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Arens Regularity of Weighted Semigroup Algebras of Totally Ordered Semigroups

Banach Algebras and Applications
Universidad de Granada
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Notation and motivation

- Given E a Banach space. We shall denote as E' its dual space and as E'' the bidual.
- Given a set S, βS is the Stone-Čech compactification of S and $S^* = \beta S \setminus S$.
- Given a set S and $s \in S$, δ_s is the characteristic function of $\{s\}$.

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Essential property of Banach spaces is that the canonical embedding $\kappa_E : E \longrightarrow E''$ is an isometric isomorphism.

Given a Banach algebra A.
Question:Is there any product in A'' such that $\kappa_A : A \longrightarrow A''$ is also an algebra homomorphism?

Two products in A'' give an affirmative answer to that question: the first and second Arens products.

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First and second Arens products

Definition

Let $M, N \in A''$. Take (a_α), (b_β) in A, such that $M = \lim_\alpha a_\alpha$ and $N = \lim_\beta b_\beta$ with respect to the weak-$*$ topology.

Then the *first and second Arens products* are

$$M \Box N = \lim_\alpha \lim_\beta a_\alpha b_\beta, \quad M \Diamond N = \lim_\beta \lim_\alpha a_\alpha b_\beta,$$

where the limits are again in the weak-$*$ topology on A''.

It follows that both $\Box$ and $\Diamond$ are associative and so $(A'', \Box)$ and $(A'', \Diamond)$ are Banach algebras containing A as a closed subalgebra. [Arens,1951]

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Topological centres

Definition

Let A be a Banach algebra.

- *Left topological centre* of A''

$$\mathfrak{Z}^{(\ell)}(A'') = \{M \in A'' : M \Box N = M \Diamond N \quad (N \in A'')\}.$$

- *Right topological centre* of A''

$$\mathfrak{Z}^{(r)}(A'') = \{M \in A'' : N \Box M = N \Diamond M \quad (N \in A'')\}.$$

Remark

When A is commutative, $M \Box N = N \Diamond M \quad (M, N \in A'')$.

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How big can these sets be?

- $A \subset \mathfrak{Z}^{(\ell)}(A'')$, (repectively $A \subset \mathfrak{Z}^{(r)}(A'')$).
- $\mathfrak{Z}^{(\ell)}(A'') \subset A''$, (repectively $\mathfrak{Z}^{(r)}(A'') \subset A''$).

Definition

Let A be a Banach algebra.

- A is *Arens regular (AR)* if $\mathfrak{Z}^{(\ell)}(A'') = \mathfrak{Z}^{(r)}(A'') = A''$;
- A is *strongly Arens irregular (SAI)* if $\mathfrak{Z}^{(\ell)}(A'') = \mathfrak{Z}^{(r)}(A'') = A$. [Dales-Lau, 2005]

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Other characterization of Arens regularity

Theorem

Let A be a Banach algebra. Then A is Arens regular if and only if $\mathcal{WAP}(A) = A'$.

This characterization leads to a different "measurement" of Arens irregularity, known as *extremely non-Arens regularity*, [Granirer, 1996].

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Weighted semigroup algebras

Definitions

- Let S be a semigroup. A function $\omega : S \longrightarrow (0, \infty)$ is a *weight* on S if

$$\omega(st) \leq \omega(s)\omega(t) \quad (s, t \in S).$$

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$$\mathcal{A}_\omega := \ell^1(S, \omega) = \left\{ \alpha = \sum_{s \in S} \alpha(s)\delta_s : ||\alpha||_\omega = \sum_{s \in S} |\alpha(s)|\omega(s) < \infty \right\},$$

with convolution product:

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Weighted semigroup algebras

Known facts

- These Banach algebras are semi-simple and they give Banach Function Algebras.
- $\mathcal{A}'_\omega := \ell^\infty(S, 1/\omega) = \left\{\lambda \in \mathbb{C}^S : \sup\{|\lambda(s)|/\omega(s) : s \in S\} < \infty\right\},$

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Definition

Let S be a semigroup and $\omega : S \to (0, \infty)$. We define

$$\Omega : S \times S \longrightarrow \mathbb{R}, \quad (s, t) \mapsto \frac{\omega(st)}{\omega(s)\omega(t)}.$$

We say that Ω $0-$clusters on $S \times S$ if, for $(x_n), (y_m)$ sequences of distinct elements of S then

$$\lim_{n\to\infty} \lim_{m\to\infty} \Omega(x_n, y_m) = \lim_{m\to\infty} \lim_{n\to\infty} \Omega(x_n, y_m) = 0$$

whenever both iterated limits exist.

Theorem (Craw-Young,1973)

Let S be a semigroup and ω a weight on S. If Ω $0-$clusters on $S \times S$ then $\ell^1(S, \omega)$ is (AR). If S is a [weakly] cancellative semigroup then $\ell^1(S, \omega)$ (AR) implies that Ω $0-$clusters.

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$$\lim_{n\to\infty} \lim_{m\to\infty} \Omega(x_n, y_m) = \lim_{m\to\infty} \lim_{n\to\infty} \Omega(x_n, y_m) = 0$$

whenever both iterated limits exist.

