Uniform ergodicity of operators associated to random walks on locally compact groups

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Joint work with Enrique Jordá (UPV, Valencia) and Alberto Rodríguez (UJI, Castellón)

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- Let G be a locally compact group. A measure $\mu \in M(G)$ defines a random walk on G. An initial distribution, a probability measure θ is given. At each step, the probability of moving from a set $A \subseteq G$ to a subset $B \subseteq G$, is given by the *transition probability* $\mu(B^{-1}A)$. The distribution of probabilities then changes after each step.

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- Vaguely speaking, random walks are ergodic when, regardless of the initial distribution, every subset is visited, with probability one, as often as its size would predict.
- More precisely, the random walk is ergodic if its corresponding Markov operator P is ergodic, i.e., if $P^*f = f$, $f \in L_\infty(G)$, implies f constant a.e..

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Let $T\colon E \to E$ a bounded operator, E a Banach space. Put $T_{[n]} = \frac{1}{n}\sum_{k=0}^{n-1} T^k$. We say that T is mean ergodic (ME) if the sequence $(T_{[n]})_n$ is convergent in the SOT topology. If $(T_{[n]})_n$ converges in the operator norm, we say that T is uniformly mean ergodic (UME).

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- **Notation:** Let $L_1^0(G) = \left\{f \in L^1(G)\colon \int f(x)\mathrm{d}m_G(x) = 0\right\}$ denote the augmentation ideal of $L_1(G)$. For $\mu \in M(G)$, denote $\lambda_1^0(\mu) = \lambda_1\big|_{L_1^0(G)}$. Thus λ_1^0 is just the convolution operator $f \mapsto \mu * f$ restricted to $L_1^0(G)$.

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- **Rewording:** A measure $\mu \in M(G)$ is ergodic iff $\lambda_1^0(\mu)$ is mean ergodic **and** $\lim_n \lambda_1^0(\mu)_{[n]} = 0$.

Some context

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5. (*The complete mixing problem, Lin*) Is it true that ergodic + strictly aperiodic =completely mixing?

The operators $\lambda_1(\mu)$ and $\lambda_1^0(\mu)$

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Theorem 4 (G.-Jordá 2021)

$\lambda_1(\mu)$ is mean ergodic if and only if H_μ is a compact group. If $\mu^* * \mu = \mu * \mu^*$, then $\lambda_1(\mu)$ is uniformly mean ergodic if and only if $\mathbf{1}$ is isolated in $\sigma(\lambda_1(\mu))$.

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If H_μ is compact, λ_1 is vague-SOT seq. continuous (and the sequence $\mu_{[n]}$ is weak* convergent).

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Question 1

What about $\lambda_1^0(\mu)$?

Uniform mean ergodicity of $\lambda_1(\mu)$ vs uniform mean ergodicity of $\lambda_1^0(\mu)$

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Obvious: $\lambda_1(\mu)$ is UME $\implies$ $\lambda_1^0(\mu)$ is UME

Lemma 5

Let $\mathfrak{A}$ be a Banach algebra and let I be an ideal of $\mathfrak{A}$. Let $M\colon \mathfrak{A} \to \mathfrak{A}$ be a multiplier of $\mathfrak{A}$. Assume that for every $a \in \mathfrak{A}$, $\|M\big|_I\| \geq \alpha > 0$. Then $\sigma_{\mathrm{ap}}(M) = \sigma_{\mathrm{ap}}\left(M\big|_I\right)$.

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Uniform mean ergodicity of $\lambda_1(\mu)$ vs uniform mean ergodicity of $\lambda_1^0(\mu)$

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Obvious: $\lambda_1(\mu)$ is UME $\implies$ $\lambda_1^0(\mu)$ is UME

Lemma 5

Let $\mathfrak{A}$ be a Banach algebra and let I be an ideal of $\mathfrak{A}$. Let $M\colon \mathfrak{A} \to \mathfrak{A}$ be a multiplier of $\mathfrak{A}$. Assume that for every $a \in \mathfrak{A}$, $\|M\big|_I\| \geq \alpha > 0$. Then $\sigma_{\mathrm{ap}}(M) = \sigma_{\mathrm{ap}}\left(M\big|_I\right)$.

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$\|\lambda_1^0(\mu)\| \geq \dfrac{1}{2}\sup_{x \in G}\|\mu - \mu * \delta_x\| \implies \|\lambda_1^0(\mu)\| = 1$ *whenever G is not compact.*

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Lemma 6 tells us Lemma 5 can be applied to $\mathfrak{A} = L_1(G)$, $I = L_1^0(G)$ and $M = \lambda_1(\mu)$.

Uniformly ergodic measures

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Some necessary conditions:

- (Lin and Witmann, 1994) If μ is ergodic, then $G = \overline{\bigcup_j \bigcup_k S_\mu^{-j} S_\mu^k}$.

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μ is said to be **spread-out** if for some n, μ^n is not singular (w.r.t. dm_G).

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This is a Corollary of a Theorem of Dunford because, in this case, $\lambda_1(\mu)$ is then a quasi-compact operator.

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If 1 is isolated in $\sigma(\mu)$ then μ is spread-out.

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Sketch of proof.

