Compact perturbations of operator semigroups

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Banach Algebras and Applications

Granada, Spain
July 18–23, 2022

Work supported by the NCN grant no. 2020/37/B/ST1/01052

Formulation of the problem

Notation:

- $\mathcal{H}$ inf. dim. separable Hilbert space
- $\mathcal{B}(\mathcal{H})$ bounded linear operators on $\mathcal{H}$
- $\mathcal{K}(\mathcal{H})$ compact operators on $\mathcal{H}$
- $\mathcal{Q}(\mathcal{H})$ Calkin algebra, i.e. $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$
- $\pi\colon \mathcal{B}(\mathcal{H}) \to \mathcal{Q}(\mathcal{H})$ quotient map
- $\mathbb{H}$ Hilbert space of density $\mathfrak{c}$, so that there is an isometric *-isomorphism from $\mathcal{Q}(\mathcal{H})$ into $\mathcal{B}(\mathbb{H})$ (Calkin, 1941)

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Problem (general formulation)

Assume $(Q(t))_{t\geqslant 0} \subset \mathcal{B}(\mathcal{H})$ is a family of (normal) operators such that

$$Q(s+t) - Q(s)Q(t) \in \mathcal{K}(\mathcal{H}) \quad \text{for all } s,t \geqslant 0.$$

Can it be, under natural circumstances, lifted to an operator semigroup? In other words, does there exist an operator semigroup $(T(t))_{t\geqslant 0} \subset \mathcal{B}(\mathcal{H})$ such that $Q(t) - T(t) \in \mathcal{K}(\mathcal{H})$ for $t \geqslant 0$?

Possible motivations

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- The problem of preserving the semigroup property while lifting leads to some modifications of the Brown–Douglas–Fillmore theory. Recall that the BDF theory provided the famous characterization of essentially normal operators that admit a normal lift.

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- The problem of preserving the semigroup property while lifting leads to some modifications of the Brown–Douglas–Fillmore theory. Recall that the BDF theory provided the famous characterization of essentially normal operators that admit a normal lift.
- Our hypothesis '*semigroup modulo compacts*' occurs for Toeplitz operators. Recall that for $\varphi \in L^\infty(\mathbb{T})$, T_φ is defined on the Hardy space H^2 by $T_\varphi f = P(\varphi f)$, where P is the orthogonal projection of $L^2(\mathbb{T})$ onto H^2. We have that $T_\varphi T_\psi - T_{\varphi\psi}$ is compact for $\varphi \in C(\mathbb{T})$ and $\psi \in L^\infty(\mathbb{T})$.

Basic definitions

Definition

A family $(T(t))_{t\geqslant 0} \subset \mathcal{B}(\mathcal{H})$ is called an *operator semigroup*, provided that

(a) $T(0) = I_{\mathcal{H}}$ (the identity operator) and

(b) $T(s+t) = T(s)T(t)$ for all $s, t \geqslant 0$.

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If additionally

$$\lim_{\varepsilon\to 0+} \|T(\varepsilon)x - x\| = 0 \quad \text{for every } x \in \mathcal{H},$$

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In general, it is an unbounded, densely defined operator.

Lifting problems

Considering the operators $q(t) = \pi Q(t) \in \mathcal{Q}(\mathcal{H})$, we may formulate our problem as follows:

Problem (precise formulation)

Assume that $(q(t))_{t\geqslant 0} \subset \mathcal{Q}(\mathcal{H})$ is a C_0-semigroup of normal elements of the Calkin algebra. Under what conditions there exists a C_0-semigroup $(T(t))_{t\geqslant 0}$ of normal operators in $\mathcal{B}(\mathcal{H})$ such that $\pi T(t) = q(t)$ for every $t \geqslant 0$?

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In a sense, we seek for a 'semigroup variant' of the famous BDF result from

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which says that an operator $T \in \mathcal{B}(\mathcal{H})$ is of the form 'normal plus compact' if and only if it is essentially normal and $\mathrm{ind}(\lambda I - T) = 0$ for every $\lambda \notin \sigma_{\mathrm{ess}}(T)$.

A counterexample

Fredholm operators

Recall that an operator $T \in \mathcal{B}(\mathcal{H})$ is called a **Fredholm operator**, provided that both $\alpha(T) := \dim \mathrm{Ker}(T)$ and $\beta(T) := \mathrm{codim}\,\mathrm{Ran}(T)$ are finite.

