A Hilbert space approach to singularities of functions

Zinaida Lykova

Newcastle University, UK

Jointly with J. Agler (UCSD), N. J. Young (Newcastle)

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Outline of the talk

- Introduction - pseudomultipliers
- Examples of pseudomultipliers
- Ambiguous vectors and the ambiguous space
- Definable vectors, polar vectors and the polar space
- Decomposition of the singular space

Reproducing Kernel Hilbert spaces

There is a long tradition of using Hilbert space methods to study problems in complex analysis; this is sometimes called "operator analysis". One of the challenges in this area is to translate function theoretic properties into Hilbert space concepts, typically involving reproducing kernel Hilbert spaces (RKHS).

There has been a substantial development of the theory of reproducing kernel Hilbert spaces, see N. Aronszajn [4] and S. Saitoh [5]. In this talk we approach singularities of functions from a RKHS perspective.

By a *Hilbert function space on a set* Ω we mean a Hilbert space $\mathcal{H}$ whose elements are complex-valued functions on Ω such that, for each $\lambda \in \Omega$, the linear functional $f \mapsto f(\lambda)$ is continuous on $\mathcal{H}$. It follows that, for each $\lambda \in \Omega$ there exists an element of $\mathcal{H}$, which we shall denote by k_λ, such that $f(\lambda) = \langle f, k_\lambda \rangle$ for every $f \in \mathcal{H}$. We call k_λ the *kernel* corresponding to λ, and we call the function $k : \Omega \times \Omega \to \mathbb{C}$ given by

$$k(\mu, \lambda) = k_\lambda(\mu) \qquad \text{for all } \lambda, \mu \in \Omega$$

the *reproducing kernel* of $\mathcal{H}$.

The Hardy space H^2

Before giving a formal definition of a pseudomultiplier, let us consider an archetypal example. The Hardy space H^2 is the space of analytic functions f on the open unit disc $\mathbb{D}$ such that

$$\sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta < \infty.$$

When endowed with pointwise addition and scalar multiplication and the inner product

$$\langle f, g \rangle = \lim_{r \to 1-} \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) \overline{g(re^{i\theta})} \, d\theta$$

H^2 is a Hilbert function space, whose reproducing kernel is the *Szegő kernel* k, defined by

$$k(\lambda, \mu) = \frac{1}{1 - \bar{\mu}\lambda} \quad \text{for all } \lambda, \mu \in \mathbb{D}.$$

Multipliers

Definition 1. *A* multiplier *of a Hilbert function space $\mathcal{H}$ on a set Ω is defined to be a function $\varphi : \Omega \to \mathbb{C}$ such that $\varphi f \in \mathcal{H}$ for every $f \in \mathcal{H}$. Here φf denotes the pointwise product of φ and f.*

Multipliers of H^2

One can show directly from the definitions that every function in the space H^∞ of bounded analytic functions on $\mathbb{D}$ is a multiplier of H^2.

On the other hand, if φ is a multiplier of H^2, then since the constant function $\mathbf{1} \in H^2$, we have $\varphi = \varphi\mathbf{1} \in H^2$, and so φ is analytic on $\mathbb{D}$. Moreover, the operator X_φ on H^2 defined, for all $f \in H^2$, by

$$(X_\varphi f)(\lambda) = \varphi(\lambda) f(\lambda), \qquad \text{for all } \lambda \in \Omega \tag{1}$$

is linear and is easily seen to have a closed graph, and hence X_φ is a bounded linear operator on H^2. The calculation

$$\begin{aligned} \left\langle X_\varphi^* k_\lambda, f \right\rangle &= \langle k_\lambda, X_\varphi f\rangle = \langle k_\lambda, \varphi f\rangle \\ &= \overline{(\varphi f)(\lambda)} = \overline{\varphi(\lambda)}\, \langle k_\lambda, f\rangle \text{ for all } \lambda \in \mathbb{D} \text{ and all } f \in H^2 \end{aligned}$$

shows that $X_\varphi^* k_\lambda = \overline{\varphi(\lambda)} k_\lambda$, so that $\overline{\varphi(\lambda)}$ is an eigenvalue of X_φ^*, and therefore $|\varphi(\lambda)| \le \|X_\varphi\|$ for all $\lambda \in \mathbb{D}$, which is to say that φ is bounded on $\mathbb{D}$. Thus the multipliers of H^2 are precisely the elements of H^∞.

