Interpolation in algebras of multipliers on the ball

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based on joint works with Ken Davidson and Nikolaos Chalmoukis

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Peak interpolation

Peak interpolation in the disc algebra

The disc algebra is

$$A(\mathbb{D}) = \left\{ f \in C(\overline{\mathbb{D}}) : f\big|_{\mathbb{D}} \text{ is holomorphic} \right\}.$$

Theorem (Rudin–Carleson, 1950s)

Let $E \subset \partial\mathbb{D}$ be a compact set with Lebesgue measure zero and let $g \in C(E) \setminus \{0\}$. Then there exists $f \in A(\mathbb{D})$ with

(1) $f\big|_E = g$, and

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(2) $|f(z)| < \|g\|_\infty$ for $z \in \overline{\mathbb{D}} \setminus E$.

E is called a peak interpolation set for $A(\mathbb{D})$.

In particular, E is peak set, i.e. there exists $f \in A(\mathbb{D})$ with $f\big|_E = 1$ and $|f(z)| < 1$ for $z \in \overline{\mathbb{D}} \setminus E$.

Peak interpolation in the ball algebra

Let $\mathbb{B}_d = \{z \in \mathbb{C}^d : \|z\|_2 < 1\}$ and

$$A(\mathbb{B}_d) = \{f \in C(\overline{\mathbb{B}_d}) : f\big|_{\mathbb{B}_d} \text{ is holomorphic}\}.$$

Theorem (Bishop, 1962)

Let $E \subset \partial\mathbb{B}_d$ be compact and totally null and let $g \in C(E) \setminus \{0\}$. Then there exists $f \in A(\mathbb{B}_d)$ with

(1) $f\big|_E = g$, and

(2) $|f(z)| < \|g\|_\infty$ for $z \in \overline{\mathbb{B}} \setminus E$.

More generally, Bishop considered peak interpolation in uniform algebras.

Goal for today

Goal

Find peak interpolation theorems in Banach algebras of analytic functions on $\mathbb{D}$ and $\mathbb{B}_d$, not necessarily uniform algebras.

Motivation: Multivariable operator theory, classical Dirichlet space theory.

Spaces on the ball

A unitarily invariant space is a reproducing kernel Hilbert space $\mathcal{H}$ of analytic functions on $\mathbb{B}_d$ with $\mathbb{C}[z_1, \ldots, z_d] \subset \mathcal{H}$ and

$$||f \circ U|| = ||f||$$

for all $f \in \mathcal{H}$ and all unitary maps U on $\mathbb{C}^d$.

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Examples

- Hardy space $H^2(\mathbb{D}) = \{f = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{O}(\mathbb{D}) : \sum_{n=0}^{\infty} |a_n|^2 < \infty\}$
- Hardy space on the ball $H^2(\mathbb{B}_d)$
- The Dirichlet space $\mathcal{D} = \{f \in \mathcal{O}(\mathbb{D}) : f' \in L^2(\mathbb{D})\}$.

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Regularity condition

We will assume that $\lim_{n\to\infty} ||z_1^{n+1}||/||z_1^n|| = 1$.

The Drury-Arveson space

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$$\frac{1}{1 - \langle z, w \rangle}.$$

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Concrete description

$$H^2_d = \left\{ f = \sum_{\alpha \in \mathbb{N}^d} \widehat{f}(\alpha) z^\alpha \in \mathcal{O}(\mathbb{B}_d) : \sum_{\alpha \in \mathbb{N}^d} \binom{|\alpha|}{\alpha}^{-1} |\widehat{f}(\alpha)|^2 < \infty \right\}.$$

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Theorem (Drury, Müller–Vasilescu, Arveson)

Let $T = (T_1, \ldots, T_d)$ be a tuple of commuting operators on Hilbert space with $\sum_{i=1}^d T_i T_i^* \le I$. Then

$$\|p(T)\| \le \|p\|_{\mathrm{Mult}(H^2_d)} = \|f \mapsto p \cdot f\|_{H^2_d \to H^2_d}.$$

for all polynomials p.

Algebras of multipliers

Let $\mathcal{H}$ be a unitarily invariant space. The multiplier algebra is

$$\operatorname{Mult}(\mathcal{H}) = \{\varphi : \mathbb{B}_d \to \mathbb{C} : \varphi \cdot f \in \mathcal{H} \text{ whenever } f \in \mathcal{H}\},$$

equipped with the multiplier norm $\|\varphi\|_{\operatorname{Mult}(\mathcal{H})} = \|f \mapsto \varphi \cdot f\|_{\mathcal{H} \to \mathcal{H}}$.

Definition

$$A(\mathcal{H}) = \overline{\mathbb{C}[z_1, \ldots, z_d]}^{\|\cdot\|} \subset \operatorname{Mult}(\mathcal{H}).$$

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Definition

$$A(\mathcal{H}) = \overline{\mathbb{C}[z_1, \ldots, z_d]}^{\|\cdot\|} \subset \text{Mult}(\mathcal{H}).$$

Then

$$A(\mathcal{H}) \subset \text{Mult}(\mathcal{H}) \cap A(\mathbb{B}_d) \subset \mathcal{H} \cap A(\mathbb{B}_d).$$

Examples

- $A(H^2(\mathbb{D})) = A(\mathbb{D})$. More generally, $A(H^2(\mathbb{B}_d)) = A(\mathbb{B}_d)$.
- $A(H^2_d)$ is Arveson's algebra $\mathcal{A}_d$, key in multivariable operator theory.

