Some remarks on weak compactness in the dual space of a JB*-triple

Antonio M. Peralta*

Abstract

We obtain several characterizations of the relatively weakly compact subsets of the predual of a JBW*-triple. As a consequence we describe the relatively weakly compact subsets of the predual of a JBW*-algebra.

1 Introduction

The study of relatively weakly compact subsets of the predual of a von Neumann algebra is mainly due to Takesaki [24], Akemann [2], Akemann, Dodds and Gamlen [3] and Saitô [22]. Their results on characterizations of relatively weakly compact subset in the predual of a von Neumann algebras were the key tool for the description of weakly compact operators from a C*-algebra to a complex Banach space found by Jarchow [17, 18].

Every von Neumann algebra belongs to a more general class of Banach spaces known as JBW*-triples. A JB*-triple is a complex Banach space equipped with a Jordan triple product satisfying certain algebraic and geometric properties (see definition below). JB*-triples were introduced by Kaup [19] in the study of bounded symmetric domains in complex Banach spaces. The class of JB*-triples contains all C*-algebras and all JB*-algebras. A JBW*-triple is a JB*-triple which is also a dual Banach space, thus every von Neumann is a JBW*-triple.

The study of weakly compact operators from a JB*-triple to a Banach space was obtained in [9] and [21, Theorem 10 and the following remarks]. However, contrarily to the case of a C*-algebra, the characterization of weakly compact operators from a JB*-triple to a complex Banach space

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was not obtained by describing the relatively weakly compact subsets of the predual of a JBW*-triple. the objective of this paper is to describe the relatively weakly compact subsets in the predual of a JBW*-triple. Theorem 2.1 and Corollary 2.4 generalize the classical description of relatively weakly compact subsets in the predual of a von Neumann algebra to the setting of JBW*-triple preduals. The above results are particularized to JBW*-algebra preduals in Theorem 2.5.

As a consequence of our results we prove that for every norm bounded sequence (ϕ_n) in the predual of a JBW*-triple W, then for each norm-one functional $\varphi \in W_*$ and for every $\varepsilon > 0$ there exists a tripotent $e \in W$ such that $\varphi(e) > 1 - \varepsilon$ and (ϕ_n) admits a subsequence which converges weakly to a functional in $(W_2(e))_*$, where $W_2(e)$ is the Peirce 2-subspace associated to e. This result extends [7] to the setting of JBW*-triples.

Let X be a Banach space. Throughout the paper, B_X and X^* denote the closed unit ball of X and the dual space of X, respectively. If X is a dual Banach space, X_* will stand for the predual of X_*.

2 Weakly compact sets in the dual of a JB*-triple

A *JB*-triple* is a complex Banach space E equipped with a continuous triple product

$$\{.,.,.\} : E \otimes E \otimes E \to E$$

$$(x, y, z) \mapsto \{x, y, z\}$$

which is bilinear and symmetric in the outer variables and conjugate linear in the middle one and satisfies:

(a) (*Jordan Identity*)

$$L(x, y)\{a, b, c\} = \{L(x, y)a, b, c\} - \{a, L(y, x)b, c\} + \{a, b, L(x, y)c\},$$

for all $x, y, a, b, c \in E$, where $L(x, y) : E \to E$ is the linear mapping given by $L(x, y)z = \{x, y, z\}$;

(b) The map $L(x, x)$ is an hermitian operator with non-negative spectrum for all $x \in E$;

(c) $\| \{x, x, x\} \| = \|x\|^3$ for all $x \in E$.

Every C*-algebra is a JB*-triple with respect to the triple product

$$\{x, y, z\} = 2^{-1}(xy^*z + zy^*x).$$

Every JB*-algebra is a JB*-triple with triple product given by

$$\{a, b, c\} = (a \circ b^*) \circ c + (c \circ b^*) \circ a - (a \circ c) \circ b^*.$$

The Banach space $B(H, K)$ of all bounded linear operators between two complex Hilbert spaces H, K is also an example of a JB*-triple with product $\{R, S, T\} = 2^{-1}(RS^*T + TS^*R)$.

