Subdifferentiability of the norm and the Banach-Stone Theorem for real and complex JB*-triples

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2000 Mathematics Subject Classification: 46B04, 46L05 and 46L70.

Abstract

We study the points of strong subdifferentiability for the norm of a real JB*-triple. As a consequence we describe weakly compact real JB*-triples and rediscover the Banach-Stone Theorem for complex JB*-triples.

1 Introduction

Let X be a Banach space. The norm of X is said to be *strongly-subdifferentiable* at a morn-one point $x \in X$ whenever the limit

$$\lim_{\alpha \to 0^+} \frac{\|x + \alpha y\| - 1}{\alpha}$$

exists uniformly for y in the closed unit ball of X. The points of strong subdifferentiability for the norm of a C*-algebra were characterized by Contreras, Payá and Werner in [6]. Recently, Becerra-Guerrero and Rodríguez-Palacios

*Authors Partially supported by I+D MCYT projects no. BFM2002-01529 and BFM2002-01810, and Junta de Andalucía grant FQM 0199

have completely described the points of strong subdifferentiability for the norm of a (complex) JB*-triple [3]. In the latter work, the authors show that the norm of a JB*-triple $\mathcal{E}$ is strongly subdifferentiable at a norm-one point x if and only if 1 is an isolated point of the triple spectrum of x, if and only if the support tripotent of x in the bidual, $\mathcal{E}^{**}$, of $\mathcal{E}$ lies in $\mathcal{E}$. As a consequence, the authors show that the JB*-triples whose norms are strongly subdifferentiable at every point of their unit spheres are precisely the so-called weakly compact JB*-triples.

The aim of the present paper is to describe the points of strong subdifferentiability of the norm of a real JB*-triple (see definition bellow). In our main result (Theorem 2.4) we prove that the norm of a real JB*-triple E is strongly subdifferentiable at a norm-one point x if and only if the (unique) norm becoming its complexification a complex JB*-triple is strongly subdifferentiable at x. As a consequence we characterize, in Corollary 2.5, the points of strong subdifferentiability for the norm of a real JB*-triple, extending the description provided by Becerra-Guerrero and Rodríguez-Palacios in the complex setting.

In [23] Werner showed that the characterization of the points of strong subdifferentiability for the norm of C*-algebra can be applied to obtain an alternative proof of the non-commutative Banach-Stone Theorem provided by Kadison: "The linear surjective isometries from a unital C*-algebras A onto another unital C*-algebra B are precisely of the form $x \mapsto u\Phi(x)$, where u is a unitary element of B and Φ is a Jordan isomorphism from A onto B. In the already quoted paper (see [23, Remarks 3.]) the author establishes without proof that it is possible to extend the method he applied to the more general setting of JB*-triples. In the last section of this paper we include a complete extension of Werner's method to the setting of real and complex JB*-triples and the appropriated version of the Banach-Stone theorem for JB*-triples.

2 Main result

Given a Banach space X, we denote by B_X, S_X, and X^* the closed unit ball, the unit sphere, and the dual space of X, respectively.

Let x be a norm one element in a Banach space X. The set $D(X,x)$ of all states of X relative to x is define by

$$D(X,x) := \{f \in S_{X^*} : f(x) = \|x\|\}.$$

A *JB*-triple* is a complex Banach space $\mathcal{E}$ equipped with a continuous triple product

$$\{.,.,.\} : \mathcal{E} \otimes \mathcal{E} \otimes \mathcal{E} \to \mathcal{E}$$
$$(x, y, z) \mapsto \{x, y, z\}$$

which is bilinear and symmetric in the outer variables and conjugate linear in the middle one and satisfies:

(a) (*Jordan Identity*)

$$L(x, y) \{a, b, c\} = \{L(x, y)a, b, c\} - \{a, L(y, x)b, c\} + \{a, b, L(x, y)c\},$$

for all $x, y, a, b, c \in \mathcal{E}$, where $L(x, y) : \mathcal{E} \to \mathcal{E}$ is the linear mapping given by $L(x, y)z = \{x, y, z\}$;

(b) The map $L(x, x)$ is an hermitian operator with non-negative spectrum for all $x \in \mathcal{E}$;

(c) $\| \{x, x, x\} \| = \|x\|^3$ for all $x \in \mathcal{E}$.

Every C*-algebra is a JB*-triple with respect to $\{x, y, z\} = 2^{-1}(xy^*z + zy^*x)$, every JB*-algebra is a JB*-triple with triple product $\{a, b, c\} = (a \circ b^*) \circ c + (c \circ b^*) \circ a - (a \circ c) \circ b^*$, and the Banach space $B(H, K)$ of all bounded linear operators between two complex Hilbert spaces H, K is also an example of a JB*-triple with respect to $\{R, S, T\} = 2^{-1}(RS^*T + TS^*R)$.

Let X be a complex Banach space with a conjugation (conjugate linear isometry of period two) τ on X. We will denote by X^τ the real Banach subspace of X of all τ-fixed points in X. In this case we will say that X^τ is a real form of X.

