☞Inertial methods for finding minimum-norm solutions of the split variational inequality problem beyond monotonicity

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Outline

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☞ Acknowledgement

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- A Special Issue can be proposed at any time to Symmetry. For details, see https://www.mdpi.com/journal/symmetry.

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INYUVESI YAKWAZULU-NATALI

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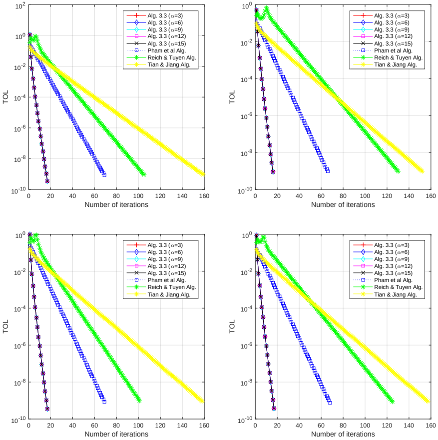

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- Thus, we define the metric projection $P_{\mathcal{C}}$ as:

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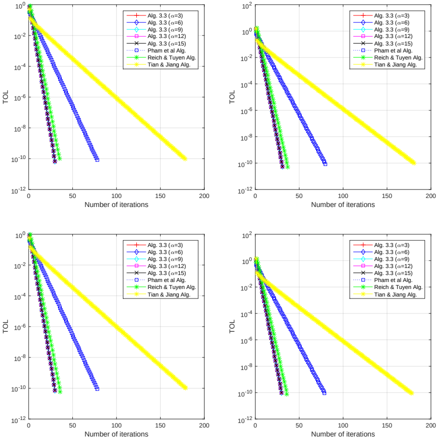
Example 2: Errors against number of iterations



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END OF TAKL

THANKS FOR LISTEN

References contd

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Thank You!