Quantum Hypergraph Homomorphisms and Operator Algebras

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Outline

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Functional analytic methods $\rightsquigarrow$ study combinatorial structures

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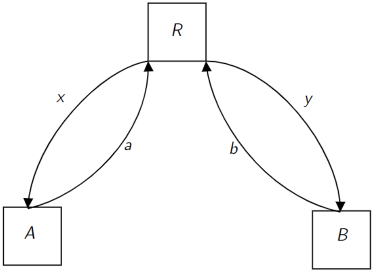
- Non-local game: discrete object we study using operator algebraic techniques
- Similarity between non-local games? Strategy transport? Allow comparison for chance of "winning"?
- Quantum homomorphisms of discrete structures $\rightsquigarrow$ studied for graphs, but few others
  - Can we do the same for non-local games?

Non-local games

- The (classical) definition of a **non-local game** is a tuple (X, Y, A, B, λ) where X, Y, A, B are finite sets and $\lambda : X \times Y \times A \times B \to \{0, 1\}$ is a "verifier" function which encodes the rules of the game.



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- Reformulate non-local games: a non-local game on (V_2, W_1, V_1, W_2) is a hypergraph $\Lambda \subseteq V_2 W_1 \times V_1 W_2$.
  - So Λ corresponds to the support of λ in classical definition.

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- A channel $\mathcal{E} : \mathcal{D}_V \to \mathcal{D}_W$ defines a hypergraph

$$E_{\mathcal{E}}(w) = \{(v, w) \in V \times W : \ \mathcal{E}(w|v) > 0\}.$$

(Here $\mathcal{E}(w|v) = \langle \mathcal{E}(\epsilon_{v,v}), \epsilon_{w,w} \rangle$ where $\langle \cdot, \cdot \rangle$ is the trace of the matrix product and $\epsilon_{v,v}$ are the basis elements for $\mathcal{D}_V$.)

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  - Channel $\mathcal{E}$, hypergraph $E_{\mathcal{E}} \iff$ channel $\mathcal{E}^*$, hypergraph $E_{\mathcal{E}^*} = (E_{\mathcal{E}})^*$

Correlations

Let V_i, W_i be finite sets with $i = 1, 2$. A **no-signalling (NS) correlation** on the quadruple (V_2, W_1, V_1, W_2) is an information channel $\Gamma : \mathcal{D}_{V_2 W_1} \to \mathcal{D}_{V_1 W_2}$ for which marginal channels

$$\Gamma_{V_2 \to V_1} : \mathcal{D}_{V_2} \to \mathcal{D}_{V_1},\ \Gamma_{V_2 \to V_1}(v_1|v_2) := \sum_{w_2 \in W_2} \Gamma(v_1, w_2 | v_2, w_1'),$$

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The collection of no-signalling correlations is denoted by $\mathcal{C}_{\rm ns}$; other classes of correlations $(\mathcal{C}_{\rm loc}, \mathcal{C}_{\rm q}, \mathcal{C}_{\rm qa}, \mathcal{C}_{\rm qc})$ are defined by additional restrictions we place on $\Gamma \in \mathcal{C}_{\rm ns}$.

Local correlations: $\Gamma \in \mathcal{C}_{\rm loc}$ is a convex combination of correlations

$$\Gamma(v_1 w_2 | v_2 w_1) = \Gamma_1(v_1|v_2)\Gamma_2(w_2|w_1),$$

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Quantum commuting: $\Gamma \in \mathcal{C}_{\mathrm{qc}}$ if

$$\Gamma(v_1 w_2 | v_2 w_1) = \langle E_{v_2 v_1} F_{w_1 w_2} \xi, \xi \rangle$$

for mutually commuting **POVM's** $(E_{v_2 v_1})_{v_1 \in V_1}, (F_{w_1 w_2})_{w_2 \in W_2}$ acting on $\mathcal{H}$ and $\xi \in \mathcal{H}$ is a unit vector.