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- The Fredholm index is invariant under compact perturbations, that is, $\mathrm{ind}(T + K) = \mathrm{ind}(T)$ for every $K \in \mathcal{K}(\mathcal{H})$.
- For any $S \in \mathcal{B}(\mathcal{H})$, the **essential spectrum** $\sigma_{\mathrm{ess}}(S)$ is defined as the set of those $\lambda \in \mathbb{C}$ for which $\lambda I - S$ is not Fredholm, and we have $\sigma_{\mathrm{ess}}(S) = \sigma(\pi(S))$.

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The BDF theory

Brief overview 1

Let X be a compact metric space. By an *extension* of $C(X)$ (*by* $\mathcal{K}(\mathcal{H})$) we mean any pair $(\mathcal{A}, \varphi)$, where $\mathcal{A}$ is a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ containing the compact operators and $\varphi\colon \mathcal{A} \to C(X)$ is a *-homomorphism such that

$$0 \longrightarrow \mathcal{K}(\mathcal{H}) \stackrel{\iota}{\longrightarrow} \mathcal{A} \stackrel{\varphi}{\longrightarrow} C(X) \longrightarrow 0$$

is an exact sequence, where ι is the inclusion map.

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The collection $\mathrm{Ext}(X)$ of all equivalence classes of extensions of $C(X)$ forms a **group** (nontrivial!) when equipped with an operation $+$ defined in terms of *-monomorphisms $C(X) \to \mathcal{Q}(\mathcal{H})$ as $[\tau_1] + [\tau_2] = [\tau_1 \oplus \tau_2]$.

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We identify $\mathcal{H}\oplus\mathcal{H}\cong\mathcal{H}$ and $\mathbb{M}_2(\mathcal{Q}(\mathcal{H}))\cong\mathcal{Q}(\mathcal{H})$, as $\mathbb{M}_2(\mathcal{K}(\mathcal{H}))$ is mapped onto $\mathcal{K}(\mathcal{H})$.

The BDF theory
Brief overview 2

Given two compact metric spaces X and Y, and a continuous map $f\colon X \to Y$, there is an induced map $f_*\colon \mathrm{Ext}(X) \to \mathrm{Ext}(Y)$ defined as

$$f_*(\tau)(g) = \tau(g \circ f) \oplus \sigma(g) \quad (g \in C(Y)),$$

where σ is any *-monomorphism corresponding to the trivial extension of $C(Y)$. We add that second direct sum summand in order to guarantee that the resulting map $f_*(\tau)$ is injective.

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The zero element of $\mathrm{Ext}(X)$ can be constructed as follows: Take any infinite direct sum decomposition $\mathcal{H} = \bigoplus_{i=1}^{\infty} \mathcal{H}_i$, where each $\mathcal{H}_i$ is infinite-dimensional, pick a countable dense subset $\{\xi_i\colon i \in \mathbb{N}\}$ of X and define $\sigma\colon C(X) \to \mathcal{B}(\mathcal{H})$ by

$$\sigma(g) = \bigoplus_{i=1}^{\infty} g(\xi_i) I_i,$$

where I_i is the identity operator on $\mathcal{H}_i$.

In $C(X)$ consider the relation of homotopy equivalence and let $\mathcal{G}_0(C(X))$ be the equivalence class of the constant one function. By $\pi^1(X)$ we denote the group $\mathcal{G}(C(X))/\mathcal{G}_0(C(X))$ of homotopy classes of invertible functions.

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Theorem (Brown, Douglas, Fillmore, 1977)

For any compact set $X \subset \mathbb{C}$, there is a well-defined map

$$\gamma\colon \mathrm{Ext}(X) \longrightarrow \mathrm{Hom}(\pi^1(X), \mathbb{Z}), \quad \gamma[\tau]([f]) = \mathrm{ind}\, \tau(f)$$

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which is a group isomorphism.

This leads to the famous characterization of 'liftable' essentially normal operators. More generally: two essentially normal operators T_1 and T_2 are unitarily equivalent modulo compacts iff $\sigma_{\mathrm{ess}}(T_1) = \sigma_{\mathrm{ess}}(T_2)$ and $\mathrm{ind}(\lambda I - T_1) = \mathrm{ind}(\lambda I - T_2)$ for every $\lambda \notin \sigma_{\mathrm{ess}}(T_1)$.