It is worth mentioning that a real JB*-triple is a norm-closed real subtriple of a JB*-triple [16, Definition 2.1]. Let E be a real JB*-triple. By [16, Proposition 2.8], there exists a unique complex JB*-triple structure on the algebraic complexification $E \oplus iE$ (denoted by $\widehat{E}$) and a conjugation τ on $E + iE$ such that $E = \widehat{E}^\tau := \{z \in \widehat{E} : \tau(z) = z\}$, i.e., every real JB*-triple is a real form of its complexification, which is a complex JB*-triple. Every real C*-algebra, every real Hilbert space, every complex JB*-triple (when is regarded as a real Banach space) and the Banach space of all bounded linear operators between real Hilbert spaces are examples of real JB*-triples (cf. [16]).

By a real or complex JBW*-triple we mean a real or complex JB*-triple which is also a dual Banach space whose triple product is separately weak*-continuous [16, §4]. By [20] and [1] we know that the assumption of the separate weak*-continuity is redundant. The bidual E^{**} of every real or complex

JB*-triple is a JBW*-triple with triple product extending the product of E (cf. [8] and [16, Lemma 4.2], respectively).

Let U be a real or complex JB*-triple and let e be a tripotent in U (i.e. $\{e, e, e\} = e$). It is known that U admits the following decomposition in terms of the eigenspaces of $L(e, e)$,

$$U = U_0(e) \oplus U_1(e) \oplus U_2(e),$$

where $U_k(e) := \{x \in U : L(e, e)x = \frac{k}{2}x\}$ is a subtriple of U $(k : 0, 1, 2)$. The natural projection of U onto $U_k(e)$ will be denoted by $P_k(e)$. This decomposition is the so-called Peirce decomposition with respect to the tripotent e and the natural projections are the so-called Peirce projections. The following rules, known as Peirce rules, are also satisfied

$$\{U_k(e), U_l(e), U_m(e)\} \subseteq U_{k-l+m}(e),$$

$$\{U_0(e), U_2(e), U\} = \{U_2(e), U_0(e), U\} = 0,$$

where $U_{k-l+m}(e) = 0$ whenever $k - l + m \neq 0, 1, 2$.

A tripotent e in a real or complex JB*-triple U is called minimal whenever $U^1(e) = \mathbb{R}e$, where $U^1(e) = \{x \in U : Q(e)(x) = x\}$.

Let E be a real JB*-triple with complexification $\widehat{E}$ and let e be a tripotent in E. It is clear that the Peirce projections of E with respect to e coincide with the restrictions to E of the Peirce projections of $\widehat{E}$ with respect to e. Therefore, the following result follows from [14, Lemma 1.3 and Lemma 1.6].

Lemma 2.1. *Let e be a tripotent in a real JB*-triple E. Then we have*

(a) $\|P_2(e)(x) + P_0(e)(x)\| = \max\{\|P_2(e)(x)\|, \|P_0(e)(x)\|\}$, *for all $x \in E$;*

(b) $\|P_2(e)^*(f) + P_0(e)^*(f)\| = \|P_2(e)^*(f)\| + \|P_0(e)^*(f)\|$, *for all $f \in E^*$.*

(c) *If x is a norm-one element in E with $P_2(e)(x) = e$, then $P_1(e)(x) = 0$, thus $x = e + P_0(e)(x)$.*

Let X be a complex Banach space with a conjugation τ on X. Then τ can be extended to a conjugation $\tilde{\tau}$ on X^* in the following way

$$\tilde{\tau} : X^* \to X^*$$

$$\tilde{\tau}(f)(x) = \overline{f(\tau(x))} \quad (f \in X^*).$$

In this case, it is also known that $(X^*)^{\tilde{\tau}}$ is isometric to $(X^\tau)^*$ via $f \mapsto f|_{X^\tau}$. Under this identification, X^* admits the decomposition $X^* = (X^\tau)^* + i(X^\tau)^*$. Let $x \in S_{X^\tau} \subset S_X$ and $f \in D(X, x)$. Then

$$\tilde{\tau}(f)(x) = \overline{f(\tau(x))} = \overline{f(x)} = 1,$$

which shows that $\tilde{\tau}(f) \in D(X, x)$. Therefore $\tilde{\tau}(D(X, x)) = D(X, x)$ and hence $D(X^\tau, x) = D(X, x)^{\tilde{\tau}}$.

Having in mind that every tripotent in a real JB*-triple is clearly a tripotent in its complexification and the strong subdifferentiability at a norm-one point is inherited by real subspaces, the next corollary is a consequence of Theorem [3, Theorem 2.7].

Corollary 2.2. *Let E be a real JB*-triple. Then the norm of E is strongly subdifferentiable at every tripotent in E.*

Let x be a norm-one element in a real or complex JBW*-triple U. The set $D(U, x) \cap U_*$ is a (possibly empty) proper closed face of B_{U_*}, and therefore, by [11, Theorem 3.7 and Lemma 2.1], there exists a unique tripotent u (possibly equal to zero) in U so that $D(U, x) \cap U_* = D(U, u) \cap U_*$. Such a tripotent is called *the support tripotent of* x and will be denoted by $u(U, x)$. Let E be a real JBW*-triple and let $x \in S_E$. As we have seen above, $E = \widehat{E}^\tau$ where $\widehat{E}$ is the complexification of E and τ is a conjugation on $\widehat{E}$. By [9, Lemma 3.4], $u(\widehat{E}, x)$ is the limit in the weak*-topology of the sequence (x^{2n+1}), where x^{2n+1} is inductively defined by $x^3 = \{x, x, x\}$ and $x^{2n+1} = \{x, x^{2n-1}, x\}$. Since the canonical conjugation on $\widehat{E}$ is weak*-continuous and preserves the triple product we have $\tau(u(\widehat{E}, x)) = u(\widehat{E}, x)$. Now, [11, Theorem 3.7] ascertains that $u(E, x) = u(\widehat{E}, x)$.

