Insights into the *Invariant Subspace Problem* for compact perturbations of normal operators

Eva A. Gallardo Gutiérrez

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- **An intrinsic difficulty:** The lack of well-known examples (Halmos)

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- $\{e_n\}_{n\geq 1}$ canonical bases in ℓ^2

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Classical Beurling Theory:

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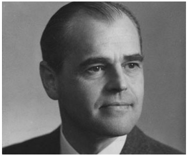

Arne Beurling (1905-1986)

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  - ⋆ 1966, Bernstein y Robinson (Hilbert spaces).
  - ⋆ 1967, Halmos.
  - ⋆ 1960's Gillespie, Hsu, Kitano, Pearcy, . . .

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Let T be a linear bounded operator on $\mathcal{H}$, $T \neq \mathbb{C}Id$. If T commutes with a non-zero compact operator, then T has a non-trivial closed invariant subspace.

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Theorem (Hadwin, Nordgren, Radjavi, Rosenthal; 1980)

There exists a "quasi-analytic" shift S on a weighted ℓ^2 space which has the following property: if K is a compact operator which commutes with a nonzero, non scalar operator in the commutant of S, then $K = 0$.

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Invariant subspace problem: current status

Invariant subspace problem

*Given any linear bounded operator T acting on a separable infinite-dimensional **reflexive** complex Banach space, does there exist a non-trivial closed invariant subspace?*

An attempt to find a examples: quasitriangular operators

Based on the work of Aronszajn and Smith (1954), Halmos (1968) introduced the concept of **quasitriangular operators**.

Definition (Halmos, 1968)

An operator $Q : H \to H$ acting on a separable infinite-dimensional complex Hilbert space is said to be **quasitriangular** whenever there exists an increasing sequence $(P_n)_{n\in\mathbb{N}}$ of finite-rank projections converging strongly to the identity I and such that

$$\|QP_n - P_nQP_n\| \to 0, \quad \text{as } n \to \infty.$$

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- Note that, given a triangular operator $T : H \to H$, there exists an increasing sequence $(P_n)_{n\in\mathbb{N}}$ of finite-rank projections converging strongly to the identity I and satisfying

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- **An example of non-quasitriangular operator:** Shift operator acting on $\ell^2(\mathbb{N})$.

Quasitriangularity and invariant subspaces

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- **Initial goal:** Understand **quasitriangular operators** from the standpoint of view of invariant subspaces.

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Question

It is still unknown if every rank-one perturbation of a diagonal operator ($T = D + u \otimes v$)*, has non-trivial invariant subspaces (problem explicitly posed by Pearcy in 1979).*

An old problem

- The study of the existence of nontrivial closed invariant subspaces for the perturbation of a Hermitian (self-adjoint) operator A by a compact operator of a Schatten class $\mathcal{C}_p$, $1 \leq p < \infty$ (1960's)

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Theorem (Radjabalipour and Radjavi, 1975)

Let $T = N + K$ be a bounded linear operator in a complex Hilbert space, where N is a normal operator with spectrum on a C^2 Jordan curve γ and K a compact operator belonging to a Schatten class $\mathcal{C}_p$ for $1 \leq p < \infty$. Then T is ***decomposable*** *if and only if $\sigma(T)$ does not fill the interior of γ.*

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- The situation turns out to be drastically different if the assumption on the spectra being contained in a curve is dropped off since, in such a case, it is still an open question if every compact perturbation of a normal operator has non-trivial closed invariant subspaces. Even, in particular, it is still open if every rank-one perturbation of a normal operator whose eigenvectors span the Hilbert space H has non-trivial closed invariant subspaces.

Question

It is still unknown if every rank-one perturbation of a diagonal operator ($T = D + u \otimes v$)*, has non-trivial invariant subspaces (problem explicitly posed by Pearcy in 1979).*

If $D_\Lambda \in \mathcal{L}(H)$ is a diagonal operator, that is, there exists an orthonormal basis $(e_n)_{n\geq 1}$ of H and a bounded sequence of complex numbers $\Lambda = (\lambda_n)_{n\geq 1} \subset \mathbb{C}$ such that

$$D_\Lambda e_n = \lambda_n e_n,$$

a rank-one perturbations of D_Λ can be written as

$$T = D_\Lambda + u \otimes v, \tag{1}$$

where u, and v are non-zero vectors in H and $u \otimes v(x) = \langle x, v \rangle \, u$ for every $x \in H$.