What kind of result we expect?

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STEP 1: With every such $(q(t))_{t\geqslant 0}$ we associate an extension of $C(\Omega)$, where Ω is a certain compact metric space defined exclusively in terms of $\sigma(A)$.

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STEP 4: Once having a section witnessing the triviality of our extension, we use a lifting procedure, similar as in the classical BDF case, to produce an operator semigroup lift; sometimes we can even obtain a C_0-semigroup.

Building an extension

(a) Since there exists $\gamma < \infty$ such that $\mathrm{Re}\,\lambda \leqslant \gamma$ for each $\lambda \in \sigma(A)$, all the sets

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Plainly, $q(s), q(t), q(t)^*$ commute for all $s, t \geqslant 0$, thus

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The **joint spectrum** of the set $\{q(2^{-n})\colon n = 0, 1, \dots\}$ is a compact subset of $\mathbb{C}^\infty$ defined by

$$\sigma_{\mathcal{A}_0}(q(2^{-n})\colon n = 0, 1, \dots) = \big\{(\varphi(q(2^{-n})))_{n=0}^\infty \colon \varphi \in \Delta\big\}.$$

Building an extension

Then, the map

$$\Delta \ni \varphi \longmapsto (\varphi(q(2^{-n})))_{n=0}^{\infty}$$

is a homeomorphism between Δ and $\sigma_{\mathcal{A}_0}(q(2^{-n})\colon n = 0, 1, \ldots)$.

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A sequence $\boldsymbol{\lambda} = (\lambda)_{n=1}^{\infty} \in \mathbb{C}^{\infty}$ belongs to $\sigma_{\mathcal{A}_0}(q(2^{-n})\colon n \in \mathbb{N})$ if and only if

$$q(\boldsymbol{\lambda}) \coloneqq \sum_{n=0}^{\infty} 2^{-n} \frac{(\lambda_n I - q(2^{-n}))^*(\lambda_n I - q(2^{-n}))}{\|\lambda_n I - q(2^{-n})\|^2} \tag{1}$$

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Indeed, as each summand is a positive operator, we infer that for every linear multiplicative functional $\varphi \in \Delta$ we have $\varphi(q(\boldsymbol{\lambda})) = 0$ iff $\varphi(q(2^{-n})) = \lambda_n$ for each $n = 0, 1, \ldots$ Hence, if $q(\boldsymbol{\lambda})$ is not invertible we pick $\varphi \in \Delta$ so that $\varphi(q(\boldsymbol{\lambda})) = 0$ to see that $\boldsymbol{\lambda}$ belongs to the joint spectrum. Conversely, if $q(\boldsymbol{\lambda})$ is invertible, then we have $\varphi(q(\boldsymbol{\lambda})) \neq 0$ for every $\varphi \in \Delta$, thus $\boldsymbol{\lambda}$ is not in the joint spectrum.

Building an extension

Fix $\boldsymbol{\lambda} = (\lambda_n)_{n=0}^{\infty} \in \mathbb{C}^{\infty}$. The operator $(\lambda_n I - q(2^{-n}))^*(\lambda_n I - q(2^{-n}))$ corresponds via functional calculus to the map $\phi_n \in L_\infty(E^A)$ given by

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For every $z \in \sigma(A)$, we have $\mathrm{Re}\, z \leqslant \gamma$ and hence

$$\|\phi_n\|_\infty \leqslant \left(e^{2^{-n}\gamma} + |\lambda_n|\right)^2$$

which implies that each denominator in formula (1) is majorized by a constant (cannot become arbitrarily large after applying functional calculus and varying z over $\sigma(A)$).

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Hence, $q(\boldsymbol{\lambda})$ is noninvertible if and only if 0 lies in the closure of the range of the map

$$\sigma(A) \ni z \longmapsto \sum_{n=0}^{\infty} 2^{-n} \frac{\phi_n(z)}{\|\phi_n\|_\infty}$$

which implies that each λ_n must belong to the closure of $\exp(2^{-n}\sigma(A))$ which is denoted by Ω_n.

Building an extension

Moreover, for any $n = 0, 1, 2, \ldots$ we pick $z \in \sigma(A)$ so that both $\phi_n(z)$ and $\phi_{n+1}(z)$ are arbitrarily close to zero. Since $\exp(2^{-n-1}z)^2 = \exp(2^{-n}z)$, we infer that for $q(\boldsymbol{\lambda})$ being noninvertible we also must have $\lambda_{n+1}^2 = \lambda_n$ $(n = 0, 1, \ldots)$.