Having the above facts in mind, the same arguments given in [3, Theorem 2.5] can be adapted to obtain the following corollary.

Corollary 2.3. *Let E be a real JBW*-triple, and let x be in S_E. The norm is strongly subdifferentiable at x if and only if $D(E, x) \cap E_*$ is weak*-dense in $D(E, x)$.*

We can now establish our main result.

Theorem 2.4. *Let E be a real JB*-triple with complexification $\widehat{E}$ and let x be in S_E. Then the norm of E is strongly subdifferentiable at x if and only if the same conclusion holds for the norm of $\widehat{E}$.*

Proof. We have already mentioned that the strong subdifferentiability of the norm at a norm-one point is inherited by real subspaces. In order to see the other implication let us suppose that the norm of E is subdifferentiable at $x \in S_E$.

Let $u = u(E^{**}, x)$ be the support tripotent of x in E^{**}. Then

$$D(E, x) = D(E^{**}, u) \cap E^*.$$

Since the norm of E is strongly subdifferentiable at x we deduce that the same conclusion remains true for the norm of E^{**} [15, Corollary 2.1], and hence, by Corollary 2.3, we get

$$\overline{D(E, x)}^{w^*} = D(E^{**}, x). \tag{1}$$

By Corollary 2.2 and Corollary 2.3 we also have

$$\overline{D(E^{**}, u) \cap E^*}^{w^*} = D(E^{**}, u). \tag{2}$$

It follows by (1) and (2) that $D(E^{**}, x) = D(E^{**}, u)$. By [11, Theorem 3.9] it follows that $x \in u + B_{E_0^{**}(u)}$. We claim that $\|P_0(u)(x)\| < 1$. Otherwise, by Lemma 2.1 and the Hahn-Banach theorem, there exist an element in $D(E^{**}, x) \setminus D(E^{**}, u)$, which is impossible. Therefore $x = u + P_0(u)(x)$ with $\|P_0(u)(x)\| < 1$. Finally, by Peirce rules, $x^{2n+1} = u + (P_0(u)(x))^{2n+1}$, thus,

$$\|x^{2n+1} - u\| \leq \|P_0(u)(x)\|^{2n+1} \to 0.$$

Therefore $u \in E$. Since $u = u(E^{**}, x) = u((\widehat{E})^{**}, x) \in E \subseteq \widehat{E}$, Theorem [3, Theorem 2.7, (4) $\Rightarrow$ (1)] implies that the norm of $\widehat{E}$ is strongly subdifferentiable at x. $\square$

Let x be an element in a complex JB*-triple $\mathcal{E}$, and denote by $\mathcal{E}(x)$ the JB*-subtriple of $\mathcal{E}$ generated by x. It is known that there exists a locally compact subset S_x of $(0, +\infty)$ such that $S_x \cup \{0\}$ is compact and $\mathcal{E}(x)$ is JB*-triple isomorphic to the C*-algebra $C_0(S_x)$ under a triple isomorphism Ψ, which satisfies $\Psi(x)(t) = t$ $(t \in S_x)$ (cf. [17, 4.8], [18, 1.15] and [14]). The subset S_x is called the *triple spectrum* of x. When x is an element in a real JB*-triple E, the *complex triple spectrum* of x is the triple spectrum of x when x is regarded as an element in the complexification, $\widehat{E}$, of E.

In the case of a real JB*-triple we can obtain the following characterizations of the strong subdifferentiability of its norm at a point of the unit sphere, similar to those obtained for (complex) JB*-triples in [3, Theorem 2.7].

Corollary 2.5. *Let E be a real JB*-triple. The following assertions are equivalent for an element x in the unit sphere of E:*

(a) *The norm of E is strongly subdifferentiable at x,*

(b) *The norm of the complexification of E is strongly subdifferentiable at x,*

(c) 1 *is an isolated point of the complex triple spectrum of x,*

(d) *There exists a unique tripotent u in E such that $x \in E_u$, where*

$$E_u = \{y \in S_E : \{u,u,y\} = u, \quad \{u,y,u\} = u \text{ and } \|y-u\| < 1\},$$

(e) *$u(E^{**},x)$ belongs to E,*

Proof. Let $\widehat{E}$ denote the complexification of E (which is a complex JB*-triple) and let τ the canonical conjugation on $\widehat{E}$ satisfying $\widehat{E}^{\tau} = E$. By Theorem 2.4 we already know that $(a) \Leftrightarrow (b)$ and [3, Theorem 2.7, (1) $\Leftrightarrow$ (2)] shows the equivalence $(b) \Leftrightarrow (c)$.