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$$\mathcal{C}_{\mathrm{loc}} \subset \mathcal{C}_{\mathrm{q}} \subset \mathcal{C}_{\mathrm{qa}} \subset \mathcal{C}_{\mathrm{qc}} \subset \mathcal{C}_{\mathrm{ns}}$$

Simulation paradigm

For an NS correlation Γ on (V_2, W_1, V_1, W_2) and a channel $\mathcal{E} : \mathcal{D}_{V_1} \to \mathcal{D}_{W_1}$, the map $\Gamma[\mathcal{E}] : \mathcal{D}_{V_2} \to \mathcal{D}_{W_2}$ defined by

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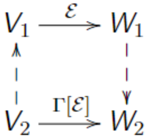
- We "wire" the output for the marginal channel $\Gamma_{V_2 \to V_1}$ to the input for $\mathcal{E}$, and the output of $\mathcal{E}$ back into Γ.



When $\Gamma[\mathcal{E}] \in \mathcal{C}(V_2 \times W_2)$ where Γ is the simulator, we write $(V_1 \mapsto W_1) \xrightarrow{\Gamma} (V_2 \mapsto W_2)$.

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  - For ${\rm t} \neq {\rm ns}$, $\Gamma \in \mathcal{C}_{\rm t}^{\rm bi}$ now has slight additional restriction: POVM's $(E_{v_2,v_1})_{v_1,v_2} \in V$ and $(F_{w_1,w_2})_{w_1,w_2 \in W}$ are **magic squares**.

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- **Note:** $\mathcal{C}_{\mathrm{t}}(\Lambda) = \mathcal{C}(\Lambda) \cap \mathcal{C}_{\mathrm{t}}, \mathcal{C}_{\mathrm{t}}^{\mathrm{bi}}(\Lambda) = \mathcal{C}(\Lambda) \cap \mathcal{C}_{\mathrm{t}}^{\mathrm{bi}}$.

Definition

We say that

- E_1 is t-homomorphic to E_2 (denoted $E_1 \to_{\mathrm{t}} E_2$) if $\mathcal{C}_{\mathrm{t}}(E_1 \leftrightarrow E_2) \neq \emptyset$.
- E_1 is t-isomorphic to E_2 (denoted $E_1 \simeq_{\mathrm{t}} E_2$) if $V_1 = V_2$, $W_1 = W_2$ with $\mathcal{C}_{\mathrm{t}}^{\mathrm{bi}}(E_1 \leftrightarrow E_2) \neq \emptyset$.

Local homomorphisms

A map $f\colon V_2 \to V_1$ is a (classical) **homomorphism between hypergraphs** E_1 and E_2 if pre-images under f preserve edge relations; that is, f is a homomorphism if there exists a map $g : W_1 \to W_2$ so that

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  - If Γ is perfect for $E_1 \to E_2$, assume Γ is an extreme point in $\mathcal{C}_{\mathrm{loc}}$ + no-signalling $\rightsquigarrow$ homomorphism (f, g).
  - If (f, g) a homomorphism between hypergraphs, let $\Phi : \mathcal{D}_{V_2} \to \mathcal{D}_{V_1}, \Psi : \mathcal{D}_{W_1} \to \mathcal{D}_{W_2}$ where $\Phi(v_1|v_2) = \delta_{v_1, f(v_2)}$ and $\Psi(w_2|w_1) = \delta_{w_2, g(w_1)}$. Then $\Gamma = \Phi \otimes \Psi \in \mathcal{C}_{\mathrm{loc}}(E_1 \leftrightarrow E_2)$.

An operator system approach

Start with a finite set V, and a block operator matrix $U = (u_{v,v'})_{v,v' \in V}$ such that U and U^{t} are isometries. Let $\mathcal{V}_V$ be the (universal) ternary ring of operators generated by $u_{v,v'}$ for $v, v' \in V$ and the relations

$$\sum_{a \in V} [u_{a'',x''}, u_{a,x}, u_{a,x'}] = \delta_{x,x'} u_{a'',x''}, \quad \sum_{x \in V} [u_{a'',x''}, u_{a,x}, u_{a',x}] = \delta_{a,a'} u_{a'',x''}.$$

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- For a faithful ternary representation $\theta : \mathcal{V}_V \to \mathcal{B}(\mathcal{H}, \mathcal{K})$ (where $\mathcal{H}, \mathcal{K}$ are Hilbert spaces), for the right C^*-algebra $\mathcal{C}_V$ we have $\mathcal{C}_V \simeq \overline{\text{span}}(\theta(\mathcal{V}_V)^* \theta(\mathcal{V}_V))$.
- Write $e_{v_1, v_1', v_2, v_2'} := u^*_{v_2, v_1} u_{v_2', v_1'}, \quad v_1, v_2, v_1', v_2' \in V$.
  - The C^*-algebra $\mathcal{C}_V$ is generated by elements $e_{v_1, v_1', v_2, v_2'}$ for $v_i, v_i' \in V, i = 1, 2$.