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Remark

Rank-one perturbations of normal operators whose eigenvectors span H belongs are unitarily equivalent to those expressed by (1).

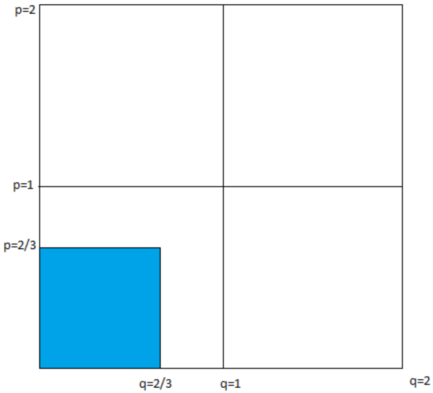
Theorem (Foias, Ko, Jung and Pearcy, JFA 2007)

Let $T = D_\Lambda + u \otimes v$ *in* $\mathcal{L}(H) \setminus \mathbb{C}I$ *where* $u = \sum_{n=1}^{\infty} \alpha_n e_n$, $v = \sum_{n=1}^{\infty} \beta_n e_n$ *and*

$$\sum_{n=1}^{\infty} |\alpha_n|^{2/3} + |\beta_n|^{2/3} < \infty.$$

Then, T has non-trivial hyperinvariant subspaces.

Note that if $\{\alpha_n\} \in \ell^p$ and $\{\beta_n\} \in \ell^q$, Foias, Jung, Ko and Pearcy Theorem can be "seen":



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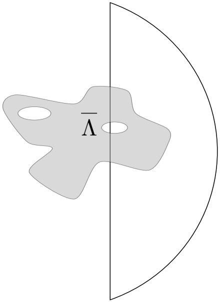

Figure: Spectrum of T.

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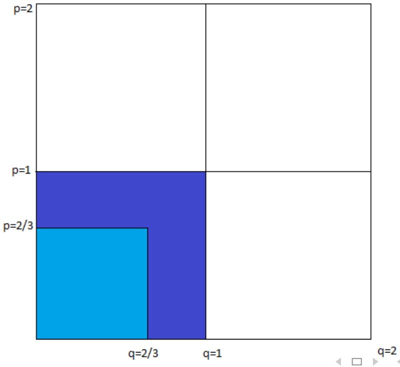

• The authors show the decomposability for a subclass of the rank-one perturbations that satisfy the summability assumption.

• An operator $T \in \mathcal{L}(H)$ is **decomposable** if for every open cover $U_1, U_2 \subset \mathbb{C}$ such that $\sigma(T) \subset U_1 \cup U_2$ there exists invariant subspaces M, N for T such that $H = M + N$ and $\sigma(T\mid_M) \subset U_1$ and $\sigma(T\mid_N) \subset U_2$.

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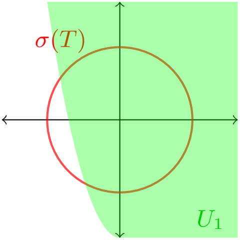

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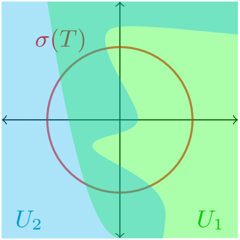

Figure: A decomposable operator.

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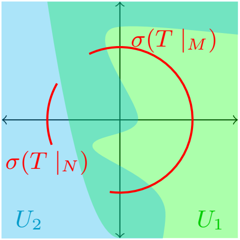

Figure: A decomposable operator.