A JBW*-triple is a JB*-triple which is also a dual Banach space. The bidual, E^{**}, of every JB*-triple, E, is a JBW*-triple with triple product extending the product of E (cf. [11]).

Let E be a JB*-triple. An element $e \in E$ is said to be a tripotent if $\{e, e, e\} = e$. The set of all tripotents of E is denoted by $\mathrm{Tri}(E)$. Given a tripotent $e \in E$ there exist a decomposition of E in terms of the eigenspaces of $L(e, e)$ given by

$$E = E_0(e) \oplus E_1(e) \oplus E_2(e), \tag{1}$$

where $E_k(e) := \{x \in \mathcal{E} : L(e, e)x = \frac{k}{2}x\}$ is a subtriple of E $(k : 0, 1, 2)$. The natural projection of E onto $E_k(e)$ will be denoted by $P_k(e)$. The following rules are also satisfied

$$\{E_k(e), E_l(e), E_m(e)\} \subseteq E_{k-l+m}(e),$$

$$\{E_0(e), E_2(e), E\} = \{E_2(e), E_0(e), E\} = 0,$$

where $E_{k-l+m}(e) = 0$ whenever $k - l + m$ is not in $\{0, 1, 2\}$. It is also known that $E_2(e)$ is a unital JB*-algebra with respect to the product and involution given by $x \circ y = \{x, e, y\}$ and $x^* = \{e, x, e\}$, respectively. When E is a JBW*-triple then $E_2(e)$ is a JBW*-algebra.

For background about JB- and JBW-algebras the reader is referred to [14]. We recall that JB-algebras (respectively, JBW-algebras) are nothing but the self-adjoint parts of JB*-algebras (respectively, JBW*-algebras) [26] (respectively, [12]).

Two tripotents e, f in a JB*-triple E are said to be orthogonal if e belongs to $E_0(f)$ and f belongs to $E_0(e)$. Let $e, f \in E$. Following [20, §5], we say that $e \leq f$ if and only if $f - e$ is a tripotent which is orthogonal to e. It is also known that $e \leq f$ if and only if e is a symmetric projection in $E_2(f)$.

Let W be a JBW*-triple and let φ a norm-one element in W_*. Let z be a norm-one element in W such that $\varphi(z) = 1$. By [4] the mapping

$(x,y) \mapsto \varphi\{x,y,z\}$ defines a positive sesquilinear form on W which does not depend on the element z. Thus the law $x \mapsto \|x\|_\varphi := (\varphi\{x,x,z\})^{\frac{1}{2}}$ $(x \in W)$ defines a prehilbert seminorm on W. If E is a JB*-triple and φ is a norm-one element in E^* then $\|.\|_\varphi$ is a prehilbertian seminorm on E^{**} and hence on E. The strong*-topology of W, introduced by Barton and Friedman in [5], is the topology on W generated by the family of seminorms $\{\|.\|_\varphi : \varphi \in S_{W_*}\}$. We use the symbol $S^*(W,W_*)$ to denote the strong*-topology of W. When φ_1, φ_2 are two norm-one functionals in W_*, then we write $\|.\|_{\varphi_1,\varphi_2}$ for the hilbertian semi-norm defined by

$$\|x\|^2_{\varphi_1,\varphi_2} := \|x\|^2_{\varphi_1} + \|x\|^2_{\varphi_2}.$$

If A is a JBW*-algebra regarded as a JB*-triple, then the $S^*(A,A_*)$ coincides with the *algebra strong*-topology* of A generated by all the seminorms of the form $x \mapsto \sqrt{\phi(x \circ x^*)}$, where ϕ is any normal state in A. Consequently, when a von Neumann algebra M is regarded as a JBW*-triple then the $S^*(M,M_*)$ coincides with the strong*-topology on M (see [23, Definition 1.8.7]).

A JB*-triple E is said to be abelian if for every $x,y,a,b \in E$, the operators $L(x,y)$ and $L(a,b)$ commute. Every abelian JBW*-triple is triple isomorphic (and hence isometric) to a von Neumann algebra.

Let W be a JBW*-triple with predual W_*. Since the triple product of W is separately weak*-continuous (compare [6]), then every maximal abelian subtriple is weak*-closed and hence a JBW*-subtriple of W.