Pseudomultipliers

Definition 2. Let $\mathcal{H}$ be a Hilbert function space on a set Ω. We say that a function

$$\varphi : D_\varphi \subset \Omega \to \mathbb{C}$$

is a *pseudomultiplier* of $\mathcal{H}$ if

(1) D_φ is a set of uniqueness for $\mathcal{H}$;

(2) the subspace E_φ of $\mathcal{H}$, defined to be

$$\{h \in \mathcal{H} : \text{there exists } g \in \mathcal{H} \text{ such that } g(\lambda) = \varphi(\lambda)h(\lambda) \text{ for all } \lambda \in D_\varphi\}$$

is closed in $\mathcal{H}$, and

(3) E_φ has finite codimension in $\mathcal{H}$.

If $\mathcal{H}$ is a Hilbert function space on a set Ω then we say that a subset D of Ω is a *set of uniqueness for* $\mathcal{H}$ if, for any functions $f, g \in \mathcal{H}$, if $f(\lambda) = g(\lambda)$ for all $\lambda \in D$ then $f = g$.

The regular space E_φ and the singular space $\mathcal{S}_\varphi$ of a pseudomultiplier φ

The space E_φ will be called the *regular space of* φ, while the space $E_\varphi^\perp$ will be called the *singular space* of φ and will be denoted by $\mathcal{S}_\varphi$.

When conditions (1) to (3) hold we define the operator $X_\varphi : E_\varphi \to \mathcal{H}$ by $X_\varphi h = g$ where g is the element of $\mathcal{H}$ (necessarily unique, by condition (1)) such that $g(\lambda) = \varphi(\lambda)h(\lambda)$ for all $\lambda \in D_\varphi$.

The idea of a pseudomultiplier of the Hilbert function space H^2 came from papers of Adamyan, Arov and Krein [1] and their forerunner by Akhiezer [3]. One can construct meromorphic functions in the disc with a prescribed number of poles by the spectral analysis of Hankel operators on H^2. In this approach a function with singularities determines a multiplier not on the whole of H^2 but on a closed subspace of finite codimension.

Examples of Pseudomultipliers

Example 1. *A pseudomultiplier of H^2.* Consider the function $\varphi(z) = 1/z$, defined on the set $\mathbb{D} \setminus \{0\}$.

This function is clearly not a multiplier of H^2, since it is unbounded on $\mathbb{D}$, nor even defined on the whole of $\mathbb{D}$. On the other hand it is close to being a multiplier in the sense that there is a finite-codimensional closed subspace of H^2, namely zH^2, which is a closed subspace of codimension 1 in H^2 with the property that, for every $f \in zH^2$, φf is the restriction to $\mathbb{D} \setminus \{0\}$ of a function in H^2.

For the pseudomultiplier $\varphi(z) = 1/z$ on H^2 of Example 1, $D_\varphi = \mathbb{D} \setminus \{0\}$ and E_φ is the closed subspace zH^2 of H^2, which has codimension 1 in H^2. Thus $\mathcal{S}_\varphi = H^2 \ominus zH^2$, the subspace of constant functions in H^2. The operator $X_\varphi : zH^2 \to H^2$ is given by $X_\varphi z f = f$ for all $f \in H^2$.

