Ideals in the algebra of operators on Baernstein's space

James Smith

Lancaster University, UK

Lancaster University

Summary

Introduction

(i) Let X be a Banach space over $\mathbb{R}$ or $\mathbb{C}$. A sequence $(b_n)_{n\in\mathbb{N}}$ is a *basis* if, for each $x \in X$, there exist unique constants $\alpha_n \in \mathbb{R}$ (or $\mathbb{C}$) such that $x = \sum_{n\in\mathbb{N}} \alpha_n b_n$.

(ii) A basis is *unconditional* if $x = \sum_{n\in\mathbb{N}} \alpha_{\pi(n)} b_{\pi(n)}$ where π is any permutation of $\mathbb{N}$.

(iii) Two bases $(b_n), (d_n)$ are *equivalent* if convergence of $\sum_n \alpha_n b_n$ is equivalent to convergence of $\sum_n \alpha_n d_n$.

Consider the Banach algebra:

$$\mathscr{B}(X) = \{T : X \to X, T \text{ is bounded and linear}\}$$

Objective: to understand the lattice of closed, (two-sided) ideals of $\mathscr{B}(X)$.

General Ideal Structure

If $X \neq \{0\}$, $\mathscr{F}(X) = \{T \in \mathscr{B}(X) : \text{dim}(TX) < \infty\}$ is the smallest non-zero ideal of $\mathscr{B}(X)$.

$\mathscr{F}(X) \subseteq \mathscr{K}(X) = \{T \in \mathscr{B}(X) : T(B_1(0))$ is relatively compact$\}$, which is also an ideal.

If X has a basis, then $\overline{\mathscr{F}(X)} = \mathscr{K}(X)$.

Some classical operator ideal results

Theorem

(Calkin, Ann. of Math. 1941). Let H *be a separable Hilbert space with dim*$(H) = \infty$*. Then* $\mathscr{B}(H)$ *has a single non-trivial closed ideal:*

$$(0) \subsetneq \mathscr{K}(H) \subsetneq \mathscr{B}(H).$$

Theorem

(Gohberg-Markus-Feldman, Bul. Akad. Štiince RSS Moldoven 1960). Let $X = c_0$ *or* $X = l_p$*, where* $1 \leq p < \infty$*. Then* $\mathscr{B}(X)$ *has a single non-trivial closed ideal:*

$$(0) \subsetneq \mathscr{K}(X) \subsetneq \mathscr{B}(X).$$

Schreier space

Schreier space origin: Banach and Saks proved that every weakly convergent sequence in $L_p([0,1])$, there exists a subsequence whose arithmetic means converge in norm. Schreier proved in 1930, this property fails to hold in $C[0,1]$. A product of this investigation was the Schreier space.

We say a finite subset $M \subset \mathbb{N}$ is **admissible** if $|M| \leq \min M$. Let S_1 be the collection of admissible sets. For $(x_n) \in c_{00}$:

$$||(x_n)||_{S_1} := \sup_{M \in S_1} \sum_{m \in M} |x_m|.$$

Then the Schreier space $X[S_1]$ is the completion of c_{00} with respect to this norm.

properties of $X[S_1]$

Proofs of the following can be found in Casazza and Shura's book 'Tsirelson's Space':

Lemma

The standard unit vector basis is an unconditional basis for $X[S_1]$.

c_0 embeds into $X[S_1]$; in fact, every infinite dimensional subspace of $X[S_1]$ contains a complemented copy of c_0.

in particular, $X[S_1]$ is not reflexive.

l_1 does not embed into $X[S_1]$.

higher order Schreier spaces

Definition: (Alspach and Argyros, Diss. Math. 1992). Let:

$$S_0 := \{\{k\} : k \in \mathbb{N}\}.$$

We say $E < F$ when $\max E < \min F$. For $n \in \mathbb{N}_0$ recursively define the $(n+1)$th Schreier family by:

$$S_{n+1} := \left\{ \bigcup_{j=1}^{m} E_j : m \in \mathbb{N}, E_1, \ldots, E_m \in S_n, m \leq \min E_1, E_1 < \cdots < E_m \right\}$$

For $(x_n) \in c_{00}$:

$$||(x_n)||_{S_n} := \sup_{M \in S_n} \sum_{m \in M} |x_m|.$$

Then the Schreier space $X[S_n]$ is the completion of c_{00} with respect to this norm.