To see $(b) \Rightarrow (d)$, let us suppose that the norm of the complexification, $\widehat{E}$, of E is strongly subdifferentiable at x. By [3, Theorem 2.7, (1) $\Leftrightarrow$ (3)] there exists a tripotent u in $\widehat{E}$ such that $\{u,u,x\} = \{u,x,u\} = u$ and $\|x-u\| < 1$. We claim that such a tripotent is unique. Indeed, suppose that w is another tripotent in $\widehat{E}$ satisfying $\{w,w,x\} = \{w,x,w\} = w$ and $\|x-w\| < 1$. By [14, Lemma 1.6] we have $x = u + P_0(u)(x)$ and $x = w + P_0(w)(x)$. Moreover, we also have $\|P_0(u)(x)\| = \|x-u\| < 1$ and $\|P_0(w)(x)\| = \|x-w\| < 1$. By Peirce rules it may be concluded that $x^{2n+1} = u + (P_0(u)(x))^{2n+1}$ and $x^{2n+1} = w + (P_0(w)(x))^{2n+1}$. Then we conclude that

$$\|u-w\| \leq \|u-x^{2n+1}\| + \|w-x^{2n+1}\| \leq \|P_0(u)(x)\|^{2n+1} + \|P_0(w)(x)\|^{2n+1} \to 0,$$

which shows that $u = w$. Therefore, there exists a unique tripotent $u \in \widehat{E}$ such that satisfying $\{u,u,x\} = \{u,x,u\} = u$ and $\|x-u\| < 1$. Since τ is a conjugate-linear triple isomorphism, it follows that $\tau(u)$ is also a tripotent in $\widehat{E}$ satisfying the same conditions of u, thus, by the uniqueness, we have $u = \tau(u)$. This shows that u is a tripotent in E and $x \in E_u$. A similar reasoning to that just developed for the complexification shows the uniqueness of the tripotent u in E.

$(d) \Rightarrow (e)$ Let u be a tripotent in E such that $x \in E_u$ and let $v = u(E^{**},x)$ be the support tripotent of x in E^{**}. By the same reasonings given in the proof of Theorem 2.4 and the implication $(b) \Rightarrow (d)$ above it may be concluded that

$$\|v - u\| \leq \|v - x^{2n+1}\| + \|u - x^{2n+1}\| \to 0,$$

which shows that $v = u(E^{**}, x) = u \in E$.

$(e) \Rightarrow (b)$ As we have seen in the comments preceding Corollary 2.3, $u(E^{**}, x) = u(\widehat{E}^{**}, x)$, which belongs to $E \subseteq \widehat{E}$ by hypothesis. Now [3, Theorem 2.7 (4) $\Rightarrow$ (1)] ascertains that the norm of $\widehat{E}$ is strongly subdifferentiable at x. $\square$

We recall that a Banach space X is said to be *smooth* at a norm-one point u whenever $D(X, u)$ reduces to a singleton, and X is Frechet-smooth at u whenever exists the limit $\lim_{\alpha \to 0} \frac{\|u+\alpha x\|-1}{\alpha}$ for every $x \in X$ and is uniformly for $x \in B_X$. It is known that X is Frechet-smooth at u if and only if the norm of X is strongly subdifferentiable at u and X is smooth at u.

The following corollary is an extension to real JB*-triples of the main result of [10] for complex JB*-triples.

Corollary 2.6. *Let E be a real JBW*-triple and let $x \in S_E$. Then E is Frechet-smooth at x if and only if E is smooth at x.*

Proof. Suppose that E is smooth at x and let C denote the real JBW*-subtriple of E generated by x. By [5, Theorems 3.3, 3.6 and 3.7] there exist two compact hyperstonean Ω_1 and Ω_2 such that C is isometric to

$$C(\Omega_1, \mathbb{R}) \oplus^{\ell_\infty} C(\Omega_2, \mathbb{C})_{\mathbb{R}}.$$

Since C is smooth at $x = (x_1, x_2)$ (with $x_1 \in C(\Omega_1, \mathbb{R})$ and $x_2 \in C(\Omega_2, \mathbb{C})_{\mathbb{R}}$) then it is easy to see that $C(\Omega_1, \mathbb{R})$ is smooth at x_1 and $\|x_1\| = 1 > \|x_2\|$ or $C(\Omega_2, \mathbb{C})_{\mathbb{R}}$ is smooth at x_2 and $\|x_2\| = 1 > \|x_1\|$, which implies that $C(\Omega_1, \mathbb{C})$ is smooth at x_1 and $\|x_1\| = 1$ or $C(\Omega_2, \mathbb{C})$ is smooth at x_2 and $\|x_2\| = 1$. By [24, Theorem] we conclude that $C(\Omega_1, \mathbb{C})$ (and hence $C(\Omega_1, \mathbb{R})$) is Frechet-smooth at x_1 or $C(\Omega_2, \mathbb{C})$ (and hence $C(\Omega_2, \mathbb{C})_{\mathbb{R}}$) is Frechet-smooth at x_2. It can be easily seen that C is Frechet-smooth at x. Finally the equivalence $(a) \Leftrightarrow (d)$ in Corollary 2.5 shows that E is Frechet-smooth at x. $\square$

Remark 2.7. *It should be noticed that when X is a complex Banach space with a conjugation τ and x is a norm-one element in X^τ satisfying that X^τ is smooth at x then X does not need to be smooth at x. For example let X^τ denote the real spin factor of type $IV_n^{n,0}$ in the terminology of [19, Theorem*

4.1], where we consider X, the complexification of X^τ, equipped with triple product and norm given by

$$\{xyz\} := (x|y)z + (z|y)x - (x|\sigma(z))\sigma(y)$$

and

$$\|x\|^2 := (x|x) + \sqrt{(x|x)^2 - |(x|\sigma(x))|^2},$$

respectively, for all x, y, z in X, where $\sigma(a+ib) = a-ib$ $(a, b \in X^\tau)$ (compare [18, Theorem 4.1]). It is easy to see that any norm-one element in X^τ is a minimal tripotent in X^τ and hence X^τ is smooth at such a point. However, every norm-one point x in X^τ is a tripotent that is not minimal in X (the complexification of X^τ), that is, there exist two orthogonal tripotents e and f in X such that $x = e + f$. This implies that X is not smooth at x.