Theorem 2.1. *Let W be a JBW*-triple with predual W_* and let K be a subset in W_*. Then the following are equivalent:*

- (a) *K is relatively weakly compact.*
- (b) *There exist norm-one elements $\varphi_1, \varphi_2 \in W_*$ having the following property: given $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x \in W$ with $\|x\| \leq 1$ and $\|x\|_{\varphi_1,\varphi_2} < \delta$, then $|\phi(x)| < \varepsilon$ for each $\phi \in K$.*
- (c) *The restriction $K|_C$ of K to each maximal abelian subtriple C of W is relatively $\sigma(C_*,C)$-compact.*

Proof. $(a) \Rightarrow (b)$ We assume that $K \subset W_*$ is relatively weakly compact. We may also assume that $K \subseteq B_{W_*}$. Let us fix $\varepsilon > 0$. Let $D = \overline{|\mathrm{co}|}^w(K)$, be the weakly closed absolutely convex hull of K in W_*. Then D is an absolutely convex weakly compact subset of W_*. Let Y denote the Banach space $\ell_1(D)$ and F the bounded linear operator from Y to W_* given by

$$F(\{\lambda_\varphi\}_{\varphi\in D}) := \sum_{\varphi\in D} \lambda_\varphi \varphi.$$

Clearly $F(B_Y) = D$. Since D is weakly compact then F (and hence F^*) is a weakly compact operator. By [21, Theorem 10] there exist norm-one elements $\varphi_1, \varphi_2 \in W_*$ and a function $N : (0, +\infty) \to (0, +\infty)$ such that

$$\|F^*(x)\| \leq N(\varepsilon)\|x\|_{\varphi_1,\varphi_2} + \varepsilon\|x\|,$$

for all $x \in W$ and $\varepsilon > 0$.

Let x be an element in W. It is clear that

$$\sup_{\phi\in D} |\phi(x)| = \sup_{y\in B_Y} |F(y)(x)| = \sup_{y\in B_Y} |F^*(x)(y)| \leq \|F^*(x)\|$$

$$\leq N\left(\frac{\varepsilon}{2}\right) \|x\|_{\varphi_1,\varphi_2} + \frac{\varepsilon}{2}\|x\|.$$

Finally, taking $\delta = N\left(\frac{\varepsilon}{2}\right)^{-1} \frac{\varepsilon}{2}$, we conclude that for every $x \in W$ with $\|x\| \leq 1$ and $\|x\|_{\varphi_1,\varphi_2} \leq \delta$ we have $|\phi(x)| \leq \varepsilon$ for each $\phi \in K$.

$(b) \Rightarrow (c)$ Suppose that there exists a maximal abelian subtriple C of W such that $K|_C$ is not relatively $\sigma(C_*, C)$-compact. Since C is a maximal abelian subtriple then C is weak*-closed and thus C is isomorphic (and hence isometric) to a von Neumann algebra when the latter is considered as a JB*-triple. By [2, Theorem II.2] (see also [25, Theorem 5.4]) there exists an orthogonal sequence (p_n) of symmetric projections in C and a sequence $(\varphi_n) \subseteq K$ satisfying

$$|\varphi_n(p_n)| \geq \Theta > 0. \tag{2}$$

By hypothesis, there are norm-one elements φ_1, φ_2 in W_* and $\delta > 0$ such that for every $x \in W$ with $\|x\| \leq 1$ and $\|x\|_{\varphi_1,\varphi_2} < \delta$, then $|\phi(x)| < \frac{\Theta}{2}$ for each $\phi \in K$.

Let ψ be a normal state of C. Since $\psi(p_n p_n^* + p_n^* p_n) = 2\ \psi(p_n)$ tends to zero, it follows that (p_n) is a strong*-null sequence in C. By [8, Corollary] we conclude that $(p_n) \to 0$ in the $S^*(W, W_*)$-topology of W. In particular $\|p_n\|_{\varphi_1,\varphi_2} \to 0$. Therefore, there exists $N \in \mathbb{N}$ such that for every $n \in NN$, $n \geq N$ we have

$$\|p_n\|_{\varphi_1,\varphi_2} < \delta.$$

As consequence, $|\phi(p_n)| < \frac{\Theta}{2}$, for each $\phi \in K$, which contradicts (2).

