The doubles of a numerical semigroup

A.M. Robles-Pérez, J.C. Rosales, and P. Vasco

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First definitions

Definition

A numerical semigroup is a subset of $\mathbb{N}$ (nonnegative integers) that is closed under addition, contains the zero element, and has finite complement in $\mathbb{N}$.

Definition

$\frac{S}{p} = \{x \in \mathbb{N} \mid px \in S\}$ *is the* quotient *of S by p.*

- $\frac{S}{2}$ is the half of S
- $\frac{S}{4}$ is the quarter of S
- S is a double of $\frac{S}{2}$

Sets of doubles

Definition

Set of doubles of a numerical semigroup.

$$\mathrm{D}(S) = \left\{ \bar{S} \mid \frac{\bar{S}}{2} = S \right\}$$

Definition

Set of doubles of a numerical semigroup with fixed smallest odd integer.

$$\mathrm{D_m}(S) = \left\{ \bar{S} \in \mathrm{D}(S) \mid m \text{ is the smallest odd integer of } \bar{S} \right\}$$

More notation

- $\mathrm{G}(S)$ is the set of element of $\mathbb{N} \backslash S$: set of gaps of S.
- $\mathrm{g}(S)$ is the cardinality of $\mathrm{G}(S)$: genus of S or singularity degree of S.
- $\mathrm{F}(S)$ is the greatest integer that does not belong to S: Frobenius number of S.

- $A + B = \{a + b \mid a \in A \text{ and } b \in B\}$
- $2A = \{2a \mid a \in A\}$ (not to be confused with $A + A$)

Example

$$S = \{0,3,5,6,8,\rightarrow\} = \{0,3,5,6\} \cup \{z \in \mathbb{Z} \mid z \geq 8\}$$

$$\frac{S}{2} = \left\{\frac{0}{2},\frac{6}{2},\frac{8}{2},\rightarrow\right\} = \{0,3,\rightarrow\}$$

$$\frac{S}{4} = \left\{\frac{0}{4},\frac{8}{4},\rightarrow\right\} = \{0,2,\rightarrow\}$$

$$\mathrm{G}(S) = \{1,2,4,7\}$$

$$\mathrm{g}(S) = 4$$

$$\mathrm{F}(S) = 7$$

"Basic" double

Definition

Let S be a numerical semigroup and let m be an odd integer of S.

$$S(m) = 2S \cup (2S + \{m\})$$

Lemma

Let S be a numerical semigroup and let m be an odd integer of S.

1. $S(m) \in \mathrm{D}_m(S)$.
2. $S(m) \subseteq \bar{S}$ *if* $\bar{S} \in \mathrm{D}_m(S)$.

Gaps of the "basic" double

Lemma

Let $\mathrm{G}(S) = \left\{h_1 < h_2 < \ldots < h_{g(S)}\right\}$. *Then*

$$\mathrm{G}(S(m)) = \mathrm{G}_0(S(m)) \cup \mathrm{G}_{m-}(S(m)) \cup \mathrm{G}_{m+}(S(m)),$$

where

1. $\mathrm{G}_0(S(m)) = 2\mathrm{G}(S) = \left\{2h_1 < 2h_2 < \ldots < 2h_{g(S)}\right\}$;
2. $\mathrm{G}_{m-}(S(m)) = \{1, 3, \ldots, m-2\}$;
3. $\mathrm{G}_{m+}(S(m)) = 2\mathrm{G}(S) + \{m\} = \left\{2h_1 + m < \ldots < 2h_{g(S)} + m\right\}$.

Example

$$S = \{0,3,5,6,8,\rightarrow\}, \quad \mathrm{G}(S) = \{1,2,4,7\}$$

$$S(5) = 2S \cup (2S+\{5\}) = \{0,6,10,12,16,...\} \cup \{5,11,15,17,21,...\} =$$

$$\{0,5,6,10,11,12,15,16,17,18,20,\rightarrow\}$$

$$\mathrm{G}(S(5)) = \{2,4,8,14\} \cup \{1,3\} \cup \{7,9,13,19\} =$$

$$\{1,2,3,4,7,8,9,13,14,19\}$$

Question

What can we remove in $\{7,9,13,19\}$?

m-upper subset

Definition

Let H *be a subset of* $\mathrm{G}(S)$.