To finish with this section we will describe those real JB*-triples whose norms are strongly subdifferentiable at every point of their unit sphere. As in the complex case, we will show that such real JB*-triples are well-studied and characterized by several previous authors (compare [3, Theorem 1.12 and Remmark 2.13]).

Lemma 2.8. *Let Ω be a locally compact Hausdorff space and let τ be a conjugation on $C_0(\Omega)$, the complex C*-algebra of continuous complex-valued functions on Ω vanishing at infinity. Then the norm of $C_0(\Omega)^\tau$ is strongly subdifferentiable at every point of $S_{C_0(\Omega)^\tau}$ if and only if Ω is discrete.*

Proof. Let us assume that the norm of $C_0(\Omega)^\tau$ is strongly subdifferentiable at every point of $S_{C_0(\Omega)^\tau}$.

By the classical Stone Theorem there exists a homomorphism $\sigma : \Omega \to \Omega$ and a continuous function $u : \Omega \to \mathbb{C}$ with $|u(t)| = 1$ $(t \in \Omega)$ such that

$$\tau(f)(t) = u(t)\overline{f(\sigma(t))},$$

for all $t \in \Omega$, $f \in C_0(\Omega)$. Since $\tau^2 = Id$ we have

$$u(t)\overline{u(\sigma(t))}f(\sigma^2(t)) = f(t), \tag{3}$$

for all $t \in \Omega$, $f \in C_0(\Omega)$.

Let $t_0 \in \Omega$ and let $f_0 \in C_0(\Omega)$ satisfying $f_0(t_0) = f_0(\sigma^2(t_0)) = 1$. By replacing f_0 and t_0 in (3) we deduce that

$$u(t_0)\overline{u(\sigma(t_0))} = 1.$$

Since t_0 is arbitrary it follows that

$$u(t) = u(\sigma(t)) \quad (t \in \Omega) \tag{4}$$

From (3) and (4) we have $\sigma^2 = Id_\Omega$.

Suppose first, that t_0 is a non-isolated point of Ω. We assume first that $\sigma(t_0) = t_0$. Let (U_n) be a sequence of compact neighbourhoods of t_0 with int $(U_n) \supsetneqq U_{n+1}$, $\sigma(U_n) = U_n$ and, by the continuity of u, we may also assume $|u(t) - u(t_0)| < \frac{1}{n}$ for all $t \in U_n$. Let $e_n \in C_0(\Omega)$ with $e_n(t) = 1$ for all $t \in U_n$ and $e_n(t) = 0$ for all $t \in \Omega \setminus U_{n-1}$. If $u(t_0) \neq -1$ we define $x = \sum_n \frac{1}{2^n} 2^{-1}(e_n + \tau(e_n)) \in C_0(\Omega)$. It is clear that $\tau(x) = x (\in C_0(\Omega)^\tau)$. We claim that $\lambda = \sum_{n=1}^{+\infty} \frac{1}{2^{n+1}}(1 + u(t_0))$ is a cluster point of the triple spectrum of x in $C_0(\Omega)$. Indeed, for every $m \in \mathbb{N}$ we can take $t_m \in U_m \setminus U_{m+1}$, then $\lambda_m := x(t_m) = \sum_{n=1}^{m} \frac{1}{2^{n+1}}(1 + u(t_m))$, is an element in the triple spectrum of x. We will show that λ_m converges to λ. For every $\varepsilon > 0$ there exists $m_0 \in \mathbb{N}$ such that for every $m \in \mathbb{N}$ with $m \geq m_0$ we have

$$\left| \lambda - \sum_{n=1}^{m} \frac{1}{2^{n+1}}(1 + u(t_0)) \right| < \frac{\varepsilon}{2},$$

and

$$\left| \sum_{n=1}^{m} \frac{1}{2^{n+1}}(1 + u(t_0)) - \sum_{n=1}^{m} \frac{1}{2^{n+1}}(1 + u(t_m)) \right| \leq \left(\sum_{n=1}^{m} \frac{1}{2^{n+1}} \right) \; |u(t_m) - u(t_0)|$$

$$< \left(\sum_{n=1}^{m} \frac{1}{2^{n+1}} \right) \frac{1}{m} < \frac{1}{m} < \frac{\varepsilon}{2}.$$

Therefore, for every $m \geq m_0$ it follows that $|\lambda - \lambda_m| < \varepsilon$, which shows that λ is a cluster point of the triple spectrum of x in $C_0(\Omega)$.

When $u(t_0) = -1$ we define $x = \sum_n \frac{1}{2^n} 2^{-1}(ie_n + \tau(ie_n)) \in C_0(\Omega)^\tau$. Following the same method applied in the case $u(t_0) \neq -1$ we can conclude that $\lambda = \sum_{n=1}^{+\infty} \frac{1}{2^{n+1}} i(1 - u(t_0)) = i \sum_{n=1}^{+\infty} \frac{1}{2^n}$ is a cluster point of the triple spectrum of x in $C_0(\Omega)$.