$(c) \Rightarrow (a)$ Suppose that the restriction $K|_C$ of K to each maximal abelian subtriple C of W is relatively $\sigma(C_*, C)$-compact. Let $x \in W$. The JBW*-subtriple of W generated by x is abelian, by Zorn's Lemma there exists a maximal abelian subtriple C of W containing x. By hypothesis, $K|_C$ is relatively $\sigma(C_*, C)$-compact, thus $\{\phi(x) : \phi \in K\}$ is bounded. By the uniform boundedness theorem K is bounded. Let $\widetilde{K}$ denote the $\sigma(W^*, W)$-closure of K in W^*. Since K is bounded then $\widetilde{K}$ is $\sigma(W^*, W)$-compact.

We claim $\widetilde{K} \subset W_*$. Indeed, let $\phi \in \widetilde{K}$. Let C be any maximal abelian subtriple of W. Then $\phi|_C$ is in the $\sigma(C^*, C)$-closure of $K|_C$. By assumptions, $K|_C$ is relatively $\sigma(C_*, C)$-compact and thus $\phi|_C \in C_*$. Now, by [15, Theorem 3.23], it follows that $\phi \in W_*$ as we claimed.

Since $\widetilde{K} \subset W_*$ then $\widetilde{K}$ coincides with the $\sigma(W_*, W)$-closure of K in W and hence K is relatively $\sigma(W_*, W)$-compact. □

The following corollary extends [10, Lemma 4] (see also [1, Lemma 1]) to general JBW*-triples and it is in fact the natural extension of [25, Lemma III.5.5] to the setting of JBW*-triples.

Corollary 2.2. *Let W be a JBW*-triple, let (φ_k) be a weakly convergent sequence in W_* and let (x_n) be a strong-*-null sequence in W. Then*

$$\lim_{n\to+\infty} \sup_{k\in\mathbb{N}} |\varphi_k(x_n)| = 0.$$

Proof. Suppose $(\varphi_k) \to \varphi$ weakly in W_*. The set $K = \{\varphi_k : k \in \mathbb{N}\}$ is a relative weakly compact subset of W_* by the Eberlein-Smulian theorem. Let $\varepsilon > 0$. By Theorem 2.1 there are norm-one elements $\varphi_1, \varphi_2 \in W_*$ and $\delta > 0$ such that for every $x \in W$ with $\|x\| \leq 1$ and $\|x\|_{\varphi_1,\varphi_2} < \delta$, we have $|\phi(x)| < \varepsilon$ for each $\phi \in K$. Since (x_n) is strong*-null, there exists $N \in \mathbb{N}$ such that for every $n \geq N$ it follows that $\|x_n\| \leq \delta$. Thus, for every $n \geq N$ we have $|\varphi(x_n)| \leq \varepsilon$, for all $f \in K$. □

Remark 2.3. Let W be a JBW*-triple. Suppose that $K \in W_*$ is a relatively weakly compact set. Then similar arguments to those given in the proof of Corollary 2.2 show that for each strong*-null sequence (x_n) in W we have

$$\lim_{n\to+\infty} \varphi(x_n) = 0,$$

uniformly for $\varphi \in K$.

Using Theorem 2.1 we generalize to the setting of JBW*-triples some known characterizations of weak compactness in the predual of a W*-algebra (compare [25, Theorem 5.4]).

Corollary 2.4. *Let K be a bounded subset in the predual of a JBW*-triple W. The following assertions are equivalent:*

(*a*) *K is relatively weakly compact.*

(*b*) *The restriction of K to $W_2(e)$ is relatively $\sigma((W_2(e))_*, W_2(e))$-compact in $(W_2(e))_*$, for every tripotent $e \in W$.*

(*c*) *For any monotone decreasing sequence of tripotents (e_n) in W with $(e_n) \to 0$ in the weak*-topology, we have $\lim_{n\to+\infty} \phi(e_n) = 0$ uniformly for $\phi \in K$.*

Proof. $(a) \Rightarrow (b)$ Suppose K is relatively weakly compact in W_*. Let e be a tripotent in W. Since the map: $\phi \to \phi|_{W_2(e)}$ is a weakly continuous operator from W_* to $(W_2(e))_*$, it follows that $K|_{W_2(e)}$ is relatively $\sigma((W_2(e))_*, W_2(e))$-compact in $(W_2(e))_*$.

