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On Herz's extension Theorem

by

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Abstract.

We present a self-contained proof of the following famous extension theorem due to Carl Herz. A closed subgroup H of a locally compact group G is a set of p-synthesis in G if and only if, for every $u \in A_p(H) \cap C_{00}(H)$ and every $\varepsilon > 0$, there is $v \in A_p(G) \cap C_{00}(G)$, an extension of u, such that

$$\|v\|_{A_p(G)} < \|u\|_{A_p(H)} + \varepsilon.$$

1 Introduction

We present a self-contained proof of the following famous extension theorem due to Carl Herz (see [3], Proposition 2, p. 94)

Theorem 1. Let G be a locally compact group, let H a closed subgroup and let $1 < p < \infty$. The following properties are equivalent:

1) The subgroup H is a set of p-synthesis in G;

2) For every $u \in A_p(H) \cap C_{00}(H)$ and for every $\varepsilon > 0$, there is a function $v \in A_p(G) \cap C_{00}(G)$ with $\mathrm{Res}_H v = u$ and such that

$$\|v\|_{A_p(G)} < \|u\|_{A_p(H)} + \varepsilon.$$

This important theorem is a consequence of two results which are both highly interesting (Proposition 2 and Theorem 8). We fill a gap (see Remark 5) and correct a mistake (see Remark 7).

In [1], we proved a more simple result (also due to C. Herz): for every $\varepsilon > 0$ and for every $\varphi \in A_p(H)$, there is $\psi \in A_p(G)$ with $\mathrm{Res}_H \psi = \varphi$ and such that

$$\|\psi\|_{A_p(G)} < \|\varphi\|_{A_p(H)} + \varepsilon\,.$$

We denote by $\mathcal{A}_p(G)$ the set of all pairs $\big((k_n),(l_n)\big)$ where (k_n) is a sequence of $\mathcal{L}^p(G)$ and (l_n) is a sequence of $\mathcal{L}^{p'}(G)$ $(1/p + 1/p' = 1)$ such that $\sum N_p(k_n) N_{p'}(l_n) < \infty$. This permits to give the definition of the Banach space $A_p(G)$: $A_p(G)$ is the set

$$\Big\{ u : G \to \mathbb{C} : \text{ there is } ((k_n),(l_n)) \in \mathcal{A}_p(G) \text{ such that}$$

$$u = \sum \overline{k_n} * \check{l_n} \Big\}$$

where $\check{l_n}(x) = l_n(x^{-1})$. For $u \in A_p(G)$ we put:

$$\|u\|_{A_p(G)} = \inf \Big\{ \sum N_p(k_n) N_{p'}(l_n) : ((k_n),(l_n)) \in \mathcal{A}_p(G)$$

$$\text{such that } u = \sum \overline{k_n} * \check{l_n} \Big\}.$$

We refer to [2] Chap 3, section 3.1, pages 33-44.

2 The fact that a closed subgroup is a set of synthesis being equivalent with some sort of Reiter-Glicksberg property.

The following result is a special case of a proposition due to Herz (see [3] Proposition 4 p. 103). We recall that a closed subset F of a locally compact group G is said to be a set of p-synthesis in G if, for every $u \in A_p(G)$ with $\mathrm{Res}_F u = 0$ and for every $\varepsilon > 0$ there is $v \in A_p(G) \cap C_{00}(G)$ with $\mathrm{supp} v \cap F = \emptyset$ and $\|u - v\|_{A_p(G)} < \varepsilon$.

Proposition 2. Let G be a locally compact group, let H be a closed subgroup and let $1 < p < \infty$. The following properties are equivalent:

(1) The H is a set of p-synthesis in G;

(2) For every $g \in A_p(G) \cap C_{00}(G)$ and for every $\varepsilon > 0$, there is a function $h \in A_p(G) \cap C_{00}(G)$ such that $g = h$ on a neighborhood of H in G and with

$$\|h\|_{A_p(G)} < \|\mathrm{Res}_H g\|_{A_p(H)} + \varepsilon.$$

Inspired by Reiter [4] p. 181 we introduce the following definition.