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An operator system approach

Start with a finite set V, and a block operator matrix $U = (u_{v,v'})_{v,v' \in V}$ such that U and U^{t} are isometries. Let $\mathcal{V}_V$ be the (universal) ternary ring of operators generated by $u_{v,v'}$ for $v, v' \in V$ and the relations

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- Consider $\mathcal{J} = \mathrm{span}\{e_{v_2,v_1} \otimes f_{w_1,w_2} : (v_2, w_1, v_1, w_2) \notin E_1 \leftrightarrow E_2\}$ as a subspace in $\mathcal{S}_V \otimes \mathcal{S}_W$.

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Theorem (H.-Todorov, in prep. 2022)

The map $s \mapsto \Gamma_s$ is an affine surjective correspondence between

- *the states of $\mathcal{S}_V \otimes_{\max} \mathcal{S}_W$ which annihilate $\mathcal{J}$ and the perfect ns-strategies of $E_1 \leftrightarrow E_2$.*
- *the states of $\mathcal{S}_V \otimes_{\mathrm{c}} \mathcal{S}_W$ which annihilate $\mathcal{J}$ and the perfect qc-strategies of $E_1 \leftrightarrow E_2$.*
- *the states of $\mathcal{S}_V \otimes_{\min} \mathcal{S}_W$ which annihilate $\mathcal{J}$ and the perfect qa-strategies of $E_1 \leftrightarrow E_2$.*

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$$\Gamma_s(v_1, w_2 | v_2, w_1) = s(e_{v_2, v_1} \otimes f_{w_1, w_2})$$

gives us the correspondence with perfect t-strategies on $E_1 \leftrightarrow E_2$ (for $\mathrm{t} \in \{\mathrm{qa}, \mathrm{qc}, \mathrm{ns}\}$).

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Remark: When $\mathcal{H}$ is a Hilbert space, a **quantum magic square** over V on $\mathcal{H}$ is a block operator matrix $(E_{v_2,v_1})_{v_1,v_2 \in V}$ with positive entries, and

$$\sum_{v_2' \in V} E_{v_1,v_2'} = \sum_{v_1' \in V} E_{v_1',v_2} = I, \quad v_1, v_2 \in V.$$

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Operator system $\mathcal{S}_V$ is universal for quantum magic squares:

ucp maps $\phi : \mathcal{S}_V \to \mathcal{B}(\mathcal{H}) \leftrightarrow$ quantum magic square $(E_{v_1,v_2})_{v_1,v_2 \in V}$ via $E_{v_1,v_2} = \phi(e_{v_1,v_2})$

Assume $V_i = W_i = V, i = 1, 2$. A bicorrelation $\Gamma \in \mathcal{C}_{\mathrm{t}}^{\mathrm{bi}}$ is **faithful** if

$$\Gamma(v_1 w_2 | v_2 w_1) = 0 \text{ if } (v_1 = w_1 \ \& \ v_2 \neq w_2) \text{ or } (v_1 \neq w_1 \ \& \ v_2 = w_2).$$

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Let $t \in \{\mathrm{loc}, \mathrm{q}, \mathrm{qc}\}$. TFAE:

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- *there exists a unitary matrix $P = (P_{v,v'})_{v,v' \in V}$ where entries $P_{v,v'} \in \mathcal{B}(\mathcal{H})$ are projections, such that*

$$P(\mathsf{A}_{E_1} \otimes I_{\mathcal{H}}) = (\mathsf{A}_{E_2} \otimes I_{\mathcal{H}})P$$

where A_{E_i} is the incidence matrix for $E_i, i = 1, 2$.

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Note: The ideas for this proof were adapted from Atserias et. al ([5] 2019), where a similar result was shown for graphs only.

