The centre of the bidual of Fourier algebras

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$\mathcal{A}$ Banach algebra $\qquad \mathcal{A}''$ second dual

(first) Arens product:

$a'' = \text{w*-}\lim a_\alpha,\ b'' = \text{w*-}\lim b_\beta$ (bounded nets, $a_\alpha, b_\beta \in \mathcal{A}$)

$a'' \,\Box\, b'' = \text{w*-}\lim_\alpha (\text{w*-}\lim_\beta\ a_\alpha\, b_\beta)$

$a'' \mapsto a'' \,\Box\, b''$ w*–continuous

$Z_t(\mathcal{A}'') = \{a'' : \quad b'' \mapsto a'' \,\Box\, b'' \text{ is w*–continuous}\}$

(left) *topological centre* of $\mathcal{A}''$

$\mathcal{A}$ (left) *strongly Arens irregular* if $Z_t(\mathcal{A}'') = \mathcal{A}$

For $\mathcal{A}$ commutative, $Z_t(\mathcal{A}'')$ coincides with the algebraic centre of $(\mathcal{A}'', \Box)$.

G locally compact group

$A(G)$ Fourier algebra of G (coefficients of the left regular representation on $L^2(G)$; $x \mapsto (\delta_x * f \mid g)$)

Banach algebra of functions on G pointwise multiplication

for G abelian Lau, L. 1988 $A(G)$ $(\cong L^1(\widehat{G}))$ always strongly Arens irregular

for G solvable, connected, second countable Lau, L. 1993 $A(G)$ strongly Arens irregular

for G discrete with $G \supseteq F_2$ (free group) L. 2016 $Z_t(A(G)'') \neq A(G)$

Semisimple Lie groups, finite centre

spherical functions:

Iwasawa decomposition $G = KS$, K compact, S solvable

$A_{\#}(G)$ radial (zonal, bi-invariant) functions in $A(G)$

$$u(k_1 x k_2) = u(x) \text{ for } k_1, k_2 \in K , \ x \in G$$

$L^1_{\#}(G)$ commutative subalgebra of $L^1(G)$

characters of $L^1_{\#}(G)$: spherical functions

Example: $G = SL(2, \mathbb{R})$

$$K = \left\{ \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \right\} \quad S = \left\{ \begin{pmatrix} a & 0 \\ b & \frac{1}{a} \end{pmatrix} : a > 0, \ b \in \mathbb{R} \right\}$$

φ_t $(t \geq 0)$ positive definite spherical functions

(coefficients of the spherical (zonal, first) principal series of representations of G)

$M(A(G))$ multipliers of $A(G)$ $\qquad \varphi \cdot A \subseteq A$

$\|\varphi\|_M$ operator norm

$\|\varphi_s - \varphi_t\|_M = O(|s-t|)$ (Haagerup: $\|\varphi\|_M = \|\varphi|S\|_B$ for radial functions)

$B_\rho(G)$ reduced Fourier-Stieltjes algebra (coefficients of representations that are weakly contained in the left regular representation)

$C^*_\rho(G)$ reduced group C*-algebra (norm closure of the image of $L^1(G)$ in $\mathcal{B}(L^2(G))$) $\qquad B_\rho(G) \cong C^*_\rho(G)'$

(but $\|\varphi_s - \varphi_t\|_B = 2$ for $s \neq t$)

for fixed $N > 0$ $\{\varphi_t : \; 0 \leq t \leq N\}$ is compact in the multiplier topology, the same for its w*-closed convex hull P $(\subseteq B_\rho(G))$

on P w*-topology coincides with the multiplier topology multiplication defines a left action of $\mathcal{A}$ on $\mathcal{A}'$ (using the dual operators) extending to an action of $\mathcal{A}''$ $\quad a'' \cdot a'$ (multipliers on $\mathcal{A}'$)

$a'' \in Z_t(\mathcal{A}'')$ holds iff there exists a net $(a_\alpha) \subseteq \mathcal{A}$ such that $a'' = \text{w*-}\lim a_\alpha$ and $a_\alpha \cdot a' \underset{\|\ \|}{\longrightarrow} a'' \cdot a'$ for each $a' \in \mathcal{A}'$ (i.e., the multipliers defined by a_α are converging in the *strong* topology of operators on $\mathcal{A}'$)