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Let $T = D_\Lambda + u \otimes v \in \mathcal{L}(H) \setminus \mathbb{C}\, Id_H$ *be any rank-one perturbation of a diagonal normal operator respect to an orthonormal basis* $(e_n)_{n\geq 1}$ *where* $u = \sum_{n=1}^{\infty} \alpha_n e_n$ *and* $v = \sum_{n=1}^{\infty} \beta_n e_n$. *If either*

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then T *has non-trivial closed hyperinvariant subspaces.*

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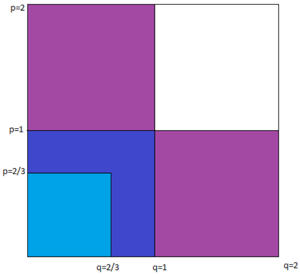

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If T *satisfies any of the previous conditions,* T *has a non-trivial closed hyperinvariant subspace.*

The class $(\mathcal{RO})$

Definition (Class $(\mathcal{RO})$)

Fixed an orthonormal basis $\mathcal{E} = (e_n)_{n\geq 1}$ of H and consider a bounded sequence of complex numbers $\Lambda = (\lambda_n)_{n\geq 1} \subset \mathbb{C}$. If D_Λ denotes the diagonal operator associated to Λ respect to $\mathcal{E}$, the rank-one perturbation of D_Λ

$$T = D_\Lambda + u \otimes v$$

with $u = \sum_{n=1}^{\infty} \alpha_n e_n$, $v = \sum_{n=1}^{\infty} \beta_n e_n$ nonzero vectors in H, belongs to the class $(\mathcal{RO})$ if:

(i) $\alpha_n \beta_n \neq 0$ *for every* $n \in \mathbb{N}$;

(ii) *the map* $n \in \mathbb{N} \mapsto \lambda_n \in \Lambda$ *is injective;*

(iii) *the derived set* Λ' *is not a singleton.*

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$$\sum_{n=1}^{\infty} \frac{c_n}{z_n - z} \equiv 0$$

whenever $|z| > \sup |z_n|$ for some non-trivial $\{c_n\} \in \ell^1$ if and only if there exists a closed invariant subspace for the diagonal operator D having eigenvalues $\{z_n\}$ which is not invariant for the adjoint D^*.

Spectrum $\sigma(T)$ and point spectrum $\sigma_p(T)$ of operators $T \in (\mathcal{RO})$

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Theorem (Ionascu, 2001)

Let $T = D_\Lambda + u \otimes v \in (\mathcal{RO})$. Then $z \in \mathbb{C}$ belongs to $\sigma_p(T)$ if and only if

(i) $z \notin \Lambda$,

(ii) $\sum_{n=1}^{\infty} \frac{|\alpha_n|^2}{|z-\lambda_n|^2} < \infty$,

(iii) $f_T(z) + 1 = 0$.

Moreover,

$$\sigma(T) = \Lambda' \cup \left\{ z \in \mathbb{C} \setminus \overline{\Lambda} : f_T(z) + 1 = 0 \right\},$$

and the essential spectrum

$$\sigma_{ess}(T) = \Lambda'.$$

Rank-One Perturbations Of Diagonal Operators

Theorem (GG,González-Doña, 2021)

Let $T = D_\Lambda + u \otimes v \in \mathcal{L}(H) \setminus \mathbb{C}\, Id_H$ *be any rank-one perturbation of a diagonal normal operator respect to an orthonormal basis* $(e_n)_{n\geq 1}$ *where* $u = \sum_{n=1}^{\infty} \alpha_n e_n$ *and* $v = \sum_{n=1}^{\infty} \beta_n e_n$. *If either*

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then T has non-trivial closed hyperinvariant subspaces. Moreover, for those $T \in (\mathcal{RO})$ with $\sigma(T)$ connected and $\sigma_p(T) \cup \sigma_p(T^*) = \emptyset$, it follows that they do have non-zero spectral subspaces which are no longer dense.

A closer look to the invariant subspaces: local spectral subspaces

Local spectral theory and local spectral manifolds

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Recall that a linear bounded operator T on a Banach space X has the **single-valued extension property** (SVEP) if for every connected open set $G \subset \mathbb{C}$ and every analytic function $f : G \to X$ such that

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- For every operator $T \in \mathcal{L}(X)$ and $x \in X$, the local spectrum $\sigma_T(x)$ is a compact subset of $\sigma(T)$.