$(b) \Rightarrow (c)$ Let (e_n) be a monotone decreasing sequence in W with $(e_n) \to 0$ in the $\sigma(W, W_*)$-topology. Since for each natural n, we have $e_1 \geq e_n$, it follows that (e_n) is a monotone decreasing sequence of projections in $W_2(e_1)$ with $(e_n) \to 0$ in the $\sigma(W_2(e_1), (W_2(e_1))_*)$-topology. It is not hard to see that $(e_n) \to 0$ in the strong-* topology of $W_2(e_1)$. Since, by assumptions, $K|_{W_2(e_1)}$ is relatively $\sigma((W_2(e_1))_*, W_2(e_1))$-compact, we conclude from Remark 2.3 that $\lim_{n\to+\infty} \phi(e_n) = 0$ uniformly for $\phi \in K$.

$(c) \Rightarrow (a)$ To obtain a contradiction, suppose that K is not relatively weakly compact. By Theorem 2.1 there exists a maximal abelian JBW*-subtriple C of W such that $K|_C$ is not relatively $\sigma(C_*, C)$-compact. As we have commented above, C is triple isomorphic to an abelian von Neumann algebra when the latter is regarded as a JBW*-triple. By [25, Theorem 5.4] there exists a monotone decreasing sequence (p_n) of projections in C with $(p_n) \to 0$ in the $\sigma(C, C_*)$-topology and $\lim_{n\to+\infty} \phi(p_n) \neq 0$ uniformly for $\phi \in K|_C$. Therefore there exists a monotone decreasing sequence (p_n) of tripotents in W with $(p_n) \to 0$ in the weak*-topology of W and $\lim_{n\to+\infty} \phi(p_n) \neq 0$ uniformly for $\phi \in K$, which is a contradiction. □

We do not know if the semi-norm $\|.\|_{\varphi_1,\varphi_2}$ appearing in Theorem 2.1 (b) could be replace by a semi-norm of the form $\|.\|_{\varphi}$ for a suitable norm-one functional $\varphi \in W_*$. This problem is connected with the problem on Grothendieck's inequalities for JB*-triples (compare [21, Remark 3]). We next show a positive answer to the above problem in the particular case of a JBW*-algebra.

Let M be a JBW*-algebra with predual M_*. Let φ_1, φ_2 be two norm-one functionals in M_*. For each $i \in \{1,2\}$ we take a tripotent $e_i \in M$ such that $\varphi_i(e_i) = 1$. Let ψ_i denote the norm-one functional in M_* given by $\psi_i(x) := \varphi_i(x \circ e_i)$ $(\forall x \in M)$. From the expression

$$\{x, x, e_i\} + \{x^*, x^*, e_i\} = 2e_i \circ (x \circ x^*),$$

we conclude that ψ_i is a positive normal state of M. Moreover, the identity

$$\|x\|^2_{\varphi_i} + \|x^*\|^2_{\varphi_i} = 2\psi_i(x \circ x^*) = 2\|x\|^2_{\psi_i}$$

holds for all $x \in M$. Set $\psi = \frac{1}{2}(\psi_1 + \psi_2)$. Then ψ is a normal state of M satisfying

$$\|x\|_{\varphi_1,\varphi_2} \leq 2\ \|x\|_\psi,$$

for all $x \in M$. We can now reformulate Theorem 2.1 to the setting of JBW*-algebras.