Definition 1. Let G be a locally compact group, let H be a closed subgroup and let $1 < p < \infty$. For every $g \in A_p(G) \cap C_{00}(G)$, we put

$$I_H^p(g) = \inf\Big\{\|h\|_{A_p(G)} : h \in A_p(G) \cap C_{00}(G),$$

$$h = g \text{ on a neighborhood of } H \text{ in } G\Big\}.$$

Corollary 3. Let G be a locally compact group, let H be a closed subgroup and let $1 < p < \infty$. Then:

1) $\|\mathrm{Res}_H g\|_{A_p(H)} \leq I_H^p(g)$ for every $g \in A_p(G) \cap C_{00}(G)$.

2) The following two statements are equivalent :

(i) The subgroup H is a set of p-synthesis in G;

(ii) $\|\mathrm{Res}_H g\|_{A_p(H)} = I_H^p(g)$ for every $g \in A_p(G) \cap C_{00}(G)$.

3 Proof of the first part of Theorem 1

Theorem 4. Let G be a locally compact group, let H be a closed subgroup and let $1 < p < \infty$. Suppose that H is a set of p-synthesis in G. Then for every $u \in A_p(H) \cap C_{00}(H)$ and for every $\varepsilon > 0$, there is $v \in A_p(G) \cap C_{00}(G)$ with $\mathrm{Res}_H v = u$ such that

$$\|v\|_{A_p(G)} < \|u\|_{A_p(H)} + \varepsilon.$$

Remark 5. There is a very serious gap in Herz's proof of Prop. 2 of [3], see [5].

4 Every closed subgroup of a locally compact group is a set of local $p$synthesis

Herz's proof of 2) implies 1) of Theorem 1 requires that H is a set of local p-synthesis in G.

Lemma 6. Let G be a locally compact goup, let H be a closed subgroup, let K be a compact subset of H, and W be a compact neighborhood of e in G. There is a positive integer $N(= N(K,W))$ depending of K and W such that, for every open neighborhood Ω of e in G, there is an open neighborhood V of e in G with

1) $V \subset W$,

2) $HV \subset H\Omega$,

3) $m_G(KV) \leq N m_G(V)$.

Remark 7. The relation $V \subset \Omega$ ([3] page 100 line 8) is not correct.

Theorem 8. Let G be a locally compact group, let H be a closed subgroup and let $1 < p < \infty$. Then H is a set of local p-synthesis in G.

Proof. Let u be a function of $A_p(G) \cap C_{00}(G)$ vanishing on H. Let ε be a positive real number. It suffices to prove the existence of $v \in A_p(G) \cap C_{00}(G)$ with $\mathrm{supp} v \cap H = \emptyset$ and $\|u - v\|_{A_p(G)} < \varepsilon$. The Lemma 6 together with the use of the dual of $A_p(G)$ permit to construct v explicitely. This proof (four pages and one half) is a jewel! (See [5]). □

5 Proof of the second part of Theorem 1

Theorem 9. Let G be a locally compact group, let H be a closed subgroup and let $1 < p < \infty$. Suppose that for every $u \in A_p(H) \cap C_{00}(H)$ and for every $\varepsilon > 0$ there is $v \in A_p(G) \cap C_{00}(G)$ with $\mathrm{Res}_H v = u$ and such that $\|v\|_{A_p(G)} < \|u\|_{A_p(H)} + \varepsilon$. Then H is a set of p-synthesis in G.

Proof. Let f be a function of $A_p(G)$ vanishing on H and ε a positive real number. We choose $g_1 \in A_p(G) \cap C_{00}(G)$ such that

$$\|f - g_1\|_{A_p(G)} < \frac{\varepsilon}{4}.$$

This implies

$$\|\mathrm{Res}_H g_1\|_{A_p(H)} < \frac{\varepsilon}{4}.$$

By assumption there is $g_2 \in A_p(G) \cap C_{00}(G)$ with $\mathrm{Res}_H g_2 = \mathrm{Res}_H g_1$ and

$$\|g_2\|_{A_p(G)} < \|\mathrm{Res}_H g_1\|_{A_p(H)} + \frac{\varepsilon}{4}.$$

We have

$$\|g_2\|_{A_p(G)} < \frac{\varepsilon}{4}.$$

The set H being of local p-synthesis there is $\varphi \in A_p(G) \cap C_{00}(G)$ with $\mathrm{supp}\varphi \cap H = \emptyset$ and such that

$$\|g_1 - g_2 - \varphi\|_{A_p(G)} < \frac{\varepsilon}{4}.$$

We finally get

$$\|f - \varphi\|_{A_p(G)} < \varepsilon.$$

□

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