$VN(G)$ $(\supseteq C^*_\rho(G))$ group von Neumann algebra (generated by the left regular representation on $L^2(G)$) $\quad VN(G) \cong A(G)'$

for a probability measure μ on $[0, N]$ $\quad \varphi = \int \varphi_t \, d\mu(t)$ (w*-integral) defines an element of P, for μ absolutely continuous (with respect to Lebesgue measure) one gets elements of $P \cap A(G)$

for an arbitrary probability measure μ: approximating μ by absolutely continuous measures gives a sequence $(\psi_n) \subseteq P \cap A(G)$ such that $\varphi = \text{w}^*\text{-}\lim \psi_n$, i.e. for $\sigma(B_\rho(G), C^*_\rho(G))$
Let $\Psi \in A(G)''$ $(= VN(G)')$ be some accumulation point of the sequence (ψ_n) with respect to the w*-topology $\sigma(VN(G)', VN(G))$.
Then $\Psi | C^*_\rho(G) = \varphi$
The multipliers on $A(G)$ defined by (ψ_n) are convergent in the norm topology (for operators on $A(G)$). By duality, the same holds for the corresponding multipliers (dual operators) on $VN(G)$ and it follows that $\Psi \in Z_t(A(G)'')$.

By approximation, it follows that
if $\varphi \in B_{\rho,\#}(G)$ is positive definite and $\Psi \in A(G)''$ is any extension with $\|\Psi\| = \|\varphi\|$ then $\Psi \in Z_t(A(G)'')$.

Taking linear combinations, one gets that for any $\varphi \in B_{\rho,\#}(G)$ there exists $\Psi \in Z_t(A(G)'')$ with $\Psi|C^*_\rho(G) = \varphi$.

Using uniform continuity of $t \mapsto \varphi_t$, further (non-zero) $\Psi \in Z_t(A(G)'')$ can be obtained, by considering w*-accumulation points of $\psi_n = \int \varphi_t \, d\mu_n(t)$ for a sequence of absolutely continuous measures μ_n with $\|\mu_n\| = 1$, $\mu_n([0,\infty[) = 0$ and $\operatorname{supp} \mu_n \subseteq [t_n, t_n+\epsilon_n]$ where $t_n \to \infty$, $\epsilon_n \to 0$.
These are "located at infinity" and satisfy
$\Psi \,\square\, a'' = a'' \,\square\, \Psi = 0$ for all $a'' \in A(G)''$. annihilator of $A(G)''$

spherical Fourier transform

$A_{\#}(G) \cong L^1(\mathbb{R}^+, t\tanh(\pi t)dt)$ $\qquad$ $\mathbb{R}^+ = [0, \infty[$

$C^*_{\rho,\#}(G) \cong C_0(\mathbb{R}^+)$

$B_{\rho,\#}(G) \cong M(\mathbb{R}^+)$

multiplication in $A_{\#}(G)$ (and $B_{\rho,\#}(G)$) transfers to a "dual convolution" on $\mathbb{R}^+$

$f \star_d g\,(t) = \int\int f(r)\,g(s)\,a(r,s,t)\,dr\,ds$

weak form of a hypergroup

groups with infinite centre:

Example: $\widetilde{G} = \widetilde{SL}(2, \mathbb{R})$ universal covering group

Iwasawa decomposition $\widetilde{G} = \widetilde{K}S$ where $\widetilde{K} \cong \mathbb{R}$

(universal covering group of K)

for $\varphi \in B_\rho(\widetilde{G})$ its restriction to cosets $x\widetilde{K}$ defines elements of $B(\widetilde{K})$. Extension to $Z_t(A(\widetilde{G})'')$ is possible only if these restrictions belong to $A(\widetilde{K})$. E.g., radial functions cannot be extended to $Z_t(A(\widetilde{G})'')$. Thus one has to consider also functions having similar invariance properties with respect to other characters of $\widetilde{K}$.

general connected groups:

generalized Levi decomposition $G = j(M)\,Q$
with Q solvable normal closed (not necessarily connected)
M semisimple, $j : M \to G$ continuous injective homomorphism
($j(M)$ not necessarily closed)
$G \cong (Q \rtimes M)/D$ with D discrete central
Example: $\widetilde{G} = \mathbb{R}^2 \rtimes SL(2, \mathbb{R})$
again, not all functions can be extended and some modifications are necessary to get elements of $Z_t(A(G)'')$ when M is not compact.

Theorem. *If G is a connected locally compact group containing a non-compact semisimple group, then $A(G)$ is not strongly Arens irregular.*

Corollary. *Let G be a connected locally compact group having no non-trivial compact quotients. Then $A(G)$ is strongly Arens irregular iff G is solvable.*

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