Theorem 2.5. *Let M be a JBW*-algebra. Let K be a norm bounded subset in M_*. The following assertions are equivalent:*

- (a) *K is relatively weakly compact,*
- (b) *The restriction $K|_C$ of K to each maximal associative subalgebra C of M is relatively $\sigma(C_*, C)$-compact.*
- (c) *there exists a normal state $\psi \in M_*$ having the following property: given $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x \in W$ with $\|x\| \leq 1$ and $\|x\|_\psi < \delta$, then $|\phi(x)| < \varepsilon$ for each $\phi \in K$.*
- (d) *For any monotone decreasing sequence of projections (e_n) in W with $(e_n) \to 0$ in the weak*-topology, we have $\lim_{n\to+\infty} \phi(e_n) = 0$ uniformly for $\phi \in K$.* □

3 Applications

Let (ϕ_n) be a bounded sequence in the predual of a JBW*-triple W. It is known that, in general, (ϕ_n) needs not admit a weakly convergent subsequence. In the setting of von Neumann algebras we can say more about bounded sequences of normal functionals. Indeed, in a recent paper, Brooks, Saitô and Wright [7] have shown that each bounded sequence in the predual of a von Neumann algebra has a subsequence which is nearly weakly convergent. More concretely, for each bounded sequence (ϕ_n) in the predual of a von Neumann algebra M, for each normal state ψ and for each $\varepsilon > 0$ there exists a projection $e \in M$ such that $\psi(1-e) \leq \varepsilon$ and the restriction of (ϕ_n) to eMe has a subsequence which converges weakly to a normal functional on eMe. The aim of this section is to obtain an analogue of the above fact in the setting of JBW*-triples.

The following lemma provides sufficient conditions to assure relative weak compactness in the predual of a JBW*-triple. It is also the natural extension of [7, Lemma 2] to the setting of JBW*-triples.

Lemma 3.1. *Let (ϕ_n) be a bounded sequence in the predual of a JBW*-triple W. Let φ be a norm-one element in W_* such that the following property holds: for each $c > 0$ there exists $\eta > 0$ such that for every tripotent $e \in W$ with $\|e\|_\varphi < \eta$ the set*

$$\{m \in \mathbb{N} : \exists u \in \mathit{Tri}(W) \text{ with } u \leq e \text{ and } |\phi_m(u)| \geq c\}$$

is finite. Then $\{\phi_n : n \in \mathbb{N}\}$ is relatively weakly compact in W_*.

Proof. Let (e_n) be a weak*-null, monotone decreasing sequence of tripotents in W. Let $c > 0$ and let $\eta > 0$ be the positive given by the property.

Since for each $n \in \mathbb{N}$, $e_1 \geq e_n$ we conclude that (e_n) is a weak*-null, monotone decreasing sequence of projections in $W_2(e_1)$. As we have commented in the above section, it is not hard to see that (e_n) is strong*-null in $W_2(e)$ and from [8, Corollary] (e_n) is strong*-null in W. In particular $\|e_n\|_\varphi \to 0$. Then there exists $m_1 \in \mathbb{N}$ such that for each $n \geq m_1$ we have $\|e_n\|_\varphi < \eta$. Since the set

$$\{m \in \mathbb{N} : |\phi_m(e_n)| \geq c \text{ for some } n \geq m_1\}$$

is finite by hypothesis, we conclude that there exists $m_0 \in \mathbb{N}$ such that for each $m \geq m_0$ we have $|\phi_m(e_n)| < c$ for every natural $n \geq m_1$.

Since for each $j : 1 \dots m_0$ the sequence $(\phi_j(e_n))_{n\in\mathbb{N}}$ tends to zero we deduce that there exists $m_2 \in \mathbb{N}$ such that for each $n \geq m_2$ and $j : 1, \dots, m_0$ we have $|\phi_j(e_n)| < c$. Therefore, for each $n \geq \max\{m_1, m_2\}$, we have $|\phi_m(e_n)| < c$ for all $m \in \mathbb{N}$. Corollary 2.4 gives the desired statement. □

When in the proof of Lemma 3.1, Theorem 2.5 replaces Theorem 2.1 we obtain the following.