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Conversely, any sequence $\boldsymbol{\lambda} = (\lambda_n)_{n=0}^{\infty} \in \mathbb{C}^{\infty}$ satisfying $\lambda_n \in \Omega_n$ and $\lambda_{n+1}^2 = \lambda_n$ for $n = 0, 1, \ldots$ produces a noninvertible operator $q(\boldsymbol{\lambda})$.

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Conclusion: The identity map

$$\mathrm{id}\colon \sigma_{\mathcal{A}_0}(q(2^{-n})\colon n = 0, 1, \ldots) \longrightarrow \varprojlim \Omega_n$$

is bijective and hence a homeomorphism, as both topologies are the product topology. Consequently, Δ is homeomorphic to the projective (inverse) limit $\{\Omega_n, p_n\}_{n \geqslant 0}$, where $p_n(z) = z^2$ for each $n = 0, 1, 2, \ldots$

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Recall that the projective (inverse) limit of an inverse system $\{X_n, f_n\}_{n\geqslant 0}$, that is, a sequence of topological spaces and continuous maps $f_n\colon X_{n+1}\to X_n$, is defined as

$$\varprojlim X_n = \Big\{\mathbf{x} = (x_n)_{n=0}^{\infty} \in \prod_{n=0}^{\infty} X_n\colon\, f_n(x_{n+1}) = x_n \;\text{ for } n\geqslant 0\Big\}.$$

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Of course, $\mathcal{K}(\mathcal{H})$ forms an ideal in $\mathcal{E}$. For every $T\in\mathcal{E}$, we have $\pi(T)\in\mathcal{A}_0$ and each element in $\mathcal{A}_0$ is of this form. Hence, the formula $\theta(T)=\widehat{\pi(T)}$ yields a *-homomorphism onto $C(\Delta)$. Obviously, $T\in\ker\theta$ iff $\pi(T)=0$, i.e. $T\in\mathcal{K}(\mathcal{H})$.

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Summarizing, what we have proved is the following:

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Proposition

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(a) Let A be the generator of $(q(t))_{t\geqslant 0}$ and define

$$\Omega_n = \overline{\exp(2^{-n}\sigma(A))} \quad (n=0,1,2,\ldots)$$

Then, $\mathcal{A}_0$ is a commutative C^*-algebra and its maximal ideal space Δ is homeomorphic to the projective limit of the inverse system $\{\Omega_n, p_n\}_{n\geqslant 0}$, where $p_n(z)=z^2$ for each $n=0,1,2,\ldots$

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Proposition (continued)

(b) The C^*-algebra $\mathcal{E}$ contains $\mathcal{K}(\mathcal{H})$ as an ideal and there is an exact sequence

$$0 \longrightarrow \mathcal{K}(\mathcal{H}) \xrightarrow{\iota} \mathcal{E} \xrightarrow{\theta} C(\Delta) \longrightarrow 0,$$

where $\theta(T) = \widehat{\pi(T)}$ and $\mathcal{A}_0 \ni q \longmapsto \widehat{q} \in C(\Delta)$ is the Gelfand transform.

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This accomplishes Step 1 of our strategy: *With every $(q(t))_{t\geqslant 0}$ as before we associate an extension of $C(\Omega)$, where Ω is a certain compact metric space defined exclusively in terms of $\sigma(A)$.*

Proceeding to Step 2

We know that every normal C_0-semigroup $(q(t))_{t\geqslant 0}$ in $\mathcal{Q}(\mathcal{H})$ generates an extension of $C(\Delta)$ by $\mathcal{K}(\mathcal{H})$, where Δ is a compact metric space depending only on the generator A of $(q(t))_{t\geqslant 0}$. Recall that

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Suppose $\{X_n, p_n\}_{n=0}^{\infty}$ is an inverse system of compact metric spaces. Let $X = \varprojlim X_n$ and $q_n\colon X \to X_n$ stand for the coordinate maps, for $n \in \mathbb{N}_0$, so that $p_n q_{n+1} = q_n$. Hence, we have another inverse system of groups $\{\mathrm{Ext}(X_n), p_{n*}\}_{n=0}^{\infty}$. Since $p_{n*}q_{(n+1)*} = q_{n*}$, we can define an *induced map* $P\colon \mathrm{Ext}(X) \to \varprojlim \mathrm{Ext}(X_n)$ by the formula

$$P(\tau) = (q_{n*}\tau)_{n=0}^{\infty}.$$

The induced map is always **surjective**, but in general not injective.