If μ is not spread-out, $r_{M(G)/L_1(G)}(\mu) = 1$. Pick $\chi \in \Delta(M(G)/L_1(G))$ with $|\langle \chi, \mu \rangle| = 1$. Šreĭder's theory of generalized characters is then used to find $\tilde{\chi}$ with $\langle \tilde{\chi}, \mu \rangle = 1$.

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Theorem 11

Let $H_\mu = G$ (i.e., μ is adapted). TFAE:

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μ is uniformly completely mixing when $\lim_n \|\lambda_1^0(\mu^n)\| = 0$.

Theorem 12 (The uniformly complete mixing problem)

Let G be a compact Abelian group and let μ be adapted. TFAE:

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There are easy examples of sequences of UME operators T with $\mathrm{SOT} - \lim_n T^n = 0$ convergent but $\|T^n\|$ is constant: Pick a sequence $\mathbf{a} := (a_n)_n \in \ell_1(\mathbb{Z})$ with $\lim_{n\to\pm\infty} = -1$ but $|a_n| < 1$ for every n. Then the multiplication operator $M_a\colon \ell_1(\mathbb{Z}) \to \ell_1(\mathbb{Z})$ is UME and $\mathrm{SOT}\text{-}\lim_n M_a^n = 0$ but $\|M_a^n\| = 1$.

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Let G be a compact Abelian group and let μ be adapted. TFAE:

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2. μ is *strictly aperiodic* and *uniformly ergodic.*
3. μ is *strictly aperiodic* and *spread out .*
4. μ is *completely mixing* and *uniformly ergodic.*

Corollary 13

If G is connected then, TFAE:

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If G is connected then, TFAE:

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μ is uniformly completely mixing when $\lim_n \|\lambda_1^0(\mu^n)\| = 0$.

Theorem 12 (The uniformly complete mixing problem)

Let G be a compact Abelian group and let μ be adapted. TFAE:

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In this case μ uniformly ergodic implies that μ is strictly aperiodic.

Definition 14

If G is a locally compact *Abelian* group, the **Fourier algebra**, $\mathbf{A(G)}$, is defined as

$$A(G) = \left\{ \widehat{f} \colon G \to \mathbb{C} \, : \, f \in L_1(\widehat{G}) \right\} \quad \widehat{f} \text{ is the Fourier transform of } f.$$

Since $\widehat{f * g} = \widehat{f} \cdot \widehat{g}$, pointwise multiplication and the norm $\|\widehat{\mathbf{f}}\|_{\mathbf{A(G)}} = \|\mathbf{f}\|_{\mathbf{1}}$, turn $A(G)$ into a **Banach algebra** naturally isomorphic to $L_1(\widehat{G})$.

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If G is a locally compact *Abelian* group the **Fourier-Stieltjes algebra**, **B(G)** is defined as:

$$B(G) = \left\{ \widehat{\mu} \colon G \to \mathbb{C} : \ \mu \in M(\widehat{G}) \right\} \quad \widehat{\mu} \text{ is the Fourier-Stieltjes transform of } \mu.$$

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Both can be seen as algebras of functions on G made of matrix coefficients of unitary representations. All of them in the case of $B(G)$ and those of the left regular representation in the case of $A(G)$. $A(G)$ is a closed ideal in $B(G)$.

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Question: under which conditions are ϕ and the operators $M(\phi))$ and $M^0(\phi)$ mean or uniformly mean ergodic?

$P^1(G) := \{\phi \in B(G) \colon \phi \text{ positive definite}, \phi(e) = 1\}$.

Definition 17

Let G be a locally compact group, $\mu \in M(G)$ and $\phi \in B(G)$. We define

$$H_\mu := \overline{\langle \mathrm{supp}(\mu) \rangle} \quad \text{smallest closed subgroup of } G \text{ containing the support of } \mu.$$

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Theorem 18

Let G be a locally compact group and let $\mu \in M(G)$ be a probability measure. Then

$$\lambda_1(\mu) \text{ is mean ergodic} \iff H_\mu \text{ is compact.}$$

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Theorem 19

Let G be a locally compact group and let $\phi \in P^1(G)$. Then,

$$M(\phi) \text{ is mean ergodic} \iff H_\phi \text{ is open.}$$

G a locally compact group. μ prob. measure, $\phi \in P^1(G)$.

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Let $\mu \in M(G)$ and $\phi \in B(G)$. Then:

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Key fact: $\|\lambda_1^0(\mu)\| \geq \frac{1}{2} \sup_{x\in G} \|\mu - \mu * \delta_x\|$.

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Key fact: $\|\lambda_1^0(\mu)\| \geq \frac{1}{2} \sup_{x \in G} \|\mu - \mu * \delta_x\|$. It follows that if G is not compact, then $\|\lambda_1^0(\mu)\| = 1$ for every probability measure μ.

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Let G be amenable. If ϕ is uniformly ergodic, then G is discrete.

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$$\sigma_{M(G)}(\mu) = \sigma\left(\lambda_1^0(\mu)\right) \cup \{1\}$$

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μ *is uniformly ergodic* **if and only if** *there is n_0 s.t. μ^{n_0} is not singular (μ is spread-out).*

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Theorem 25

μ *is uniformly ergodic* **if and only if** *there is n_0 s.t. μ^{n_0} is not singular (μ is spread-out).*

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Some pending tasks

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- Is it true that $\sigma(\phi) = \sigma(M(\phi))$ when G is **not** amenable?

THANKS FOR YOUR ATTENTION