Lemma 3.2. *Let M be a JBW*-algebra and let (ϕ_n) be a bounded sequence in M_*. Let φ be a normal state of M such that the following property holds: for each $c > 0$ there exists $\eta > 0$ such that for every projection $e \in M$ with $\|e\|_\varphi < \eta$ the set*

$$\Big\{m \in \mathbb{N} : \text{ there exists a projection } p \in M \text{ with } p \leq e \text{ and } |\phi_m(p)| \geq c\Big\}$$

is finite. Then $\{\phi_n : n \in \mathbb{N}\}$ is relatively weakly compact in M_*. □

Let M be a JBW*-algebra. Let φ be a positive normal functional on M and let (ϕ_n) be a norm-bounded sequence in M_*. We shall denote by Δ the set of all $c \in \mathbb{R}^+$ such that for each $\eta > 0$ there exists a projection $e_\eta \in W$ such that $\|e_\eta\|_\varphi < \eta$ and the set

$$\Big\{m \in \mathbb{N} : \text{ there exists a projection } p \in M \text{ with } p \leq e_\eta \text{ and } |\phi_m(p)| \geq c\Big\}$$

is infinite. Following [7, Definition in page 162], we call Δ the *anti-compactness set* of (ϕ_n) with respect to the functional φ. It is clear that Δ is bounded.

Remark 3.3. Let M be a JBW*-algebra. Let φ be a positive functional in M_*, (ϕ_n) a norm-bounded sequence in M_*, and Δ the anti-compactness set of (ϕ_n) with respect to φ. We claim that (ϕ_n) is relatively weakly compact in M_* whenever $\Delta = \emptyset$. Indeed, let $c \in \mathbb{R}^+$. Since $c \notin \Delta$ there exists $\eta > 0$ such that for every projection $e \in M$ with $\|e\|_\varphi < \eta$ the set

$$\Big\{m \in \mathbb{N} : \text{ there exists a projection } p \in M \text{ with } p \leq e \text{ and } |\phi_m(p)| \geq c\Big\}$$

is finite. We conclude from Lemma 3.2 that (ϕ_n) is relatively weakly compact in W_*.

We recall that a positive functional ψ of a JB*-algebra A is said to be faithful if and only if $\psi(x) > 0$ for every positive element $x \in A\backslash\{0\}$. Suppose that a JBW*-algebra M has a faithful normal state ψ. Then the strong*-topology in the closed unit ball of M is metrized by the distance

$$d_\psi(a, b) := (\psi((a - b) \circ (a - b)^*))^{\frac{1}{2}} .$$

More precisely, a bounded net $(x_i)_{i \in I}$ in M converges in the strong*-topology of M to an element $x \in M$ if and only if $d_\psi(x_i, x) \to 0$ (compare [16, page 200]). When M is regarded as a JBW*-triple we have $d_\psi(a, b) = \|a - b\|_\psi$.

The following lemma is a verbatim extension of [7, Lemmma 3] to the setting of JBW*-algebras.

Lemma 3.4. *Let M be a JBW*-algebra having a faithful positive normal functional ψ. Let (ϕ_n) be a norm bounded sequence in M_* and let Δ be the anti-compactness set of (ϕ_n) with respect to ψ, considering M as a JBW*-triple. Then (ϕ_n) is relatively weakly compact in M_* if and only if $\Delta = \emptyset$.*

We sketch the main ideas of the proof for completeness. We have already shown that $\Delta = \emptyset$ implies (ϕ_n) is relatively weakly compact in M_* (compare Remark 3.3).

To prove the if-implication we suppose, contrary to our claim, that $\Delta \neq \emptyset$. There is no loss of generality in assuming $\|\psi\| = 1$. Let $c \in \Delta$. Then for each $k \in \mathbb{N}$ there exists a tripotent $e_k \in M$ satisfying $\|e_k\|_\psi < 2^{-k}$ and the set

$$\Big\{ m \in \mathbb{N} : \exists u \in \mathrm{Tri}(M) \text{ with } u \leq e_k \text{ and } |\phi_m(u)| \geq c \Big\} \tag{3}$$

is infinite. Thus (e_k) is a bounded sequence in M satisfying

$$d_\psi(e_k, 0) = \|e_k\|_\psi \to 0.$$

Since ψ is a faithful normal state of M, and the strong*-topology of M is determined by the metric d_ψ, we deduce that (e_k) tends to zero in the strong*-topology of M.