Suppose now that $\sigma(t_0) \neq t_0$. Let (U_n) be a sequence of compact neighbourhoods of t_0 with $\mathrm{int}(U_n) \supsetneqq U_{n+1}$, $\sigma(U_n) \cap U_n = \emptyset$ and $|u(t) - u(t_0)| < \frac{1}{n}$ for all $t \in U_n$. Let $e_n \in C_0(\Omega)$ with $e_n(t) = 1$ for all $t \in U_n$ and $e_n(t) = 0$ for all $t \in \Omega \setminus U_{n-1}$ and define $x = \sum_n \frac{1}{2^{n+1}}(e_n + \tau(e_n)) \in C_0(\Omega)$. Clearly $\tau(x) = x$. The same ideas developed in the case $\sigma(t_0) = t_0$ allow us to assure that $\lambda = \sum_{n=1}^{+\infty} \frac{1}{2^{n+1}}$ is a cluster point of the triple spectrum of x in $C_0(\Omega)$.

To finish the proof we claim that if a (norm-one) τ-symmetric element in $C_0(\Omega)$ has a non discrete triple spectrum in $C_0(\Omega)$ then the norm of $C_0(\Omega)$ is not strongly subdifferentiable at an element in $S_{C_0(\Omega)^\tau}$. Indeed, let x be norm-one element in $C_0(\Omega)$ with $\tau(x) = x$ and with non-discrete triple spectrum $S_x \subseteq [0,1]$. It is known that the JB*-subtriple of $C_0(\Omega)$ generated by x (denoted by C) is JB*-triple isomorphic to the C*-algebra $C_0(S_x)$ under a triple isomorphism Ψ, which satisfies $\Psi(x)(t) = id_{S_x}(t) = t$ $(t \in S_x)$. If S_x is not discrete then there exists a non-isolated point $\alpha \in S_x$. The function $g(t) := \frac{t}{t+|t-\alpha|} \in C_0(S_x)$ has an odd extension to $S_x \cup -S_x$, and can be approximated uniformly by real linear combinations of odd powers of t. Since $\Psi^{-1}(id_{S_x}) = x$ is τ-symmetric and Ψ is a triple isomorphism then $\Psi^{-1}(g)$ is norm-one and τ-symmetric. It is easy to see that α is the unique $t \in S_x$ satisfying $g(t) = 1$. Since α is not isolated in S_x it follows by [10, Lemma 2.2] that the norm of $C_0(S_x)$ (and hence the norm of C) is not strongly subdifferentiable at g (at $\Psi^{-1}(g)$), which contradicts the assumption since the strong subdifferentiability is inherited by closed subspaces. □

Following [4], a real or complex JB*-triple U is defined to be *weakly compact* if the operator $Q(a) : U \to U$ defined by $Q(a)(x) := \{a, x, a\}$ is weakly compact for every $a \in U$ and to be *compact* if $Q(a)$ is compact for all $a \in U$. Let E be a real JB*-triple with complexification $\widehat{E}$. Clearly, E is weakly compact whenever $\widehat{E}$ is. On the other side, if E is weakly compact, i.e. $Q(a) : E \to E$ is weakly compact for every a in E, we have $Q(a) : \widehat{E} \to \widehat{E}$ is weakly compact for every $a \in E$, by [21, Theorem 10]. Let $Q(a,b) : \widehat{E} \to \widehat{E}$ be the mapping given by $Q(a,b)(x) := \{a, x, b\}$. The expression $2Q(a,b) = Q(a+b, a+b) - Q(a) - Q(a)$ implies that $Q(a,b) : \widehat{E} \to \widehat{E}$ is weakly compact. Since $\widehat{E} = E + iE$ and for every $x, y \in E$ the equality $Q(x+iy) = Q(x) - Q(y) + 2iQ(x,y)$ holds, we conclude that $\widehat{E}$ is weakly compact. Therefore, E is weakly compact if and only if its complexification is.

Compact and weakly compact complex JB*-triples were completely described in [4, §3 and §4]. More recently, in [3, Theorem 2.12 and Remmark 2.13], the authors show that a complex JB*-triple is weakly compact if and only if its norm is strongly subdifferentiable at every point of its unit sphere. Our next goal is to describe those real JB*-triples whose norm is strongly subdifferentiable at every point of its unit sphere.

Theorem 2.9. *Let E be a real JB*-triple. The following assertions are equivalent:*

1. *The norm of the complexification, $\widehat{E}$, of E is strongly subdifferentiable at every point of $S_{\widehat{E}}$.*

2. *The norm of E is strongly subdifferentiable at every point of S_E.*

3. *For every x in E, the complex triple spectrum of x is discrete.*

4. *$\widehat{E}$ is weakly compact.*

5. *E is weakly compact.*

Proof. Since the strong subdifferentiability at a norm-one point is inherited by subspaces, the implication (1.) $\Rightarrow$ (2.) is clear.

(2.) $\Rightarrow$ (3.) Let $\widehat{E}$ denote the complexification of E and let τ be the canonical conjugation on $\widehat{E}$ satisfying $\widehat{E}^{\tau} = E$. By Theorem 2.4 we conclude that the norm of $\widehat{E}$ is strongly subdifferentiable at every point x in S_E ($\|x\| = 1$ and $\tau(x) = x$). Let x be a norm-one element in E and let $\widehat{E}(x)$ denote the complex JB*-subtriple of $\widehat{E}$ generated by x. Since $\tau(x) = x$ we deduce that $\tau|_{\widehat{E}(x)}$ is a conjugation on $\widehat{E}(x)$. Since the strong subdifferentiability is inherited by subspaces we conclude that the norm of $\widehat{E}(x)$ is strongly subdifferentiable at every point $x \in S_{\widehat{E}(x)^{\tau}}$. It is known that $\widehat{E}(x)$ is JB*-triple isomorphic to the C*-algebra $C_0(S_x)$, therefore Lemma 2.8 implies that S_x is discrete.