Examples of Pseudomultipliers

Example 2. *A pseudomultiplier on a Sobolev space.* Let $\mathcal{H}$ be the space $W^{1,2}[0,1]$ of absolutely continuous functions on $[0,1]$ whose weak derivatives are in L^2 and with inner product given by

$$\langle f, g \rangle = \int_0^1 f(t)\overline{g(t)}dt + \int_0^1 f'(t)\overline{g'(t)}dt.$$

It is well known that $W^{1,2}[0,1]$ is a reproducing kernel Hilbert space, with reproducing kernel k given by

$$k(\lambda, \mu) = \begin{cases} \operatorname{cosech} 1 \cosh(1-\mu) \cosh\lambda & \text{if } 0 \le \lambda \le \mu \le 1, \\ \operatorname{cosech} 1 \cosh(1-\lambda) \cosh\mu & \text{if } 0 \le \mu \le \lambda \le 1. \end{cases} \tag{2}$$

One can check that $\chi(t) = \sqrt{t}$ is a pseudomultiplier of $W^{1,2}$ with $E_\chi = k_0^\perp = \{f \in W^{1,2} \text{ such that } f(0) = 0\}$ and $\mathcal{S}_\chi = \mathbb{C}k_0$. Here

$$k_0(\lambda) = k(\lambda, 0) \quad \text{for all } \lambda \in [0,1].$$

Consider a function $f \in k_0^\perp$, so that $f \in W^{1,2}[0,1]$ and $f(0) = 0$. We have to prove that the function χf is absolutely continuous and $(\chi f)'$ is in $L^2(0,1)$. Note that

$$(\chi f)'(t) = \frac{1}{2\sqrt{t}} f(t) + \sqrt{t} f'(t).$$

To prove that $(\chi f)'$ is in $L^2(0,1)$, it is enough to show that $\frac{1}{\sqrt{t}} f(t)$ is in $L^2(0,1)$. It can be done using Hardy's inequality. Therefore $(\chi f)' \in L^1(0,1)$, and so

$$(\chi f)(x) = \int_0^x (\chi f)'(t)\, dt$$

is absolutely continuous on $[0,1]$.

The operator $X_\chi : E_\chi \to W^{1,2}$ is given by $X_\chi f = \sqrt{t} f$ for all $f \in k_0^\perp$.

Seeing a vector

Let $\mathcal{H}$ be a Hilbert function space on Ω and fix $v \in \mathcal{H}$. We shall formalise the idea of enlarging the set Ω by adjoining an extra point $p \notin \Omega$ in such a way that the vector v is the reproducing kernel associated with p. Let $\tilde{\Omega} = \Omega \cup \{p\}$, and for $f \in \mathcal{H}$ define $f_v : \tilde{\Omega} \to \mathbb{C}$ by the formula

$$f_v(\lambda) = \begin{cases} f(\lambda) & \text{if} \quad \lambda \in \Omega \\ \\ \langle f, v \rangle & \text{if } \lambda = p \ . \end{cases}$$

Let $\mathcal{H}_v = \{f_v : f \in \mathcal{H}\}$. Since $f = 0$ in $\mathcal{H}$ implies that $f_v = 0$ in $\mathcal{H}_v$, we see that the formula

$$\langle f_v, g_v \rangle_{\mathcal{H}_v} \ = \ \langle f, g \rangle_{\mathcal{H}}$$

defines an inner product on $\mathcal{H}_v$. Furthermore, with this inner product, $\mathcal{H}_v$ is a Hilbert function space on $\tilde{\Omega}$, and the restriction map

$$\mathcal{H}_v \ni f_v \mapsto f_v|\Omega = f \in \mathcal{H}$$

is a Hilbert space isomorphism.

Seeing a vector

Now let $\mathcal{H}$ be a Hilbert function space, fix $v \in \mathcal{H}$, and let φ be a pseudomultiplier of $\mathcal{H}$. Since D_φ is a set of uniqueness for $\mathcal{H}$, D_φ is also a set of uniqueness for $\mathcal{H}_v$ and thus φ is also a pseudomultiplier of $\mathcal{H}_v$. Its domain when so regarded is still just D_φ and its regular space is unchanged. One can wonder, however, whether φ can be extended to a pseudomultiplier of $\mathcal{H}_v$ with domain containing $D_\varphi \cup \{p\}$.

