. a homomorphism $T : A \to B$ with the strong Riesz property,

positive almost r-invertible r-Fredholm $\Rightarrow$ contractive upper r-Browder

- proper: $C \cap -C = \{0\}$
- inverse-closed: $a \in C \cap A^{-1} \Longrightarrow a^{-1} \in C$

Theorem (Benjamin and Mouton)

Let (A, C) be a Dedekind complete OBA

,

- Dedekind complete:
  Every non-empty order-bounded set in A has a supremum

Theorem (Benjamin and Mouton)

Let (A, C) be a Dedekind complete OBA which has a disjunctive product

,

- Dedekind complete:
  Every non-empty order-bounded set in A has a supremum

Theorem (Benjamin and Mouton)

Let (A, C) be a <u>Dedekind complete</u> OBA which has a <u>disjunctive product</u> and with closed and <u>normal</u> algebra cone C.

,

- Dedekind complete:
  Every non-empty order-bounded set in A has a supremum
- normal:
  there exists a constant α with the property that
  if $0 \leq a \leq b$, then $||a|| \leq \alpha ||b||$

Theorem (Benjamin and Mouton)

Let (A, C) be a Dedekind complete OBA which has a disjunctive product and with closed and normal algebra cone C. W.r.t. a homomorphism $T : A \to B$ with the strong Riesz property such that the spectral radius in $(A/\overline{\mathrm{N}(T)})$ is weakly monotone,

- Dedekind complete:
  Every non-empty order-bounded set in A has a supremum
- normal:
  there exists a constant α with the property that
  if $0 \leq a \leq b$, then $||a|| \leq \alpha ||b||$

Theorem (Benjamin and Mouton)

Let (A, C) be a Dedekind complete OBA which has a disjunctive product and with closed and normal algebra cone C. W.r.t. a homomorphism $T : A \to B$ with the strong Riesz property such that the spectral radius in $(A/\overline{\mathrm{N}(T)})$ is weakly monotone,

- Dedekind complete:
  Every non-empty order-bounded set in A has a supremum
- normal:
  there exists a constant α with the property that
  if $0 \leq a \leq b$, then $||a|| \leq \alpha ||b||$

Theorem (Benjamin and Mouton)

Let (A, C) be a Dedekind complete OBA which has a disjunctive product and with closed and normal algebra cone C. W.r.t. a homomorphism $T : A \to B$ with the strong Riesz property such that the spectral radius in $(A/\overline{\mathrm{N}(T)})$ is weakly monotone,

a positive almost r-invertible r-Fredholm spectrally order continuous	$\Longrightarrow$	$p_i(\mathbf{1} - p_{i-1})ap_i(\mathbf{1} - p_{i-1})$ contractive upper r-Browder

- Dedekind complete:
  Every non-empty order-bounded set in A has a supremum
- normal:
  there exists a constant α with the property that
  if $0 \leq a \leq b$, then $||a|| \leq \alpha||b||$

Theorem (Benjamin and Mouton)

Let (A, C) be a Dedekind complete OBA which has a disjunctive product and with closed and normal algebra cone C. W.r.t. a homomorphism $T : A \to B$ with the strong Riesz property such that the spectral radius in $(A/\overline{\mathrm{N}(T)})$ is weakly monotone,

a positive almost r-invertible r-Fredholm spectrally order continuous $\implies$ $p_i(\mathbf{1} - p_{i-1})ap_i(\mathbf{1} - p_{i-1})$ contractive upper r-Browder

for all $i \in \{1, \ldots, n\}$ such that $r(p_i(\mathbf{1} - p_{i-1})ap_i(\mathbf{1} - p_{i-1})) = r(a)$.