We say that H *is an m-*upper subset of $\mathrm{G}(S)$ *if it verifies the following conditions*

(H1) $\mathrm{H}+\{m\}\subset S$ $(\Leftrightarrow (\mathrm{H}+\{m\})\cap \mathrm{G}(S)=\emptyset)$;

(H2) $\mathrm{H}+\mathrm{H}+\{m\}\subset S$ $(\Leftrightarrow (\mathrm{H}+\mathrm{H}+\{m\})\cap \mathrm{G}(S)=\emptyset)$;

(H3) *if* $h\in \mathrm{H}$, *then* $\{g\in \mathrm{G}(S)\mid h\leq_S g\}\subseteq \mathrm{H}$.

(Order relation in $\mathrm{G}(S)$: $g_1\leq_S g_2 \Leftrightarrow g_2-g_1\in S$*)*

Independence of the conditions

Example

$$S = \{0,3,5,6,8,\rightarrow\}, \quad \mathrm{G}(S) = \{1,2,4,7\}$$

$$m = 5$$

1. $\mathrm{H} = \{2,7\}$ satisfies (H2) and (H3) but not (H1).
2. $\mathrm{H} = \{1,4,7\}$ satisfies (H1) and (H3) but not (H2).
3. $\mathrm{H} = \{4\}$ satisfies (H1) and (H2) but not (H3).

In these cases $S(5) \cup (2\mathrm{H} + \{5\})$ is not a numerical semigroup.

Central lemma

Lemma

Let S be a numerical semigroup, let m be an odd integer of S, and let H be a subset of $\mathrm{G}(S)$. The following conditions are equivalent

1. $\bar{S} = S(m) \cup (2\mathrm{H} + \{m\}) \in \mathrm{D}_m(S)$.
2. H *is an m-upper subset of* $\mathrm{G}(S)$.

Definition

$$S(m,\mathrm{H}) = S(m) \cup (2\mathrm{H} + \{m\}) = S \cup (S + \{m\}) \cup (2\mathrm{H} + \{m\})$$

$$S(m) = S(m,\emptyset)$$

Example

$$S = \{0,3,5,6,8,\to\}, \quad \mathrm{G}(S) = \{1,2,4,7\}$$

$$m = 5$$

1. $S(5,\emptyset) = S(5) = \{0,5,6,10,11,12,15,16,17,18,20,\to\}.$
2. $S(5,\{7\}) = S(5) \cup (2\{7\}+\{5\}) = \{0,5,6,10,11,12,15,\to\}.$
3. $S(5,\{4,7\}) = S(5) \cup (2\{4,7\}+\{5\})$ $= \{0,5,6,10,11,12,13,15,\to\}.$

These are all the elements of $\mathrm{D}_5(S)$.

Description of $\mathrm{D}_m(S)$

Theorem

Let S be a numerical semigroup and $m \in S$ an odd integer. Then

$$\mathrm{D}_m(S) = \{S(m,\mathrm{H}) \mid \mathrm{H} \textit{ is } m\textit{-upper subset of } \mathrm{G}(S)\}.$$

Moreover, $S(m_i,\mathrm{H}_i) = S(m_j,\mathrm{H}_j)$ if and only if $m_i = m_j$ and $\mathrm{H}_i = \mathrm{H}_j$.

Remark

1. *Each double $\bar{S}$ has a unique smallest odd integer, m.*
2. *$\bar{S}$ can be in only one set $\mathrm{D}_m(S)$.*
3. *$\{\mathrm{D}_m(S) \mid m \in S \text{ is odd}\}$ is a partition of $\mathrm{D}(S)$.*

¿$\mathrm{H} = \mathrm{G}(S)$?

Proposition

$\mathrm{G}(S)$ *is an m-subset of* $\mathrm{G}(S)$ *if and only if* $m > \mathrm{F}(S)$.