Definition (Local spectral manifold)

Given an operator $T \in \mathcal{L}(X)$ and any subset $\Omega \subseteq \mathbb{C}$, the **local spectral manifold** is defined by

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- $X_T(\Omega)$ is always a (non-necessarily closed...) T-hyperinvariant linear manifold!!
- Those operators $T \in \mathcal{B}(X)$ such that $X_T(\Omega)$ is norm-closed for every closed subset $\Omega \subseteq \mathbb{C}$ are said to satisfy **Dunford property** (**C**).

SVEP for operators in $(\mathcal{RO})$

Proposition (GG, González-Doña, 2021)

Let $T = D_\Lambda + u \otimes v \in (\mathcal{RO})$, *where* $u = \sum_{n=1}^{\infty} \alpha_n e_n$, $v = \sum_{n=1}^{\infty} \beta_n e_n \in H$. *The following conditions are equivalent:*

(i) T *has the SVEP.*

(ii) $\sigma_p(T)$ *does not fill any hole of* $\overline{\Lambda}$.

(iii) $f_T + 1$ *is not constantly* 0 *on any hole of* $\overline{\Lambda}$.

Strategy: characterizing *particular* spectral subspaces

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Given $T = D_\Lambda + u \otimes v \in (\mathcal{RO})$ where $\Lambda = (\lambda_n) \subset \mathbb{C}$ and provided any set $A \subset \mathbb{C}$, we will denote by N_A the set of positive integers:

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Given an open set U, a holomorphic map g on U and $w \in U$, we define

$$\Gamma(g)(z,w) = \begin{cases} \frac{g(z)-g(w)}{z-w} & z \neq w \\ g'(w) & z = w \end{cases}$$

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$\Gamma(g)(z,w)$ is continuous in $U \times U$ and for every $w \in U$, the map $z \mapsto \Gamma(g)(z,w)$ is, indeed, holomorphic in U.

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(i) *If $x = \sum_n x_n e_n$, then*

$$x_n = g_x(\lambda_n)\alpha_n$$

for every $n \in N_{F^c}$.

(ii) *The function*

$$z \in F^c \mapsto \sum_{n \in N_{F^c}} \Gamma(g_x)(z, \lambda_n)\alpha_n e_n$$

is a vector-valued holomorphic function on F^c.

(iii) *The identity*

$$\sum_{n \in N_F} \frac{x_n \overline{\beta_n}}{\lambda_n - z} = g_x(z) \left(\sum_{n \in N_F} \frac{\alpha_n \overline{\beta_n}}{\lambda_n - z} + 1 \right) - \sum_{n \in N_{F^c}} \Gamma(g_x)(z, \lambda_n)\alpha_n \overline{\beta_n},$$

holds for every $z \in F^c$.

A few remarks

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Example

Observe that for $u = \sum_{n=1}^{\infty} \alpha_n e_n$ and $v = \sum_{n=1}^{\infty} \beta_n e_n$,

$$g_u(z) = \frac{f_T(z)}{f_T(z)+1} \quad \text{and} \quad g_v(z) = \frac{1}{f_T(z)+1} \sum_{n=1}^{\infty} \frac{|\beta_n|^2}{\lambda_n - z}$$

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for every $z \in \rho(T)$.

Theorem

Let $T = D_\Lambda + u \otimes v \in (\mathcal{RO})$ with $u = \sum_{n=1}^{\infty} \alpha_n e_n$ and $v = \sum_{n=1}^{\infty} \beta_n e_n$ nonzero vectors in H. Assume $\sigma(T)$ is connected and both $\sigma_p(T)$ and $\sigma_p(T^*)$ are empty. Let F be a non-empty closed set contained in $\sigma(T)$. Then the vector $u \in H_T(F)$ if and only if $F = \sigma(T)$.