- Dedekind complete:
  Every non-empty order-bounded set in A has a supremum
- normal:
  there exists a constant α with the property that
  if $0 \leq a \leq b$, then $||a|| \leq \alpha||b||$

Application

Upper Browder spectrum property

Let (A, C) be an OBA and $T : A \to B$ an algebra homomorphism. Then $a \in C$ has the upper Browder spectrum property if

$$r(a) \notin \sigma(Ta, B) \Longrightarrow r(a) \notin \beta_T^+(a)$$

Application

Upper Browder spectrum property

Let (A, C) be an OBA and $T : A \to B$ an algebra homomorphism. Then $a \in C$ has the upper Browder spectrum property if

$$r(a) \notin \sigma(Ta, B) \Longrightarrow r(a) \notin \beta_T^+(a)$$

$$\beta_T^+(a) := \{\lambda \in \mathbb{C} : \lambda \mathbf{1} - a \text{ is not upper Browder}\}$$

Application

Upper Browder spectrum property

Let (A, C) be an OBA and $T : A \to B$ an algebra homomorphism. Then $a \in C$ has the upper Browder spectrum property if

$$r(a) \notin \sigma(Ta, B) \implies r(a) \notin \beta_T^+(a)$$

$$\beta_T^+(a) := \{\lambda \in \mathbb{C} : \lambda \mathbf{1} - a \text{ is not upper Browder}\}$$

a upper Browder element:

Application

Upper Browder spectrum property

Let (A, C) be an OBA and $T : A \to B$ an algebra homomorphism. Then $a \in C$ has the upper Browder spectrum property if

$$r(a) \notin \sigma(Ta, B) \Longrightarrow r(a) \notin \beta_T^+(a)$$

$$\beta_T^+(a) := \{\lambda \in \mathbb{C} : \lambda\mathbf{1} - a \text{ is not upper Browder}\}$$

a upper Browder element:
$a = b + c$, where b is invertible ($0 \notin \sigma(b)$) and $c \in C \cap \mathrm{N}(T)$ commute

Application

Upper Browder spectrum property

Let (A, C) be an OBA and $T : A \to B$ an algebra homomorphism. Then $a \in C$ has the upper Browder spectrum property if

$$r(a) \notin \sigma(Ta, B) \implies r(a) \notin \beta_T^+(a)$$

$$\beta_T^+(a) := \{\lambda \in \mathbb{C} : \lambda\mathbf{1} - a \text{ is not upper Browder}\}$$

a upper Browder element:
$a = b + c$, where b is invertible ($0 \notin \sigma(b)$) and $c \in C \cap \mathrm{N}(T)$ commute

$$\underbrace{a \text{ positive almost } r\text{-invertible } r\text{-Fredholm}}_{a \in C,\, r(a) \notin \sigma(Ta, B)} \Rightarrow a \text{ contractive upper } r\text{-Browder}$$

Application

Upper Browder spectrum property

Let (A, C) be an OBA and $T : A \to B$ an algebra homomorphism. Then $a \in C$ has the upper Browder spectrum property if

$$r(a) \notin \sigma(Ta, B) \Longrightarrow r(a) \notin \beta_T^+(a)$$

$$\beta_T^+(a) := \{\lambda \in \mathbb{C} : \lambda\mathbf{1} - a \text{ is not upper Browder}\}$$

a upper Browder element:
$a = b + c$, where b is invertible ($0 \notin \sigma(b)$) and $c \in C \cap \mathrm{N}(T)$ commute

$$\underbrace{a \text{ positive almost } r\text{-invertible } r\text{-Fredholm}}_{a \in C,\ r(a) \notin \sigma(Ta, B)} \Rightarrow a \text{ contractive upper } r\text{-Browder}$$

Application

Upper Browder spectrum property

Let (A, C) be an OBA and $T : A \to B$ an algebra homomorphism. Then $a \in C$ has the upper Browder spectrum property if

$$r(a) \notin \sigma(Ta, B) \Longrightarrow r(a) \notin \beta_T^+(a)$$

$$\beta_T^+(a) := \{\lambda \in \mathbb{C} : \lambda \mathbf{1} - a \text{ is not upper Browder}\}$$

a upper Browder element:
$a = b + c$, where b is invertible ($0 \notin \sigma(b)$) and $c \in C \cap \mathrm{N}(T)$ commute

$$\underbrace{a \text{ positive almost } r\text{-invertible } r\text{-Fredholm}}_{a \in C,\, r(a) \notin \sigma(Ta, B)} \Rightarrow a \text{ contractive upper } r\text{-Browder}$$

$$\Rightarrow \quad r(a) \notin \beta_T^+(a)$$

Results: Upper Browder spectrum property

Recall:

Any finite-dimensional semisimple OBA is algebraically isomorphic to an OBA (A, C) with the property that, w.r.t. any $T : A \to B$,

positive almost r-invertible r-Fredholm $\Rightarrow$ *a* contractive upper r-Browder

Results: Upper Browder spectrum property

Recall:

Any finite-dimensional semisimple OBA is algebraically isomorphic to an OBA (A, C) with the property that, w.r.t. any $T : A \to B$,

positive almost r-invertible r-Fredholm $\Rightarrow$ *a* contractive upper r-Browder

Any finite-dimensional semisimple OBA is algebraically isomorphic to an OBA (A, C) with the property that all positive elements in A has the upper Browder spectrum property relative to arbitrary algebra homomorphisms $T : A \to B$.

Results: Upper Browder spectrum property

Recall:

Let (A, C) be an OBA with closed algebra cone C. W.r.t. any $T : A \to B$,

a positive almost r-invertible r-Fredholm		a contractive
	$\Longrightarrow$	upper
$r(a)$ simple pole of the resolvent of a		r-Browder

Results: Upper Browder spectrum property

Recall:

Let (A, C) be an OBA with closed algebra cone C. W.r.t. any $T : A \to B$,

a positive almost r-invertible r-Fredholm		a contractive
	$\Longrightarrow$	upper
$r(a)$ simple pole of the resolvent of a		r-Browder

Let (A, C) be an OBA with closed algebra cone C and $T : A \to B$ an algebra homomorphism satisfying the Riesz property.
If $a \in C$ with $r(a)$ a simple pole of the resolvent of a, then

$$r(a) \notin \sigma(Ta, B) \Longrightarrow r(a) \notin \beta_T^+(a).$$

Recall:

Let (A, C) be a Dedekind complete semisimple OBA which has a disjunctive product and with closed and normal algebra cone C. W.r.t. a homomorphism $T : A \to B$ with the strong Riesz property such that the spectral radius in $(A/\overline{\mathrm{N}(T)})$ is weakly monotone,

a positive almost r-invertible r-Fredholm spectrally order continuous $\Longrightarrow$ $p_i(\mathbf{1} - p_{i-1})ap_i(\mathbf{1} - p_{i-1})$ contractive upper r-Browder

for all $i \in \{1, \ldots, n\}$ such that $r(p_i(\mathbf{1} - p_{i-1})ap_i(\mathbf{1} - p_{i-1})) = r(a)$.

Recall:

Let (A, C) be a Dedekind complete semisimple OBA which has a disjunctive product and with closed and normal algebra cone C. W.r.t. a homomorphism $T : A \to B$ with the strong Riesz property such that the spectral radius in $(A/\overline{\mathrm{N}(T)})$ is weakly monotone,

a positive almost *r*-invertible *r*-Fredholm spectrally order continuous $\Longrightarrow$ $p_i(\mathbf{1} - p_{i-1})ap_i(\mathbf{1} - p_{i-1})$ contractive upper *r*-Browder

for all $i \in \{1, \ldots, n\}$ such that $r(p_i(\mathbf{1} - p_{i-1})ap_i(\mathbf{1} - p_{i-1})) = r(a)$.

Let (A, C) be a Dedekind complete semisimple OBA which has a disjunctive product and with closed and normal algebra cone C. Also, suppose that $T : A \to B$ is an algebra homomorphism satisfying the strong Riesz property such that the spectral radius function in $(A/\overline{\mathrm{N}(T)})$ is weakly monotone. If $a \in C$ is a spectrally order continuous element, then

$$r(a) \notin \sigma(Ta, B) \Longrightarrow r(a) \notin \bigcup_{i=1}^{n} \beta_T^+(p_i(\mathbf{1} - p_{i-1})ap_i(\mathbf{1} - p_{i-1}))$$

Thank you for your attention