Small sets on the boundary

A regular Borel measure μ on $\partial\mathbb{B}_d$ is called Mult($\mathcal{H}$)-Henkin if

$$\mathrm{Mult}(\mathcal{H}) \mapsto \mathbb{C}, \quad p \mapsto \int_{\partial\mathbb{B}_d} p \, d\mu \quad (p \in \mathbb{C}[z])$$

is weak-$*$ continuous.

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A Borel set $E \subset \partial\mathbb{B}_d$ is called Mult($\mathcal{H}$)-totally null if $\mu(E) = 0$ for every Mult($\mathcal{H}$)-Henkin measure μ.

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Example

- μ is $\mathrm{Mult}(H^2(\mathbb{D}))$-Henkin iff it is absolutely continuous.
- E is $\mathrm{Mult}(H^2(\mathbb{D}))$-totally null iff it has Lebesgue measure zero.

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- E is Mult($H^2(\mathbb{D})$)-totally null iff it has Lebesgue measure zero.

$\mathcal{H} = H^2(\mathbb{B}_d)$: Henkin (1968), Cole–Range (1972)
$\mathcal{H} = H^2_d$: Clouâtre–Davidson (2016)
General $\mathcal{H}$: Bickel–H.–M^{c}Carthy (2017)

Peak interpolation in the Drury–Arveson space

Theorem (Clouâtre–Davidson, 2016)

Let $E \subset \partial\mathbb{B}_d$ be compact and $\mathrm{Mult}(H^2_d)$-totally null, let $g \in C(E) \setminus \{0\}$ and let $\varepsilon > 0$. Then there exists $f \in A(H^2_d)$ with

(1) $f\big|_E = g$,

(2) $|f(z)| < \|g\|_\infty$ for all $z \in \overline{\mathbb{B}_d} \setminus E$, and

(3) $\|f\|_{\mathrm{Mult}(H^2_d)} \le (1+\varepsilon)\|g\|_\infty$.

Sharp peak interpolation on the ball

Let $\mathcal{H}$ be a unitarily invariant space on $\mathbb{B}_d$.

Theorem (Davidson–H.)

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$\mathcal{H} = H^2(\mathbb{D})$: Rudin–Carleson

$\mathcal{H} = H^2(\mathbb{B}_d)$: Bishop

$\mathcal{H} = H^2_d$: Clouâtre–Davidson with $\varepsilon = 0$

Duality

A regular Borel measure ν on $\partial\mathbb{B}_d$ is called Mult($\mathcal{H}$)-totally singular if

$$\nu \perp \mu \quad \text{for all } \mu \text{ Mult}(\mathcal{H})\text{-Henkin.}$$

Let $\text{TS}(\text{Mult}(\mathcal{H})) = \{\nu \in M(\partial\mathbb{B}_d) : \nu \text{ is Mult}(\mathcal{H})\text{-totally singular}\}$.

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Theorem (Davidson-H.)

Let $\mathcal{H}$ be a unitarily invariant space on $\mathbb{B}_d$. Then

$$A(\mathcal{H})^* = \text{Mult}(\mathcal{H})_* \oplus_1 \text{TS}(\text{Mult}(\mathcal{H})).$$

$\mathcal{H} = H^2(\mathbb{B}_d)$: Henkin and Cole–Range

$\mathcal{H} = H^2_d$: Clouâtre–Davidson

From duality to interpolation

Let $E \subset \partial\mathbb{B}_d$ be compact and $\mathrm{Mult}(\mathcal{H})$-totally null.

Goal

Show that

$$R : A(\mathcal{H}) \to C(E), \quad f \mapsto f\big|_E,$$

maps the closed unit ball onto the closed unit ball.

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The adjoint

$$R^* : M(E) \to A(\mathcal{H})^* = \mathrm{Mult}(\mathcal{H})_* \oplus_1 \mathrm{TS}(\mathrm{Mult}(\mathcal{H}))$$

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Show that $\ker(R)$ is an M-ideal to prove $\varepsilon = 0$ works.

Pick and peak interpolation

Pick's theorem

Theorem (Pick 1916, Nevanlinna 1919)

Let $z_1, \ldots, z_n \in \mathbb{D}$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$. There exists $f \in A(\mathbb{D})$ with

$$f(z_i) = \lambda_i \text{ for } 1 \leq i \leq n \quad \text{and} \quad ||f||_\infty \leq 1$$

if and only if the matrix

$$\left[\frac{1 - \lambda_i \overline{\lambda_j}}{1 - z_i \overline{z_j}}\right]_{i,j=1}^n$$

is positive.

Question

Can we solve Pick and peak interpolation problems simultaneously?