Milnor's exact sequence

J. Milnor, *On the Steenrod homology theory* (first distributed 1961), in: S. Ferry, A. Ranicki, J. Rosenberg (Eds.), *Novikov Conjectures, Index Theorems, and Rigidity*: Oberwolfach 1993, London Mathematical Society Lecture Note Series, pp. 79–96, Cambridge University Press, Cambridge 1995.

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Theorem (Milnor, 1961)

For any inverse system $\{X_n\}$ of compact metric spaces, and any $k \in \mathbb{Z}$, there exists an exact sequence

$$0 \longrightarrow \varprojlim{}^{(1)} \mathrm{Ext}_{k+1}(X_n) \longrightarrow \mathrm{Ext}_k(\varprojlim X_n) \xrightarrow{P} \varprojlim \mathrm{Ext}_k(X_n) \longrightarrow 0,$$

where $\varprojlim{}^{(1)}$ is the *first derived functor* of inverse limit.

Conditions on the kernel

Therefore, we can ask: Given a C_0-semigroup in the Calkin algebra, when does the resulting extension of Ω actually land in the kernel from the Milnor's exact sequence?

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$$P\colon \mathrm{Ext}(\Omega) \longrightarrow \varprojlim \mathrm{Ext}(\Omega_n), \quad \text{where } \Omega = \varprojlim \Omega_n \approx \Delta,$$

be the induced surjective map. Then, $(\mathcal{E}, \theta) \in \ker P$ if and only if

$$\mathrm{ind}(\lambda I - q(2^{-n})) = 0 \quad \text{for all } n \in \mathbb{N}_0,\ \lambda \notin \Omega_n.$$

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Consequently, if we start with a collection of normal operators $(T(t))_{t\geqslant 0} \subset \mathcal{B}(\mathcal{H})$ and consider the C_0-semigroup $(\pi T(t))_{t\geqslant 0}$, we automatically have an extension from the Milnor kernel.

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This accomplishes Step 2: *We show that BDF conditions imposed 'separately' on $q(t)$'s imply that we land in Milnor's kernel.*

Suspensions

Recall that for any compact metric space X, the *cone* CX over X is obtained from $X \times I$ by collapsing $X \times \{0\}$ to a single point, where $I = [0,1]$. The *suspension* SX is obtained from $X \times I$ by collapsing $X \times \{0\}$ and $X \times \{1\}$ to two distinct points.

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The extension functor is defined for ranks $q \leqslant 1$ by $\mathrm{Ext}_q(X) = \mathrm{Ext}(S^{1-q}X)$. It was shown by BDF that, analogously to Bott's periodicity in K-theory, there exist isomorphisms

$$\mathrm{Per}_*\colon \mathrm{Ext}_{q-2}(X) \longrightarrow \mathrm{Ext}_q(X) \quad (r \leqslant 1).$$

This allows us to extend the definition of Ext to all integer dimensions:

$$\mathrm{Ext}_q(X) = \begin{cases} \mathrm{Ext}(X) & \text{if } q \text{ is odd,} \\ \mathrm{Ext}(SX) & \text{if } q \text{ is even.} \end{cases}$$

The derived functor

By definition, the functor $\varprojlim^{(1)}$ applied to an inverse system of groups $\{G_n, p_n\}_{n=0}^{\infty}$ returns the cokernel of the map

$$\prod G_n \ni (a_0, a_1, \ldots) \stackrel{d}{\longmapsto} (a_0 - p_0(a_1), a_1 - p_1(a_2), \ldots)$$

defined on the full direct product of all the G_n. That is,

$$\varprojlim^{(1)} G_n = \prod G_n / d(\prod G_n).$$

The derived functor

By definition, the functor $\varprojlim^{(1)}$ applied to an inverse system of groups $\{G_n, p_n\}_{n=0}^{\infty}$ returns the cokernel of the map

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We have $(Sp_n)_*\tau(g) = \tau(g \circ Sp_n)$ for $g \in C(S\Omega_n)$. Fix any $\lambda \in \mathrm{Ext}_2(\Omega_n)$. Our goal is to find a *-monomorphism $\tau \colon C(S\Omega_{n+1}) \to \mathcal{Q}(\mathcal{H})$ such that

$$\tau(g \circ Sp_n) = \lambda(g) \quad \text{for every } g \in C(S\Omega_n), \tag{2}$$

where the equality is understood as unitary equivalence between the both sides regarded as *-homomorphisms on $C(S\Omega_n)$.