Remark

1. *If* $\bar{S} \in \mathrm{D}_m(S)$ *and* $\mathrm{G}(S)$ *is an m-upper subset of* $\mathrm{G}(S)$ *then* $\bar{S} \subseteq S(m, \mathrm{G}(S))$.
2. $\mathrm{G}_{m+}(S(m, \mathrm{G}(S))) = \emptyset$.
3. $S(m, \mathrm{G}(S)) = 2S \cup (2\mathbb{N} + \{m\})$.

Singularity degree

Lemma

- $\mathrm{G}_0(S(m,\mathrm{H})) = 2\mathrm{G}(S) = \left\{2h_1 < 2h_2 < \ldots < 2h_{g(S)}\right\};$
- $\mathrm{G}_{m-}(S(m,\mathrm{H})) = \{1,3,\ldots,m-2\};$
- $\mathrm{G}_{m+}(S(m,\mathrm{H})) = 2(\mathrm{G}(S)\backslash\mathrm{H}) + \{m\}.$

$S(m,\mathrm{H}) = S \cup (S + \{m\}) \cup (2\mathrm{H} + \{m\})$

Proposition

$\mathrm{g}(S(m,\mathrm{H})) = 2\mathrm{g}(S) + \frac{m-1}{2} - \#\mathrm{H}.$

Frobenius number

Proposition

1. $\mathrm{H} \neq \mathrm{G}(S)$,

$$\mathrm{F}(S(m,\mathrm{H})) = \max\{2\mathrm{F}(S), 2(\max(\mathrm{G}(S)\backslash\mathrm{H})) + m\}.$$

2. $\mathrm{H} = \mathrm{G}(S)$,

$$\mathrm{F}(S(m,\mathrm{H})) = \max\{2\mathrm{F}(S), m-2\}.$$

Corollary

$F(S) \notin \mathrm{H} \ \Rightarrow\ \mathrm{F}(S(m,\mathrm{H})) = 2\mathrm{F}(S) + m.$

Example

$$S=\{0,3,5,6,8,\rightarrow\},\quad \mathrm{G}(S)=\{1,2,4,7\},\quad m=5$$

1. $S(5,\emptyset)=\{0,5,6,10,11,12,15,16,17,18,20,\rightarrow\}$.
   1. $\mathrm{G}(S(5,\emptyset))=\{1,2,3,4,7,8,9,13,14,19\}$, $\mathrm{g}(S(5,\emptyset))=10$.
   2. $\mathrm{F}(S(5,\emptyset))=2\times 7+5=19$.
2. $S(5,\{7\})=\{0,5,6,10,11,12,15,\rightarrow\}$.
   1. $\mathrm{G}(S(5,\{7\}))=\{1,2,3,4,7,8,9,13,14\}$, $\mathrm{g}(S(5,\{7\}))=9$.
   2. $\mathrm{F}(S(5,\{7\}))=\max\{2\times 7,2\times 4+5\}=14$.
3. $S(5,\{4,7\})=\{0,5,6,10,11,12,13,15\rightarrow\}$.
   1. $\mathrm{G}(S(5,\{4,7\}))=\{1,2,3,4,7,8,9,14\}$, $\mathrm{g}(S(5,\{4,7\}))=8$.
   2. $\mathrm{F}(S(5,\{4,7\}))=\max\{2\times 7,2\times 2+5\}=14$.

Definitions

Definition

S is symmetric if

$$x \in \mathbb{Z} \backslash S \;\Rightarrow\; \mathrm{F}(S) - x \in S.$$

Definition

S is pseudo-symmetric if $\mathrm{F}(S)$ *is even and*

$$x \in \mathbb{Z} \backslash S \;\Rightarrow\; \mathrm{F}(S) - x \in S \quad \text{or} \quad 2x = \mathrm{F}(S).$$

Definition

S is equilibrated if $\mathrm{g}_0(S) = \mathrm{g}_1(S)$.