Theorem (Craw-Young,1973)

Let S be a semigroup and ω a weight on S. If Ω 0−clusters on $S \times S$ then $\ell^1(S,\omega)$ is (AR). If S is a [weakly] cancellative semigroup then $\ell^1(S,\omega)$ (AR) implies that Ω 0−clusters.

Closer look at $S = \mathbb{N}_\wedge$

Consider the semigroup $S := \mathbb{N}$ with the semigroup operation

$$m \wedge n = \min\{m, n\} \quad (m, n \in \mathbb{N}).$$

Every sequence $\omega : \mathbb{N} \longrightarrow [1, \infty)$ is a weight on $\mathbb{N}$.

Theorem (Dales-Dedania, 2009)

Consider the semigroup $\mathbb{N}_\wedge$ and let $\omega : \mathbb{N} \to [1, \infty)$ such that $\lim_{n\to\infty} \omega(n) = \infty$. Then $\mathcal{A}_\omega = \ell^1(\mathbb{N}_\wedge, \omega)$ is Arens regular.

Proposition (Dales-Lau-Strauss,2010)

Let consider the semigroup $\mathbb{N}_\wedge$. The Banach algebra $\ell^1(\mathbb{N}_\wedge)$ is (SAI) and it has a 2 point DTC set.

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Generalization

In the case where $S = \mathbb{N}$ we have an interesting dichotomy

- $\mathcal{A}_\omega$ is (AR) when $\lim \omega = \infty$.
- $\mathcal{A}_\omega$ is (SAI) when $\liminf \omega < \infty$.

When we are trying to generalise to a semigroup S, there are several questions.

- What does $\lim \omega = \infty$ mean?
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Definitions

Let S be an infinite set. Let $\omega : S \longrightarrow \mathbb{R}$.

- Lim $\omega = C \quad (C \in \mathbb{R})$: $\forall \varepsilon > 0$, there is a finite set F of S such that

$$|\omega(s) - C| < \varepsilon \quad (s \in S \setminus F).$$

- Lim $\omega = \infty$: $\forall M > 0$, there is a finite set F of S such that

$$\omega(s) > M \quad (s \in S \setminus F).$$

- Lim inf $\omega < \infty$ iff it is not true that Lim $\omega = \infty$, i.e. there exists $M > 0$ such that the set $\{s \in S : \omega(s) < M\}$ is inifinite.

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Do they work in $\mathbb{N}$?

When $S = \mathbb{N}$ these definitions are equivalent the classical limits.

Are they different in any other set S?

Let $S = \mathbb{Q}^{+\bullet} = \{s \in \mathbb{Q} : s > 0\}$

- $\omega : \mathbb{Q}^{+\bullet} \to [1, \infty)$ such that $\omega(s) = 1$ $(s \in [1, 2])$, $\lim_{s\to\infty} \omega(s) = \infty$. Then Lim inf $\omega < \infty$.
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Totally ordered semigroups

Definitions

- Let S be an infinite, semi-lattice and consider the semigroup operation

$$s \wedge t = \min\{s, t\} \quad (s, t \in S).$$

- A *completion of S* is a set $T \supset S$ s.t.
  - T is a totally ordered set that preserves the order in S;
  - T has a minimum and a maximum,
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Main result

Theorem (C.)

Let $(S, \wedge)$ be an infinite semi-lattice. Let ω a weight on S. Then the following conditions are equivalent:

(a) the algebra $\mathcal{A}_\omega$ is Arens regular;

(b) $\mathrm{Lim}_{s\to\infty}\,\omega(s) = \infty$;

(c) $M\Box N = M\Diamond N = 0$ $(M, N \in E_\omega^\perp)$.

Sketch of the proof

(b) $\Rightarrow$(c) $\Longleftrightarrow$(a) follows from E_ω Banach algebra predual
(a) $\Rightarrow$ (b) When Lim $\inf \omega < \infty$ we can find $p, q \in \mathcal{A}_\omega{}'' \setminus \mathcal{A}_\omega$ such that $p\Box q \neq p\Diamond q$.

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Example 1

Let $S = \mathbb{Z}$, and $T = \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$
Consider ω a weight on S such that

$$\lim_{n\to\infty} \omega(n) = \infty, \quad \sup_{n<0} \omega(n) \leq M < \infty,$$

for $M \geq 1$. Then $\mathcal{A}_\omega$ is neither Arens regular nor strongly Arens irregular.

Example 2

Let $S = \mathbb{Q}^{+\bullet} = \{p \in \mathbb{Q} : p > 0\}$.
Consider $\omega : \mathbb{Q} \to [1, \infty)$ such that $\omega(p) = 1$ $(p \in [0,1] \cap S)$ and such that $\lim_{p\to\infty} \omega(p) = \infty$. Then $\mathcal{A}_\omega$ is not strongly Arens irregular.

Example 1

Let $S = \mathbb{Z}$, and $T = \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$
Consider ω a weight on S such that

$$\lim_{n\to\infty} \omega(n) = \infty, \quad \sup_{n<0} \omega(n) \leq M < \infty,$$

for $M \geq 1$. Then $\mathcal{A}_\omega$ is neither Arens regular nor strongly Arens irregular.

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Final example

Given S such that $\mathrm{cl}_T\, S$ is not scattered, and ω on S such that $\mathcal{A}_\omega$ is strongly Arens irregular.

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Thank you for your attention

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