A few remarks

Corollary

Let $T = D_\Lambda + u \otimes v \in (\mathcal{RO})$ *with* $u = \sum_{n=1}^{\infty} \alpha_n e_n$ *and* $v = \sum_{n=1}^{\infty} \beta_n e_n$ *nonzero vectors in* H*. Assume* $\sigma(T)$ *is connected and both* $\sigma_p(T)$ *and* $\sigma_p(T^*)$ *are empty. Then*

$$\limsup_{n\to\infty} ||(D_\Lambda + u \otimes v)^n u||^{1/n} = \max\{|z| : z \in \Lambda'\} = r(T), \tag{2}$$

and, analogously,

$$\limsup_{n\to\infty} ||(D_{\Lambda^*} + v \otimes u)^n v||^{1/n} = \max\{|z| : z \in \Lambda'\} = r(T^*). \tag{3}$$

Strategy: constructing spectral subspaces

Strategy: constructing spectral subspaces

Theorem (GG, González-Doña, 2021)

Let $T = D_\Lambda + u \otimes v \in (\mathcal{RO})$ with $u = \sum_{n=1}^{\infty} \alpha_n e_n$ and $v = \sum_{n=1}^{\infty} \beta_n e_n$ nonzero vectors in H. Assume $\sigma(T)$ is connected and both $\sigma_p(T)$ and $\sigma_p(T^*)$ are empty. Assume that there exists a closed, simple, piecewise differentiable curve γ in $\mathbb{C}$ not intersecting Λ such that

(i) $\sigma(T) \cap \operatorname{int}(\gamma) \neq \emptyset$.

(ii) *The map*

$$\xi \in \gamma \to \frac{1}{1 + f_T(\xi)}$$

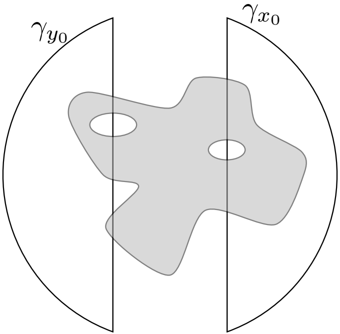
is well defined and continuous on γ.

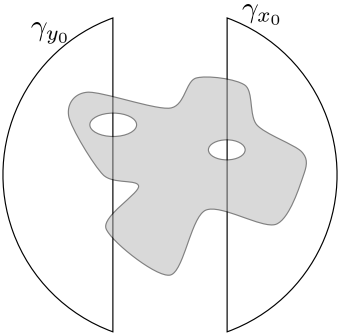
(iii)

$$\sum_{n=1}^{\infty} \left(\int_\gamma \frac{d|\xi|}{|\lambda_n - \xi|} \right)^2 |\alpha_n|^2 < \infty.$$

Then, $H_T(\overline{\operatorname{int}(\gamma)})$ is a non-zero spectral subspace.

Strategy: constructing spectral subspaces



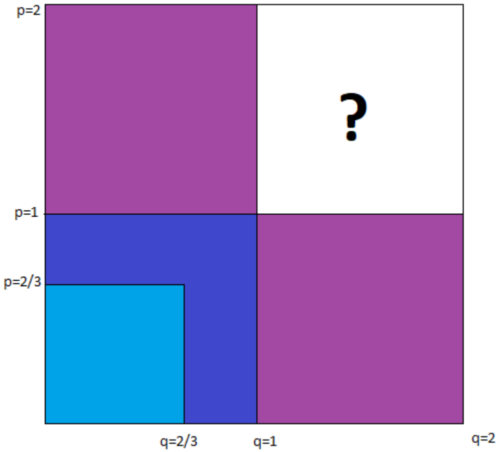


Final step: If both T and T^* enjoy the SVEP, and $F_1, F_2 \subset \mathbb{C}$ are disjoint closed sets, then

$$H_T(F_1) \subseteq H_{T^*}(F_2^*)^{\perp},$$

Question

Question



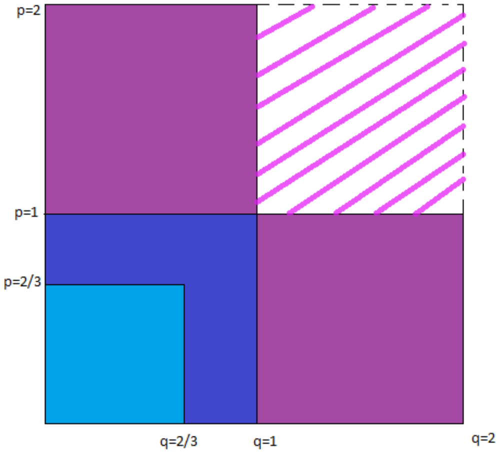
Theorem (GG,González-Doña, 2022)