$(\mathrm{g}_0(S) = \#\{h \in \mathrm{G}(S) \mid h \text{ is even}\};\; \mathrm{g}_1(S) = \#\{h \in \mathrm{G}(S) \mid h \text{ is odd}\})$

Half and quarter of numerical semigroups

Proposition

1. *Every numerical semigroup is one half of infinitely many symmetric ones.*
2. *Every numerical semigroup is one half of an equilibrated one.*
3. *Every symmetric numerical semigroup is one half of an equilibrated pseudo-symmetric one.*
4. *Every numerical semigroup is one quarter of infinitely many equilibrated pseudo-symmetric ones.*

Symmetric numerical semigroups

Definition

S is symmetric if

$$x \in \mathbb{Z} \setminus S \;\Rightarrow\; \mathrm{F}(S) - x \in S.$$

Lemma

1. *S symmetric* $\Rightarrow \mathrm{F}(S)$ *odd.*
2. *S is symmetric* $\Leftrightarrow \mathrm{g}(S) = \frac{\mathrm{F}(S)+1}{2}$.

Symmetric numerical semigroups

Definition

Set of gaps of second type.

$$\mathrm{H}_2(S) = \{x \in \mathrm{G}(S) \mid \mathrm{F}(S) - x \in \mathrm{G}(S)\}$$

Lemma

$\#\mathrm{H}_2(S) = 2\mathrm{g}(S) - \mathrm{F}(S) - 1$

Proposition

If $\mathrm{H}_2(S)$ is an m-upper subset of $\mathrm{G}(S)$ then

$S(m, \mathrm{H}_2(S))$ is a symmetric numerical semigroup.

Symmetric numerical semigroups

Lemma

$H_2(S)$ is an m-upper subset of $G(S)$ if $m > F(S)$.

Example

1. $S = \{0,4,5,7,\rightarrow\}$, $G(S) = \{1,2,3,6\}$, $F(S) = 6$
   $H_2(S) = \{3\}$ is a 5-upper subset of $G(S)$.
2. $S = \{0,4,5,8,\rightarrow\}$, $G(S) = \{1,2,3,6,7\}$, $F(S) = 7$
   $H_2(S) = \{1,6\}$ is not a 5-upper subset of $G(S)$.

Corollary

Every numerical semigroup is one half of infinitely many symmetric ones.

Equilibrated pseudo-symmetric numerical semigroups

Definition

S is pseudo-symmetric if $\mathrm{F}(S)$ is even and

$$x \in \mathbb{Z} \backslash S \;\Rightarrow\; \mathrm{F}(S) - x \in S \quad \textit{or} \quad 2x = \mathrm{F}(S).$$

Definition

S is equilibrated if

$$\mathrm{g}_0(S) = \mathrm{g}_1(S).$$

$(\mathrm{g}_0(S) = \#\{h \in \mathrm{G}(S) \mid h \textit{ is even}\};\; \mathrm{g}_1(S) = \#\{h \in \mathrm{G}(S) \mid h \textit{ is odd}\})$

Equilibrated pseudo-symmetric numerical semigroups

Lemma

1. *S is pseudo-symmetric* $\Leftrightarrow \mathrm{g}(S) = \frac{\mathrm{F}(S)+2}{2}$.
2. *S is equilibrated* $\Leftrightarrow \mathrm{g}(S) = 2\mathrm{g}\left(\frac{S}{2}\right)$.

Proposition

$\bar{S} = S(2g(S)+1, \mathrm{G}(S))$ is an equilibrated pseudo-symmetric numerical semigroup.
Moreover, $\mathrm{F}(\bar{S}) = 2\mathrm{F}(S)$ *and* $\mathrm{g}(\bar{S}) = 2\mathrm{g}(S)$.

Corollary

Every numerical semigroup is one half of an equilibrated one.

Equilibrated pseudo-symmetric numerical semigroups

Lemma

S is symmetric $\Leftrightarrow S(2g(S)+1, \mathrm{G}(S))$ is pseudo-symmetric.

Proposition

Every symmetric numerical semigroup is one half of an equilibrated pseudo-symmetric one.

Corollary

Every numerical semigroup is one quarter of infinitely many equilibrated pseudo-symmetric ones.

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