Twisting maneuver

Recall that we want: $\tau(g \circ Sp_n) = \lambda(g)$ for every $g \in C(S\Omega_n)$. Functions of the form $g \circ Sp_n$ preserve antipodal points, i.e.

$$(g \circ Sp_n)([x,t]) = (g \circ Sp_n)([-x,t]), \quad \text{whenever } [x,t], [-x,t] \in S\Omega_{n+1}.$$

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By using a 'twisting maneuver' we can reduce the requirement of preserving antipodal points to just those pairs which correspond to **just one** direction, namely, $\alpha/2$.

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- let $T_\pm$ be the 'upper'/'lower' semicircles of the unit circle $\mathbb{T}$ which are determined by the antipodal points $\pm e^{i\alpha/2}$, that is, $T_+ = \{e^{i(\alpha/2+t\pi)} \colon 0 \leqslant t < 1\}$ and $T_- = \mathbb{T} \setminus T_+$;

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- for $x \in \Omega_n$, let $\sqrt{x}$ be the set of square roots of x, and let $s_0(x) \in \sqrt{x}$ be determined by the condition

$$s_0(x) \in \begin{cases} \sqrt{|x|}\,T_+ & \text{if } \sqrt{|x|}\,T_+ \cap \Omega_{n+1} \neq \varnothing \\ \sqrt{|x|}\,T_- & \text{otherwise.} \end{cases}$$

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- Then, for $j = 0, 1$ and $f \in \mathcal{A}_0$, we set

$$\Delta_j f([x,t]) = f([s_j(x), t]) \quad ([x,t] \in S\Omega_n).$$

For any set $E \subseteq \mathbb{C}$, we denote by $\mathrm{A}(E)$ the set of those $z \in E$ for which $-z \in E$. That is, $\mathrm{A}(E)$ consists of points z which belong to E together with their antipode.

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Technical lemma

Suppose that for some $n \in \mathbb{N}_0$ we have

$$\overline{\Omega_{n+1} \setminus \mathrm{A}(\Omega_{n+1})} \cap \mathrm{A}(\Omega_{n+1}) = \varnothing. \tag{3}$$

Then, for every $f \in \mathcal{A}_0$, $\Delta_j f$ are continuous on $S\Omega_n$ $(j = 0, 1)$.

Geometric conditions

An empty direction

- Fix Calkin's *-representation $\gamma\colon \mathcal{Q}(\mathcal{H}) \to \mathcal{B}(\mathbb{H})$ and a unital *-monomorphism $\lambda\colon C(S\Omega_n) \to \mathcal{Q}(\mathcal{H})$.

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An 'empty direction' condition

Assume that condition (3) is satisfied, and there exists $\alpha \in [0, 2\pi)$ such that $\mu_i(\mathcal{S}_\alpha) = 0$ for all $i \in I$. Then, the homomorphism $(Sp_n)_*\colon \mathrm{Ext}(S\Omega_{n+1}) \to \mathrm{Ext}(S\Omega_n)$ is surjective.

Kasparov's Technical Theorem

Let E be a σ-unital C^*-algebra and $\mathscr{C}(E) = \mathscr{M}(E)/E$ be the corona algebra. Suppose that D is a separable subset of $\mathscr{C}(E)$. If $x, y \in \mathscr{C}(E) \cap D'$ satisfy $x, y \geqslant 0$ and $xy = 0$, then there exists $0 \leqslant z \leqslant 1$, $z \in \mathscr{C}(E) \cap D'$ such that $zx = 0$ and $zy = y$.