Local vs. quantum strategies

For a given finite graph G with vertex set X, we can form hypergraphs

$$E_G = \{(x, x') : \ x \sim x'\}, \quad F_G = \{((x, y), y) : x \sim_G y\}$$

in $X \times X$ and $XX \times X$, respectively.

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Local vs. quantum strategies

Proof: (Sketch)

(i) As $G_1 \cong_{\mathrm{q}} G_2$, find permutation $P \in M_X \otimes M_d$ intertwining $\mathrm{A}_{G_1} \otimes I_d$ and $\mathrm{A}_{G_2} \otimes I_d$; this implies $E_{G_1} \cong_{\mathrm{q}} E_{G_2}$.

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If $\xi \in \mathcal{H} \otimes \mathcal{H}$ is maximally entangled, set

$$p(ab, c|xy, z) = \langle (Q_{xy,ab} \otimes P^{t}_{y,c})\xi, \xi \rangle, \quad x, y, z, a, b, c \in X.$$

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Then p gives us a perfect approximately quantum strategy for $F_{G_1} \cong F_{G_2}$.

Strong no-signalling correlations

We restrict ourselves to considering non-local games as hypergraphs. We assume:

- X_i, Y_i, A_i, B_i are finite sets, $E_i \subseteq X_iY_i \times A_iB_i, i = 1, 2$ are non-local games.
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A channel $\Gamma : \mathcal{D}_{X_2Y_2 \times A_1B_1} \to \mathcal{D}_{X_1Y_1 \times A_2B_2}$ is **strongly no-signalling (SNS)** if

$$\sum_{b_2 \in B_2} \Gamma(x_1y_1, a_2b_2|x_2y_2, a_1b_1) = \sum_{b_2 \in B_2} \Gamma(x_1y_1, a_2b_2|x_2y_2, a_1b_1'), \quad b_1, b_1' \in B_1,$$

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- Operator matrix $P = (P_{xy,ab})$ is **NS** if marginal operators $P_{xa} = \sum_b P_{xy,ab}$ and $P^{yb} = \sum_a P_{xy,ab}$ are well-defined.

SNS correlation classes

A NS operator matrix $P = (P_{xy,ab})_{xy,ab}$ is **dilatable** if there is an isometry $V : \mathcal{H} \to \mathcal{K}$ between Hilbert spaces and mutually commuting POVM's $(E_{xa})_{a\in A}, (F_{yb})_{b\in B}$ on $\mathcal{K}$ with

$$P_{xy,ab} := V^* E_{xa} F_{yb} V, \quad x \in X, y \in Y, a \in A, b \in B.$$

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- local if Γ is quantum and individual entries in operator matrices P, Q commute with themselves as well.

Homomorphisms of non-local games

Theorem (H.-Todorov, in prep. 2022)

Let Γ be an SNS correlation over the quadruple $(X_2Y_2, A_1B_1, X_1Y_1, A_2B_2)$ and $\mathcal{E}$ be an NS correlation over (X_1, Y_1, A_1, B_1). Then

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$$\tilde{E}_{x_2,a_2} = \sum_{x_1 \in X_1} \sum_{a_1 \in A_1} P_{x_2,x_1} Q_{a_1,a_2} \otimes E_{x_1,a_1}, \quad \tilde{F}_{y_2,b_2} = \sum_{y_1 \in Y_1} \sum_{b_1 \in B_1} P^{y_2,y_1} Q^{b_1,b_2} \otimes F_{y_1,b_1}.$$

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We then have qc-decomposition $\Gamma[\mathcal{E}](a_2, b_2|x_2, y_2) = \langle \tilde{E}_{x_2,a_2} \tilde{F}_{y_2,b_2} (\xi \otimes \eta), \xi \otimes \eta \rangle$. (Others follow similarly).

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- Non-local games are now t-isomorphic if we can find perfect SNS strategies $\Gamma \in \mathcal{C}_{\mathrm{t}}^{\mathrm{bi}}(E_1 \leftrightarrow E_2)$.
- If $E_1 \rightarrow_{\mathrm{st}} E_2$ or $E_1 \simeq_{\mathrm{st}} E_2$, we simulate optimal strategies for E_1 using SNS bicorrelation Γ and get strategies for E_2.

Thank you for listening!

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