(3.) $\Rightarrow$ (4.) Since the implication (3.) $\Rightarrow$ (2.) follows by Corollary 2.5, we deduce that (3.) $\Leftrightarrow$ (2.). We therefore assume that the norm of E is strongly subdifferentiable at every point of S_E.

Let z be in E and let $E(z)$ denote the real JB*-subtriple of E generated by z. By Zorn's lemma there is an abelian subtriple C containing $E(z)$ which is maximal with respect to inclusion. Let $\widehat{C}$ denote the complexification of C. Since C is abelian then $\widehat{C}$ is an abelian JB*-triple. It is well known that $\widehat{C}$ is triple isomorphic (and hence isometric) to $C_0(\Omega)$ for some locally compact Hausdorff space Ω. Since the strong subdifferentiability is inherited by subspaces it follows that the norm of $\widehat{C}$ is strong subdifferentiable at every point in S_C, by Theorem 2.4. Now Lemma 2.8 implies that Ω is discrete and hence there exists a family, $\{e_\alpha\}$, of mutually orthogonal minimal tripotents in C such that every element in C can be approximated in norm by linear combinations of $\{e_\alpha\}$. We claim that every e_α is also a minimal tripotent in E. Suppose on the contrary that there exists $0 \neq x \in E^1(e_{\alpha_0}) \setminus \mathbb{R}e_{\alpha_0}$, for some α_0. Let C' denote the real JB*-subtriple generated by C and x. Since for every $\alpha \neq \beta$ we have $e_\alpha \perp e_\beta$ we conclude that C' is an abelian real JB*-triple containing C which contradicts the maximality of C.

Therefore, every element in E can be approximated in norm by linear combinations of minimal tripotents in E. Since $\widehat{E} = E + iE$ we conclude, by [22, Lemma 3.2] (see also [2, Corollary 3.5]), that every element in $\widehat{E}$ can be approximated in norm by linear combinations of minimal tripotents in $\widehat{E}$. By [4, Theorem 3.4, $(i) \Leftrightarrow (vi)$] it follows that $\widehat{E}$ is weakly compact.

As we have seen in the comments preceding this Theorem, (4) $\Leftrightarrow$ (5). Finally (4) $\Leftrightarrow$ (1), by [3, Theorem 2.12 and Remmark 2.13]. □

3 Applications

The aim of this section is to obtain an alternative proof of Kaup's Banach-Stone Theorem for JB*-triples by applying the characterization of the points of strong subdifferentiability for the norm of a JB*-triple. This provides a complete proof to the statement settled by W. Werner in [23, Remarks 3.] and extend the method developed in the already quoted paper to the more general setting of JB*-triples.

Having in mind that the bidual of every real or complex JB*-triple E is a real or complex JBW*-triple, and since every tripotent in E is also a tripotent in its bidual, the proof of the following Lemma could be derived from [11, Lemma 2.1] and [11, Theorem 3.7] in the complex and real case, respectively.

Lemma 3.1. *Let E be a real or complex JB*-triple, and let e, u be tripotents in E. If $D(E, e)$ coincides with $D(E, u)$, then $e = u$.*

The following lemma generalizes [23, Lemma 3] to the setting of real and complex JB*-triples.

Lemma 3.2. *Let E be a real or complex JB*-triple, let e be a tripotent in E and $a \in S_E$. The following statements hold:*

(a) $a \in E_e$ if, and only if, $D(E, a) = D(E, e)$.

(b) If $x \in E_e$ satisfies $\|x - b\| < 1$ for all $b \in E_e$ then $x = e$.

Proof. Suppose first that E is a complex JB*-triple.
(a) $(\Rightarrow)$ Suppose $a \in E_e$. In particular we have $\{e, a, e\} = \{e, e, a\} = e$, which implies $P_2(e)(a) = e$. Since $\|a\| = 1$, [14, Lemma 1.6] assures that $P_1(e)(a) = 0$, and hence $a = e + z_0$, where $z_0 \in E_0(e)$. Let $f \in D(E, e)$. By [14, Proposition 1] we have $f = fP_2(e)$ and $f(a) = fP_2(e)(a) = f(e) = 1$. This implies $f \in D(E, a)$, and hence $D(E, e) \subseteq D(E, a)$.

To see the other inclusion, let C be the JB*-subtriple generated by e and $a = e + z_0$ ($z_0 \in E_0(e)$, $\|z_0\| < 1$). Since, by the Peirce arithmetic, e and z_0 are orthogonal and e is a tripotent, it follows that C coincides with $\mathbb{C}e \oplus^\infty D$, where D is the JB*-subtriple generated by z_0 which is triple isomorphic (and hence isometric) to $C_0(S_{z_0})$. Therefore, C is triple isomorphic (and hence isometric) to an abelian C*-algebra.