Tsirelson's space and properties

Tsirelson's space origin: T was the first example (1974) of a Banach space lacking copies of c_0 or any ℓ_p.

Lemma

The standard unit vector basis is an unconditional basis for T.

T *is reflexive.*

Introducing spatial ideals

If X has an unconditional basis (b_n):

Define $P_E(\sum_{n\in\mathbb{N}} \alpha_n b_n) = \sum_{n\in E} \alpha_n b_n$.

An ideal $\mathscr{I}$ of $\mathscr{B}(X)$ is *spatial* if $\mathscr{I} = \overline{\langle P_E \rangle}$ (two-sided) for some $E \subseteq \mathbb{N}$.

If $E \subset \mathbb{N}$ is finite and X has a basis, $\mathscr{K}(X) = \overline{\langle P_E \rangle}$.

We call a spatial ideal $\mathscr{I}$ non-trivial if $\mathscr{K}(X) \subsetneq \mathscr{I}$.

Ideals in $\mathscr{B}(X[S_n])$ and $\mathscr{B}(T)$

The following classication is due to Beanland, Kania and Laustsen (2019):

Theorem

Let X denote either Tsirelson's space T or the Schreier space $X[S_n]$.
(i) The family of non-trivial spatial ideals in $\mathscr{B}(X)$ is non-empty and contains no minimal or maximal elements.
(ii) Let $\mathscr{I}_1 \subsetneq \mathscr{I}_2$ be spatial ideals in $\mathscr{B}(X)$. There is a family $\{\Gamma_\lambda : \lambda \in \Lambda\}$ of uncountably large chains of spatial ideals, such that:

The indexing set Λ has cardinality of the continuum;

for each $\lambda \in \Lambda$, $\mathscr{I}_1 \subsetneq J \subsetneq \mathscr{I}_2 \;\; \forall J \in \Gamma_\lambda$;

For distinct $\lambda, \mu \in \Lambda, \mathscr{L} \in \Gamma_\lambda$ and $\mathscr{M} \in \Gamma_\mu \implies \overline{\mathscr{L} + \mathscr{M}} = \mathscr{I}_2$.

(iii) The algebra $\mathscr{B}(X)$ contains at least continuum many maximal ideals.

Approach for $X[S_n]$ and T

A central property enabling this classification is the following result. For a Banach space X with basis (b_n), define $X_M := \overline{span}(b_m : m \in M)$ where $M \subseteq \mathbb{N}$.

Theorem

In $X = X[S_n]$ or T, the spaces $X_M \cong X_L$ if and only if $(e_l)_{l \in L}$ is equivalent to $(e_m)_{m \in M}$.

For $X[S_n]$ this is due to Gasparis and Leung (1999), and for T is due to Casazza, Johnson and Tzafriri.

Lemma

In $X[S_n]$ and T, the ideals $\langle P_M \rangle = \langle P_L \rangle$ iff $X_M \cong X_L$.

Question: How do we determine when (e_l) is equivalent to (e_m)?

The Gasparis-Leung Index

Let $L, M \subseteq \mathbb{N}$ be infinite and $\varphi(l_n) = m_n$. First we have the **pre-index** on finite subsets $A \subset \mathbb{N}$:

$$\tau(A) := \min\left\{ n \in \mathbb{N} : A \subseteq \bigcup_{j=1}^{n} E_j,\ E_1 < \cdots < E_n \text{ are in } S_1 \right\}.$$

This leads to the full **Gasparis-Leung index:**

$$d(L, M) := \sup\{\tau(\varphi^{-1}A) : A \in S_1 \text{ and } A \subset M\}.$$

Intuition: d measures the extent to which admissibility is lost when passing through φ^{-1}. Then:

Theorem

In $X[S_1]$, the subsequences $(e_l)_{l\in L}$ and $(e_m)_{m\in M}$ are equivalent iff $d(L, M), d(M, L)$ are finite.

pth Baernstein space

Baernstein Space origin: A Banach space purpose-built to be reflexive and fail the Banach-Saks property: fix $1 < p < \infty$. Let $x \in c_{00}$. Given some S_1 sets $E_1 < \cdots < E_m$, we define:

$$\mu_p(x, (E_j)_{j=1}^m) = \left(\sum_{j=1}^m \left(\sum_{i \in E_j} |x_i| \right)^p \right)^{\frac{1}{p}}.$$

Then $||x||_{B_p} = \sup\{\mu_p(x, (E_j)_{j=1}^m) : E_1 < \cdots < E_m$ are S_1 sets$\}$. The completion of c_{00} with repsect to this norm is B_p.
The original space was where $p = 2$.

Properties of B_p

Theorem

(Seifert. 1977.) In the pth Baernstein space:

- *Each B_p is reflexive, but fails the Banach-Saks property. But B_p^* has the Banach-Saks property.*
- *Every infinite-dimensional subspace of B_p contains a complemented copy of l_p.*
- $\mathscr{B}(B_p, B_q) = \mathscr{K}(B_p, B_q)$ *whenever $p > q$.*

The main Theorem on B_p

New result (Joint work with Niels Laustsen): the **same** classification of $X[S_n]$ and T holds for B_p:

Theorem

(i) The family of non-trivial spatial ideals in $\mathscr{B}(B_p)$ *is non-empty and contains no minimal or maximal elements.*
(ii) Let $\mathscr{I}_1 \subsetneq \mathscr{I}_2$ *be spatial ideals in* $\mathscr{B}(B_p)$*. There is a family* $\{\Gamma_\lambda : \lambda \in \Lambda\}$ *of uncountably large chains of spatial ideals, such that:*

The indexing set Λ *has cardinality of the continuum;*

for each $\lambda \in \Lambda$, $\mathscr{I}_1 \subsetneq J \subsetneq \mathscr{I}_2 \ \forall J \in \Gamma_\lambda$;

For distinct $\lambda, \mu \in \Lambda, \mathscr{L} \in \Gamma_\lambda$ *and* $\mathscr{M} \in \Gamma_\mu \implies \overline{\mathscr{L} + \mathscr{M}} = \mathscr{I}_2$.

(iii) The algebra $\mathscr{B}(B_p)$ *contains at least continuum many maximal ideals.*

Approach for B_p

Surprisingly, the **Gasparis-Leung index** also characterises when subsequences of (e_n) are equivalent:

Theorem

In B_p, the subsequences $(e_l)_{l\in L}$ and $(e_m)_{m\in M}$ are equivalent iff $d(L, M), d(M, L)$ are finite.

And we can draw the same conclusion:

Theorem

If $X = B_p$, the spaces $X_M \cong X_L$ if and only if $(e_l)_{l\in L}$ is equivalent to $(e_m)_{m\in M}$.

Then, the algebraic machinery of Beanland, Kania and Laustsen can be applied to produce the main theorem.

context

(i) $X[S_n]$ and T were not the first examples of separable spaces with infinitely many closed operator ideals. The following is a special case of Porta's example (Bull. Amer. Math. Soc.1969): let $2 < p_1 < p_2 < \dots$

$$X = \left(\bigoplus_{n \in \mathbb{N}} \ell_{p_n} \right)_{\ell_2} .$$

(ii) Mankiewicz and Dales–Loy-Willis have independently constructed separable Banach spaces X such that $\mathscr{B}(X)$ admits a bounded, surjective algebra homomorphism φ onto ℓ_∞ and therefore $\{\varphi^{-1}(\mathscr{M}) : \mathscr{M} \text{ is a maximal ideal of } \ell_\infty\}$ is a family of cardinality 2^c of maximal ideals of $\mathscr{B}(X)$.

The End