Definition 3. Let $\mathcal{H}$ be a Hilbert function space on Ω, let $v \in \mathcal{H}$, let $c \in \mathbb{C}$, and let φ be a pseudomultiplier of $\mathcal{H}$. We say that φ *sees* v *with value* c if there exists a pseudomultiplier ψ of $\mathcal{H}_v$ such that ψ extends φ (that is, $D_\varphi \subseteq D_\psi$ and $\varphi = \psi|D_\varphi$), $p \in D_\psi, E_\varphi = E_\psi|\Omega$, and $\psi(p) = c$. We say φ *sees* v if there exists $c \in \mathbb{C}$ such that φ sees v with value c; in this case we also say that v is *visible* to φ.

Examples - Seeing a vector

Example 3. Let $\mathcal{H}$ be a Hilbert function space on Ω and let φ be a pseudomultiplier of $\mathcal{H}$. For any $c \in \mathbb{C}, \varphi$ sees the zero vector in $\mathcal{H}$ with value c.

Example 4. Let $\mathcal{H}$ be a Hilbert function space on Ω and let φ be a pseudomultiplier of $\mathcal{H}$. If $\lambda \in D_\varphi$, then φ sees k_λ with value $\varphi(\lambda)$.

Example 5. *What vectors does $1/z$ see in H^2?* Consider the pseudomultiplier φ of Example 1: $\varphi(z) = 1/z$ on H^2. Here, $D_\varphi = \mathbb{D} \setminus \{0\}$ and $E_\varphi = zH^2$. One can show that the vectors in H^2 visible to φ are precisely the scalar multiples of the kernels k_λ, for $\lambda \in \mathbb{D} \setminus \{0\}$.

Ambiguous vector and the ambiguous space for a pseudomultiplier

Definition 4. Let φ be a pseudomultiplier of $\mathcal{H}$ and let $v \in \mathcal{H}$. We say that *v is an ambiguous vector for* φ if φ sees v with *arbitrary value*, and we define $\mathcal{A}_\varphi$ to be the set of ambiguous vectors for φ.

Observe also that Example 3 immediately implies that, for any Hilbert function space $\mathcal{H}$, the zero vector of $\mathcal{H}$ is an ambiguous vector of every pseudomultiplier of $\mathcal{H}$.

Example 6. *$\varphi(z) = 1/z$ on H^2 has no non-zero ambiguities.* The pseudomultiplier $\varphi(z) = 1/z$ of H^2 sees only the scalar multiples of kernels k_λ for some $\lambda \in \mathbb{D} \setminus \{0\}$. Moreover, for such λ, φ sees k_λ only with value $\varphi(\lambda)$. Thus φ does not see any non-zero vector with arbitrary value, which is to say that φ has no ambiguous vectors other than 0.

Example 7. Consider the pseudomultiplier $\chi(t) = \sqrt{t}$ on $W^{1,2}[0,1]$ of Example 2. One can show that $\mathcal{A}_\chi = \mathbb{C}k_0$.

An operator-theoretic expression of the concepts of seeing a vector and ambiguity

Recall that if φ is a pseudomultiplier then X_φ is a bounded linear transformation from E_φ into $\mathcal{H}$. Thus X_φ^* is a bounded linear transformation from $\mathcal{H}$ into E_φ. Recall also a basic fact in the theory of multipliers that if φ is a multiplier of $\mathcal{H}$ and k_λ is the reproducing kernel for $\lambda \in \Omega$, then $X_\varphi^* k_\lambda = \overline{\varphi(\lambda)} k_\lambda$. We generalize this fact to pseudomultipliers in the following lemma.