With the notation as introduced above, the linear bounded operator $T = D_\Lambda + u \otimes v$ has non trivial closed invariant subspaces provided that either u or v have a Fourier coefficient which is zero or u and v have non zero Fourier coefficients and

$$\sum_{n=1}^{\infty} |\alpha_n|^2 \log \frac{1}{|\alpha_n|} + |\beta_n|^2 \log \frac{1}{|\beta_n|} < \infty. \tag{4}$$

Moreover, if T is not a scalar multiple of the identity, it has non trivial closed hyperinvariant subspaces.

Rank-One Perturbations Of Diagonal Operators: a step further



Let $(e_n)_{n\geq 1}$ be an orthonormal basis in H and $u_1, \cdots, u_N, v_1, \cdots v_N$ non-zero vectors in H. Let us we denote their Fourier coefficients by

$$u_k = \sum_{n=1}^{\infty} \alpha_n^{(k)} e_n, \qquad v_k = \sum_{n=1}^{\infty} \beta_n^{(k)} e_n$$

for each $1 \leq k \leq N$.

Finite Rank Perturbations Of Diagonal Operators

Let $(e_n)_{n\geq 1}$ be an orthonormal basis in H and $u_1, \cdots, u_N, v_1, \cdots v_N$ non-zero vectors in H. Let us we denote their Fourier coefficients by

$$u_k = \sum_{n=1}^{\infty} \alpha_n^{(k)} e_n, \qquad v_k = \sum_{n=1}^{\infty} \beta_n^{(k)} e_n$$

for each $1 \leq k \leq N$.

We consider operators that can be expressed by

$$T = D_\Lambda + \sum_{k=1}^{N} u_k \otimes v_k \in \mathcal{L}(H),$$

where D_Λ is a diagonal operator with respect to $(e_n)_{n\geq 1}$ with eigenvalues $\Lambda = (\lambda_n)_n$ and $N \in \mathbb{N}$ fixed.

Theorem (GG, González-Doña, 2022)

Let $T = D_\Lambda + \sum_{k=1}^{\infty} u_k \otimes v_k \in \mathcal{L}(H) \setminus \mathbb{C}\, Id_H$ *be any finite rank perturbation of a diagonal normal operator* D_Λ *with respect to an orthonormal basis* $\mathcal{E} = \{e_n\}_{n \geq 1}$ *where* $u_k = \sum_{n=1}^{\infty} \alpha_n^{(k)} e_n$ *and* $v_k = \sum_{n=1}^{\infty} \beta_n^{(k)} e_n$ *are non zero vectors in* H*. Then* T *has non trivial closed hyperinvariant subspaces provided that*

$$\sum_{n \in \mathcal{N}} \left|\alpha_n^{(k)}\right|^2 \log \frac{1}{\left|\alpha_n^{(k)}\right|} + \left|\beta_n^{(k)}\right|^2 \log \frac{1}{\left|\beta_n^{(k)}\right|} < \infty,$$

where

$$\mathcal{N} = \{n \in \mathbb{N} : \ \alpha_n^{(k)} \neq 0, \beta_n^{(k)} \neq 0 \ \textit{for}\ 1 \leq k \leq N\}.$$

Invariant Subspace Problem

Question

Given any linear bounded operator T acting on a separable infinite-dimensional Hilbert space (or reflexive Banach space), does there exist a non-trivial closed invariant subspace?

- **An intrinsic difficulty:** The lack of well-known examples

Thank you for your attention

References

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Thank you for your attention