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Extremal Pick problems have a unique solution.

Pick and peak interpolation in the disc algebra

Theorem (Izzo, 2018)

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$$\left[\frac{1 - \lambda_i \overline{\lambda_j}}{1 - z_i \overline{z_j}}\right]_{i,j=1}^{n} \geq 0.$$

Let $E \subset \partial\mathbb{D}$ be compact with Lebesgue measure zero and let $g \in C(E)$ with $\|g\|_\infty \leq 1$.

Then for each $\varepsilon > 0$, there exists $f \in A(\mathbb{D})$ with

$$f(z_i) = \lambda_i \quad (1 \leq i \leq n), \quad f\big|_E = g \quad \text{and } \|f\|_\infty \leq 1 + \varepsilon.$$

$\varepsilon > 0$ is necessary in general.

Pick and peak interpolation on the ball

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Theorem (Davidson–H.)

Let $\mathcal{H}$ be a unitarily invariant Pick space on $\mathbb{B}_d$ with kernel K.

Let $z_1, \ldots, z_n \in \mathbb{B}_d$, $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ with

$$\left[K(z_i, z_j)(1 - \lambda_i \overline{\lambda_j})\right] \geq 0.$$

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Is the totally null condition necessary?

Interpolation sets

A compact set $E \subset \partial\mathbb{B}_d$ is said to be an interpolation set if

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Theorem (Davidson–H.)

Let $\mathcal{H}$ be a unitarily invariant space that admits non-empty $\mathrm{Mult}(\mathcal{H})$-totally null sets (e.g. any space mentioned so far).

Then every interpolation set is $\mathrm{Mult}(\mathcal{H})$-totally null.

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Proposition

If $\mathcal{H}$ does not admit non-empty $\mathrm{Mult}(\mathcal{H})$-totally null sets, then there are no infinite interpolation sets.

Summary of interpolation theorems

Theorem

The following are equivalent for a compact set $E \subset \partial\mathbb{B}_d$:

(TN) E is Mult($\mathcal{H}$)-totally null;

(PI) E is a peak interpolation set;

(P) E is a peak set;

(PPI) E is a Pick-peak interpolation set.

Moreover, if there exist non-empty Mult($\mathcal{H}$)-totally null sets, then this is equivalent to

(I) E is an interpolation set.

Which sets are totally null?

Capacity zero

Consider the Dirichlet space $\mathcal{D} = \{f \in \mathcal{O}(\mathbb{D}) : f' \in L^2(\mathbb{D})\}$.

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Definition (Capacity zero, functional analysis view)

(a) A positive Borel measure μ on $\partial\mathbb{D}$ is said to have finite energy if

$$\mathcal{D} \to \mathbb{C}, \quad p \mapsto \int_{\partial\mathbb{D}} p \, d\mu \quad (p \in \mathbb{C}[z])$$

is continuous.

(b) A compact set $E \subset \partial\mathbb{D}$ has logarithmic capacity zero if it does not support a Borel probability measure of finite energy.

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(b) A compact set $E \subset \partial\mathbb{D}$ has logarithmic capacity zero if it does not support a Borel probability measure of finite energy.

Proposition

If E is $\mathrm{Mult}(\mathcal{D})$-totally null, then E has logarithmic capacity zero.

Capacity zero vs. totally null

Theorem (Chalmoukis–H.)

Let $E \subset \partial\mathbb{D}$ be compact. Then E is Mult($\mathcal{D}$)-totally null if and only if E has logarithmic capacity zero.

Similar result holds for weighted Dirichlet spaces (a.k.a. Besov–Sobolev spaces on $\mathbb{B}_d$) and capacities of Ahern and Cohn.

Peak interpolation in the Dirichlet space

Theorem (Peller–Khrushchëv, 1982)

Let $E \subset \partial\mathbb{D}$ be compact with logarithmic capacity zero. Then for every $g \in C(E)$, there exists $f \in \mathcal{D} \cap A(\mathbb{D})$ with

(1) $f\big|_E = g$, and

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Theorem (Davidson–H. + Chalmoukis–H.)

Let $E \subset \partial\mathbb{D}$ be compact with logarithmic capacity zero. Then for every $g \in C(E) \setminus \{0\}$, there exists $f \in A(\mathcal{D}) \subset \mathrm{Mult}(\mathcal{D}) \cap A(\mathbb{D})$ with

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(2) $|f(z)| < \|g\|_\infty$ for $z \in \overline{\mathbb{D}} \setminus E$, and

(3) $\|f\|_{\mathrm{Mult}(\mathcal{D})} \leq \|g\|_\infty$.

Similarly for weighted Dirichlet spaces on $\mathbb{B}_d$, improves Cohn–Verbitsky.

Summary

- Sharp peak interpolation and Pick-peak interpolation can be done on totally null sets in many algebras of multipliers on the ball.
- Conversely, mere interpolation sets are typically totally null.
- Duality plays a key role in establishing interpolation theorems.
- In Dirichlet type spaces, totally null sets and capacity zero sets agree.

Thank you!