According with the notation of [23, Theorem 4] we can see that

$$a \in C_e = F_{e,0}(C) := \{y \in S_C : ye^* = ee^* \text{ and } \|y - e\| < 1\}.$$

Therefore, by [23, Lemma 3], we have $D(C, e) = D(C, a)$. Finally, let f in $D(E, a)$. It is clear that $f|_C$ lies in $D(C, a) = D(C, e)$ and hence $f(e) = 1$. This assures that $D(E, a) \subseteq D(E, e)$.

($\Leftarrow$) Suppose now that $D(E, a) = D(E, e)$. By [3, Theorem 2.7] we conclude that the norm of E is strongly subdifferentiable at every tripotent element of E. Since the strong subdifferentiability of the norm of E at a norm-one element x depends only on the set $D(E, x)$ (compare [13, Theorem 1.2 and Proposition 3.1]) it follows that the norm of E is also strongly subdifferentiable at a.

From [3, Theorem 2.7] we conclude that there is a tripotent u in E such that a lies in E_u. By the first part of the proof we have $D(E, u) = D(E, a) = D(E, e)$. By Lemma 3.1 we get $u = e$, which gives $a \in E_e$.

(b) Let C be the JB*-subtriple generated by e and $x = e + z_0$ ($z_0 \in E_0(e)$, $\|z_0\| < 1$). As we have seen in the first part of the proof, C is an abelian C*-algebra. By hypothesis we have $x \in C_e = F_{e,0}(C)$ and $\|x - b\| < 1$ for every $b \in C_e = F_{e,0}(C)$. By [23, Lemma 3 (ii)] we get $x = e$.

Suppose now that E is a real JB*-triple. Having in mind that [14, Lemma 1.6, and Propositsix 1] remain true for real JB*-triples, the proof of (a) is a repetition of the one given in the complex case but replacing [3, Theorem 2.7] by Corollary 2.5. To see (b) let $\widehat{E}$ denote the complexification of E (which is a complex JB*-triple) and let τ denote the canonical conjugation on $\widehat{E}$ such that $\widehat{E}^\tau = E$. Suppose $x \in E_e$ satisfies $\|x - b\| < 1$ for all $b \in E_e$.

As we have seen several times $x = e + z_0$, where $z_0 \in E_0(e)$, and clearly $x \in \widehat{E}_e$. Let $c \in \widehat{E}_e$ then $c = b_1 + ib_i$ for some $b_1, b_2 \in E$. The equalities $\{e, c, e\} = \{e, e, c\} = e = \tau(e)$ give us

$$\{e, b_1, e\} = \{e, e, b_1\} = e, \quad \{e, e, b_2\} = \{e, b_2, e\} = 0.$$

Therefore $b_1 \in E^1(e) = \widehat{E}^1(e) \subset \widehat{E}_2(e)$ and $b_2 \in E_0(e) \subset \widehat{E}_0(e)$. By [14, Lemma 1.3] it follows

$$1 = \|c\| = \|b_1 + ib_2\| = \max\{\|b_1\|, \|b_2\|\},$$

$$1 > \|e - c\| = \|e - b_1 - ib_2\| = \max\{\|e - b_1\|, \|b_2\|\},$$

and then $\|e - b_1\|, \|b_2\| < 1$. It is easy to see that

$$1 = \|e\| = \| \{e, b_1, e\} \| \leq \|b_1\|,$$

which shows $b_1 \in E_e$. By hypothesis $\|x - b_1\| < 1$. Finally

$$\|x - c\| = \|x - b_1 - ib_2\| = \max\{\|x - b_1\|, \|b_2\|\} < 1,$$

for every $c \in \widehat{E}_e$. Now, the proof in the complex case assures that $x = e$. □

Corollary 3.3. *Let $\Phi : E \to F$ be a surjective isometry between two real or complex JB*-triples. Then Φ preserves tripotents.*

Proof. Let e be a tripotent in E. By Corollary 2.5, it follows that the norm of E is strongly subdifferentiable at e. Since the strong subdifferentiability is preserved by surjective isometries, we can conclude that the norm of F is strongly subdifferentiable at $\Phi(e)$. By Corollary 2.5 (d), there is a (unique) tripotent u in F such that $\Phi(e) \in F_u$.

Let $x \in E_e$. By Lemma 3.2 (a) we deduce that

$$D(F, \Phi(x)) = (\Phi^*)^{-1} D(E, x) = (\Phi^*)^{-1} D(E, e) = D(F, \Phi(e)) = D(F, u).$$

Again Lemma 3.2 (a), implies $x \in F_u$. Therefore $\Phi(E_e) \subseteq F_u$. Similar arguments show the reciprocal inclusion and the equality $\Phi(E_e) = F_u$.

Given $y \in F_u = \Phi(E_e)$, there is $x \in E_e$ with $\Phi(x) = y$, thus

$$\|y - \Phi(e)\| = \|\Phi(x) - \Phi(e)\| = \|\Phi(x - e)\| = \|x - e\| < 1.$$

Now, Lemma 3.2 (b), gives us $\Phi(e) = u$. This shows that Φ preserves tripotents. □

It is well known that the fact that any surjective isometry between complex JB*-triples preserves tripotents can be applied to give an alternative proof to Kaup's Banach-Stone Theorem (see for instance [7] or [12, Proof of Theorem 2.2]).

Corollary 3.4. *Let $\Phi : E \to F$ be a surjective isometry between two complex JB*-triples. Then Φ is a triple isomorphism.*

Acknowledgments: The authors would like to thank A. Rodríguez Palacios for fruitful comments and discussions during the preparation of this paper and to the referee for his or her interesting suggestions.

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