Lemma 1. *Let φ be a pseudomultiplier of $\mathcal{H}$ and let $v \in \mathcal{H}$. Then*
(i) *φ sees v with value c if and only if there exists $u \in E_\varphi^\perp$ such that*

$$X_\varphi^* v = \bar{c} v + u. \tag{3}$$

(ii) *φ sees v with value c if and only if*

$$X_\varphi^* v = \bar{c} P_{E_\varphi} v, \tag{4}$$

where P_{E_φ} is the orthogonal projection of $\mathcal{H}$ onto E_φ.
(iii) *If φ sees v with two distinct values then v is an ambiguous vector for φ.*

An operator-theoretic expression of the concept of the ambiguous space

As a further corollary of Lemma 1 we obtain

Proposition 1. *Let φ be a pseudomultiplier and let $v \in \mathcal{H}$. Then*

(i) v *is an ambiguous vector for* φ *if and only if* $v \in E_\varphi^\perp \cap \ker X_\varphi^*$.

(ii)

$$\mathcal{A}_\varphi = E_\varphi^\perp \cap \ker X_\varphi^* \tag{5}$$

is a finite-dimensional subspace of $\mathcal{H}$.

Definable vectors and the polar space

In addition to singularities that arise from ambiguous vectors, pseudomultipliers can possess another type of singular vector, which we call a *polar vector*. We say that a vector $d \in \mathcal{H}$ is $\textit{definable for } \varphi$ if $d \neq 0, \varphi$ sees d, and $d \perp \mathcal{A}_\varphi$. We denote by $\mathbf{D}_\varphi$ the set of vectors that are definable for φ. For any definable vector v for a pseudomultiplier φ, there is a unique $c \in \mathbb{C}$ such that φ sees v with value c. We denote this number c by $\varphi(v)$. Thus we can regard φ as a function on $\mathbf{D}_\varphi$.

Example 8. *The definable vectors for* $1/z$ *on* H^2*.* By Example 6, for the pseudomultiplier $\varphi(z) = 1/z$ of H^2, $\mathcal{A}_\varphi = \{0\}$, and so the definable vectors for φ coincide with the non-zero vectors visible to φ. Hence, by Example 5, $\mathbf{D}_\varphi$ comprises the non-zero scalar multiples of the kernels k_λ, for $\lambda \in \mathbb{D} \setminus \{0\}$.

Polar vectors and the polar space

Definition 5. Let φ be a pseudomultiplier and let $p \in \mathcal{H}$. We say p is a *polar vector for* φ if $p \neq 0$ and there exists a sequence of definable vectors $\{d_n\} \subseteq \mathbf{D}_\varphi$ such that $d_n \to p$ (with respect to $\|\cdot\|_{\mathcal{H}}$) and $\varphi(d_n) \to \infty$ as $n \to \infty$. We define $\mathcal{P}_\varphi$, the *polar space of* φ, to be the set of vectors $v \in \mathcal{H}$ such that either v is a polar vector of φ or $v = 0$.

Example 9. *The polar space of* $1/z$. In Example 8 we proved that the set of definable vectors $\mathbf{D}_\varphi$ for $\varphi(z) = 1/z$ on H^2 comprises the scalar multiples of the kernels k_λ, for $\lambda \in \mathbb{D}\backslash\{0\}$. Recall that the singular space $\mathcal{S}_\varphi = E_\varphi^\perp = H^2 \ominus zH^2$. Let us show that

$$\mathcal{P}_\varphi = \mathcal{S}_\varphi = E_\varphi^\perp = H^2 \ominus zH^2 = \{\text{constant functions on } \mathbb{D}\}.$$

It is enough to show that k_0 is in $\mathcal{P}_\varphi$. Choose $\lambda_n \in \mathbb{D}$, $n \geq 1$, such that $\lambda_n \to 0$. By Example 4, φ sees k_{λ_n} with value $\frac{1}{\lambda_n}$. Since $k_{\lambda_n} \to k_\mu$ in H^2 if and only if $\lambda_n \to \mu$ in the usual topology of $\mathbb{D}$, $\{k_{\lambda_n}\}$ is a sequence of definable vectors such that $k_{\lambda_n} \to k_0$ in H^2 and $\varphi(k_{\lambda_n}) = \varphi(\lambda_n) = \frac{1}{\lambda_n} \to \infty$. This implies that $k_0 \in \mathcal{P}_\varphi$. $\square$