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A 'cross retract' condition

Assume that condition (3) is satisfied, and there exist $\alpha, \theta \in [0, 2\pi)$, $\frac{\alpha}{2} \notin \{\theta, \theta - \pi\}$ such that each of the sections

$$\mathsf{S}_{\alpha/2} = \mathbb{R}e^{\mathrm{i}\alpha/2} \cap \Omega_{n+1}, \quad \mathsf{S}_\theta = \mathbb{R}e^{\mathrm{i}\theta} \cap \Omega_{n+1}$$

is a retract of both the corresponding left and the right part of Ω_{n+1}. Then, the homomorphism $(Sp_n)_* \colon \mathrm{Ext}(S\Omega_{n+1}) \to \mathrm{Ext}(S\Omega_n)$ is surjective.

Summary of the main results

Let $(Q(t))_{t \geqslant 0}$ be a collection of normal operators in $\mathcal{B}(\mathcal{H})$ satisfying

$$Q(s+t) - Q(s)Q(t) \in \mathcal{K}(\mathcal{H}) \quad \text{for all } s, t \geqslant 0.$$

Assume that $(q(t))_{t \geqslant 0} \subset \mathcal{Q}(\mathcal{H})$, defined by $q(t) = \pi Q(t)$ for $t \geqslant 0$, is a C_0-semigroup with respect to some faithful *-representation $\gamma \colon \mathcal{Q}(\mathcal{H}) \to \mathcal{B}(\mathbb{H})$. Let also A be its infinitesimal generator, densely defined on $\mathbb{H}$. Then:

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(A1) The spectrum of the C^*-algebra $\mathrm{C}^*(q(2^{-n}),1_{\mathcal{Q}(\mathcal{H})})$ is homeomorphic to the inverse limit $\Delta=\varprojlim\{\Omega_n,p_n\}$, where $p_n(z)=z^2$ and

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(A2) There is an extension $\Gamma\in\mathrm{Ext}(\Delta)$ such that $\Gamma=\Theta$ implies that there exists a semigroup $(Q(t))_{t\in\mathbb{D}}\subset\mathcal{B}(\mathcal{H})$, defined on positive dyadic rationals, such that $\pi Q(t)=q(t)$ for every $t\in\mathbb{D}$.

Summary of the main results

Let $(Q(t))_{t\geqslant 0}$ be a collection of normal operators in $\mathcal{B}(\mathcal{H})$ satisfying

$$Q(s+t)-Q(s)Q(t)\in\mathcal{K}(\mathcal{H})\quad\text{for all } s,t\geqslant 0.$$

Assume that $(q(t))_{t\geqslant 0}\subset\mathcal{Q}(\mathcal{H})$, defined by $q(t)=\pi Q(t)$ for $t\geqslant 0$, is a C_0-semigroup with respect to some faithful *-representation $\gamma\colon\mathcal{Q}(\mathcal{H})\to\mathcal{B}(\mathbb{H})$. Let also A be its infinitesimal generator, densely defined on $\mathbb{H}$. Then:

(A1) The spectrum of the C^*-algebra $\mathrm{C}^*(q(2^{-n}),1_{\mathcal{Q}(\mathcal{H})})$ is homeomorphic to the inverse limit $\Delta=\varprojlim\{\Omega_n,p_n\}$, where $p_n(z)=z^2$ and

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(A3) We have the Milnor exact sequence with $\Gamma\in\ker P$.

Summary of the main results (cont.)

Let $(Q(t))_{t \geqslant 0}$ be a collection of normal operators in $\mathcal{B}(\mathcal{H})$ satisfying

$$Q(s+t) - Q(s)Q(t) \in \mathcal{K}(\mathcal{H}) \quad \text{for all } s, t \geqslant 0.$$

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(A4) Assuming that for each $n \in \mathbb{N}$, $\overline{\Omega_n \setminus \mathrm{A}(\Omega_n)} \cap \mathrm{A}(\Omega_n) = \varnothing$, and that Ω_n satisfies either an 'empty direction' condition, or a 'cross retract' condition, we have $\Gamma = \Theta$.

Summary of the main results (cont.)

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(A5) If Δ is a perfect compact metric space, and γ is one of Calkin's representations of $\mathcal{Q}(\mathcal{H})$, then the obtained lifting $(Q(t))_{t\in\mathbb{D}}$ is $\textsc{sot}$-continuous and it extends to a C_0-semigroup $(Q(t))_{t\geqslant 0}\subset\mathcal{B}(\mathcal{H})$.

Summary of the main results (cont.)

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T.K., *Compact perturbations of operator semigroups*, `arXiv:2203.05635`