Since, by assumptions, (ϕ_n) is relatively weakly compact, then by Theorem 2.1 there exist norm-one functionals $\varphi_1, \varphi_2 \in M_*$ and $\delta > 0$ satisfying that for every $x \in M$ with $\|x\| \leq 1$ and $\|x\|_{\varphi_1,\varphi_2} < \delta$ we have $|\phi_n(x)| \leq \frac{c}{2}$ for all $n \in \mathbb{N}$. Since $(e_k) \to 0$ in the strong*-topology, there exists $k_0 \in \mathbb{N}$ such that $\forall k \geq k_0$ we have $\|e_k\|_{\varphi_1,\varphi_2} < \delta$. Let $k \geq k_0$. It is not hard to see that (from the orthogonality of u and $e_k - u$) for each tripotent $u \leq e_k$ we have $\|u\|_{\varphi_1,\varphi_2} \leq \|e_k\|_{\varphi_1,\varphi_2} < \delta$. Consequently, $|\phi_n(u)| \leq \frac{c}{2}$ for all $n \in \mathbb{N}$, which contradicts (3). □

Having the above facts in mind, the proof of [7, Proposition 4] can be lightly adapted to prove the following result.

Proposition 3.5. *Let M be a JBW*-algebra having a faithful positive functional ψ. Let (ϕ_n) be a norm bounded sequence in M_*. Then for every $\varepsilon > 0$ there exists a projection $p \in M$ such that $\psi(p) < \varepsilon$ and there is a subsequence of ϕ_n, (β_n), such that the sequence (β_n) restricted to $P_2(1-p)(M)$ is relatively weakly compact.* □

Let φ be a norm-one functional in the predual of a JBW*-triple W. By [13, Proposition 2], there exists a unique tripotent $e = e(\varphi) \in W$ such that $\varphi = \varphi P_2(e)$ and $\varphi|_{W_2(e)}$ is a faithful normal state of the JBW*-algebra $W_2(e)$. This unique tripotent $e = e(\varphi)$ is called the *support tripotent* of φ.

We can now state the analogue of [7, Theorem 8] in the setting of JBW*-triples.

Theorem 3.6. *Let W be a JBW*-triple. Let φ be a norm-one element in W_* and let (ϕ_n) be a norm bounded sequence in W_*. Then for each $1 > \varepsilon > 0$ there exists a tripotent $e \in W$ such that $\|e\|_\varphi > 1-\varepsilon$ and there is a subsequence $(\phi_{\sigma(n)})$ such that $(\phi_{\sigma(n)}|_{W_2(e)})$ is relatively $\sigma((W_2(e))_*, W_2(e))$-compact in $(W_2(e))_*$.*

Proof. Let $s = s(\varphi)$ be the support tripotent of φ. Let $\varepsilon > 0$. By Proposition 3.5 there exists a projection $p \in W_2(s)$ such that $\varphi(p) < \varepsilon$ and there is a subsequence $(\phi_{\sigma(n)})$ such that $(\phi_{\sigma(n)})$ restricted to $P_2(s-p)(W)$ is relatively weakly compact. We take $e = s - p$ to obtain the desired statement. $\square$

When in the proof of [7, Corollaries 9,10] we replace [7, Theorem 8] by Theorem 3.6 we obtain:

Corollary 3.7. *Let φ be a norm-one functional in the predual of a JBW*-triple W. Let (ϕ_n) be a norm bounded sequence in W_* and let $s = s(\varphi)$ be the support tripotent of φ. Then there exists a sequence of tripotents (e_k) (with $e_k \leq s$ for each $k \in \mathbb{N}$) which converges in the strong*-topology to s and there is a subsequence of (ϕ_n), $(\phi_{\sigma(n)})$, such that*

$$\lim_{n\to+\infty} \phi_{\sigma(n)} P_2(e_k)(x)$$

exists for each $x \in W$. $\square$

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Departamento de Análisis Matemático, Facultad de Ciencias,
Universidad de Granada, 18071 Granada, Spain.
e-mail: aperalta@ugr.es