Another example of the psedomultiplier on H^2

Example 10. Let $\mathcal{H} = H^2$, the classical Hardy space on $\mathbb{D}$, let $n \geq 2$, and define φ on $\mathbb{D}$ by

$$\varphi(\lambda) = \begin{cases} \frac{1}{\lambda^n} & \text{if } \lambda \neq 0 \\ 1 & \text{if } \lambda = 0. \end{cases}$$

Here

$$E_\varphi = \{f \in H^2 : f(0) = f'(0) = \cdots = f^{(n)}(0) = 0\} = z^{n+1}H^2.$$

Let us describe the space of ambiguous vectors $\mathcal{A}_\varphi$ for φ. By Proposition 1,

$$\mathcal{A}_\varphi = E_\varphi^\perp \cap \ker X_\varphi^*.$$

The operator $X_\varphi : z^{n+1}H^2 \to H^2$ is given by $X_\varphi = (S^*)^n P^*_{z^{n+1}H^2}$ where S denotes the forward shift operator on H^2 and $P_{z^{n+1}H^2} : H^2 \to z^{n+1}H^2$ is the orthogonal projection operator (so that $P^*_{z^{n+1}H^2}$ is the injection operator

$z^{n+1}H^2 \to H^2$). Note that

$$X_\varphi^* f = 0 \quad \Leftrightarrow \quad S^n f \in H^2 \ominus z^{n+1}H^2$$
$$\Leftrightarrow \quad f \text{ is constant.}$$

Hence, the space of ambiguous vectors for φ is

$$\mathcal{A}_\varphi = E_\varphi^\perp \cap \ker X_\varphi^* = (H^2 \ominus z^{n+1}H^2) \cap \mathbb{C}\mathbf{1} = \mathbb{C}\mathbf{1}.$$

Let us describe the set of definable vectors $\mathbf{D}_\varphi$ of φ. By Lemma 1, for $f \in H^2$, φ sees f with value c if and only if there exists $u \in E_\varphi^\perp$ such that

$$X_\varphi^* f = \bar{c} f + u, \tag{6}$$

equivalently, if and only if there exist $a_0, a_1, \ldots, a_n \in \mathbb{C}$ such that

$$X_\varphi^* f = \bar{c} f + a_0 + a_1 z + \cdots + a_n z^n.$$

One can show that

$$\mathbf{D}_\varphi = \left\{ f(z) = \frac{b_1 z + b_2 z^2 + \cdots + b_n z^n}{1 - \bar{c}^{-1} z^n} : |c| > 1 \text{ and } b_1, \ldots, b_n \in \mathbb{C} \text{ are not all zero} \right\}. \tag{7}$$

Now we can describe the polar space $\mathcal{P}_\varphi$ for φ. For every non-zero polar vector p there exists a convergent sequence of definable vectors

$$\left\{ d_k = \frac{b_{1k} z + b_{2k} z^2 + \cdots + b_{nk} z^n}{1 - \bar{c_k}^{-1} z^n} \right\}_{k=1}^{\infty} \text{ in } \mathbf{D}_\varphi,$$

where $b_{1k}, b_{2k}, \ldots, b_{nk} \in \mathbb{C}$, not all zero, are such that $d_k \to p$ and $\varphi(d_k) = c_k \to \infty$ as $k \to \infty$. The non-zero limits of such convergent sequences $\{d_k\}_{k=1}^{\infty}$ in H^2 coincide with $\operatorname{span}\{z, z^2, \ldots, z^n\} \setminus \{0\}$, so that $\mathcal{P}_\varphi = \operatorname{span}\{z, z^2, \ldots, z^n\}$. We may observe that

$$\mathcal{A}_\varphi \oplus \mathcal{P}_\varphi = \mathbb{C}\mathbf{1} \oplus \operatorname{span}\{z, z^2, \ldots, z^n\} = H^2 \ominus z^{n+1} H^2 = E_\varphi^{\perp}.$$

$\square$

Polar vectors

The nature of polar vectors is much more rigid than one might at first expect. This is brought out in the following proposition, which gives a clean relationship between $\mathbf{D}_\varphi$ and the set $\mathcal{P}_\varphi$ of polar vectors of φ.

Proposition 2. *Let φ be a pseudomultiplier and let $p \in \mathcal{H}$ with $p \neq 0$. The following are equivalent.*

(i) p *is a polar vector for* φ*.*

(ii) $p \in \mathbf{D}_\varphi^-$ *and for every neighborhood* U *of* p, φ *is unbounded on* $U \cap \mathbf{D}_\varphi$*.*

(iii) $p \in \mathbf{D}_\varphi^- \setminus \mathbf{D}_\varphi$*.*

(iv) $p \in \mathbf{D}_\varphi^-$ *and* $\lim\limits_{\substack{d \to p \\ d \in \mathbf{D}_\varphi}} \varphi(d) = \infty$*.*

Polar vectors

We say that a pseudomultiplier φ is *locally bounded on* $\mathbf{D}_\varphi^-$ if, for every point $d \in \mathbf{D}_\varphi^-$ there is a neighbourhood U of d in $\mathcal{H}$ such that φ is bounded on $U \cap \mathbf{D}_\varphi$.

Theorem 1. *Let φ be a pseudomultiplier of $\mathcal{H}$. Then φ is locally bounded on $\mathbf{D}_\varphi^-$ if and only if φ has no polar vectors.*

We also establish an alternative description of the polar space in terms of definable vectors.

Theorem 2. *If $\mathcal{H}$ is infinite-dimensional and φ is a pseudomultiplier of $\mathcal{H}$ then*

(i) $\mathbf{D}_\varphi$ *is non-empty.*

(ii) $\mathcal{P}_\varphi = \overline{\mathbf{D}_\varphi} \setminus \mathbf{D}_\varphi$.

Decomposition of the singular space

We prove that every singular vector of a pseudomultiplier can be represented in a unique way as the sum of a polar vector and an ambiguous vector.

Theorem 3. *Let $\mathcal{H}$ be a Hilbert function space on a set Ω. Let φ be a pseudomultiplier of $\mathcal{H}$. $\mathcal{P}_\varphi$ is a closed subspace of $\mathcal{H}$ and*

$$\mathcal{S}_\varphi = \mathcal{P}_\varphi \oplus \mathcal{A}_\varphi.$$

This theorem contains the very interesting fact that the polar space $\mathcal{P}_\varphi$, which is the set of polar vectors of a pseudomultiplier φ, together with the zero vector, constitutes a linear subspace of $\mathcal{H}$.

Decomposition of the singular space - Examples

Example 11. The pseudomultiplier $\varphi(z) = \frac{1}{z(z-\frac{1}{2})}$ on H^2 has no ambiguities $\mathcal{A}_\varphi = \{0\}$, and the polar space of φ

$$\mathcal{P}_\varphi = \operatorname{span}\{k_0, k_{\frac{1}{2}}\} = H^2 \ominus z(z - \tfrac{1}{2})H^2 = E_\varphi^\perp.$$

Example 12. If $\chi(t) = \sqrt{t}$ on $W^{1,2}[0,1]$ is the pseudomultiplier of Example 2, then, as Example 7 implies, the ambiguous space $\mathcal{A}_\chi = \mathbb{C}k_0 = \mathcal{S}_\chi$, and so we deduce that $\mathcal{P}_\chi = \{0\}$. Thus χ has no polar vectors.

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Thank you