New advances on the Grothendieck's inequality problem for bilinear forms on JB*-triples

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Abstract

1 Introduction

The results known as Grothendieck's inequalities began with the famous paper [8] in which A. Grothendieck proved the so-called "Grothendieck's inequalities" for commutative C*-álgebras. These inequalities were generalized by G. Pisier [18] and U. Haagerup [10, 9] to the setting of C*-algebras.

Every C*-algebra belongs to a more general class of Banach spaces known as JB*-triples (see definition and examples below). JB*-triples were introduced by Kaup [14] in the study of bounded symmetric domains in complex Banach spaces. The class of JB*-triples has been intensively developed in the last twenty years. In the setting of JB*-triples, Grothendieck's inequalities were studied by T. Barton and Y. Friedman [1], C.-H. Chu, B. Iochum and G. Loupias [3], A. M. Peralta [15] and A. M. Peralta and A. Rodríguez Palacios [16, 17].

The natural prehilbertian seminorms associated derived from states in a C*-algebra do not make sense in a JB*-triple because the latter needs not have, in general, a natural order structure. In the setting of JB*-triples, the prehilbertian seminorms associated to norm-one functionals are constructed as follows: Let φ be a norm-one element in the dual space of a JB*-triple

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E. Let z be a norm-one element in E (or in E^{**}) such that $\varphi(z) = 1$. By [1, Proposition 1.2] the mapping $(x, y) \mapsto \varphi\{x, y, z\}$ defines a positive sesquilinear form on E which does not depend on the element z. Thus the law $x \mapsto \|x\|_\varphi := (\varphi\{x, x, z\})^{\frac{1}{2}}$ $(x \in E)$ defines a prehilbert seminorm on E.

The main contribution of [15, 16] is the discovery that some technical result from the Banach space geometry on weak*-continuous bilinear forms is not true (see [16, Example 1 and comments before]). Therefore, previously published results on Grothendieck's inequalities for JB*-triples in [1, 3] cannot be considered fully proved. In the amendment provided in [16, Coroolaries 1 and 7] it is shown that the assertions in [1, Theorems 1.3 and 1.4] remains true when the seminorms of the form $\|.\|_\varphi$ are replaced by seminorms of the form $\|x\|_{\varphi_1,\varphi_2} = \sqrt{\|x\|^2_{\varphi_1} + \|x\|^2_{\varphi_2}}$. More precisely, there exists a universal constant $M > 0$ such that for every pair of JB*-triples (E, F) and every bounded bilinear form V on $E \times F$ there exist norm-one functionals $\varphi_1, \varphi_2 \in E^*$ and $\psi_1, \psi_2 \in F^*$ satisfying

$$|V(x, y)| \leq M \ \|V\| \ \|x\|_{\varphi_1,\varphi_2} \ \|y\|_{\psi_1,\psi_2} \tag{1}$$

for all $(x, y) \in E \times F$. However, until this moment we do not know a counterexample to the version of Grothendieck's inequality for JB*-triples established by Barton and Friedman. Therefore, it is natural to ask whether the seminorms of the form $\|x\|_{\varphi_1,\varphi_2}$ appearing in (1) can be replaced by seminorms of the form $\|x\|_\varphi$, as it is established in [1]. More concretely, let $\mathcal{G}$ denote the set of all bounded bilinear forms V on $E \times F$ such that there exist norm-one functionals $\varphi \in E^*$ and $\psi \in F^*$ satisfying

$$|V(x, y)| \leq M \ \|V\| \ \|x\|_\varphi \ \|y\|_\psi,$$

for all $(x, y) \in E \times F$. Although it is known that $\mathcal{G}$ is norm-dense in $L(^2(E \times F))$, the space of all bounded bilinear forms on $E \times F$ (see [16, Theorem 1]), we do not know if $\mathcal{G}$ coincides or not with whole space $L(^2(E \times F))$.

When E and F are JBW*-triples (JB*-triples which are dual Banach spaces) and the bilinear form ia assumed to be separately weak*-continuous it seems natural to request that the functionals appearing in (1) belong to the preduals of E and F, respectively.

In the present paper we present a big class of JB*-triples where the above problem have a positive answer. We shall show that this class includes all Cartan factors and all atomic JBW*-triples.

Let X and Y be Banach spaces. Throughout the paper, $L(X, Y)$ will stand for the Banach space of all bounded linear operators from X to Y. We usually write $L(X)$ instead of $L(X, X)$.

A *JB*-triple* is a complex Banach space E equipped with a continuous triple product

$$\{.,.,.\} : E \otimes E \otimes E \to E$$

$$(x, y, z) \mapsto \{x, y, z\}$$

which is bilinear and symmetric in the outer variables and conjugate linear in the middle one and satisfies:

(a) (*Jordan Identity*)

$$L(x, y)L(a, b) - L(a, b)L(x, y) = L(L(x, y)a, b) - L(a, L(y, x)b),$$

for all $x, y, a, b \in E$, where $L(x, y) : E \to E$ is the linear mapping given by $L(x, y)z = \{x, y, z\}$;

(b) The map $L(x, x)$ is an hermitian operator with non-negative spectrum for all $x \in E$;

(c) $\| \{x, x, x\} \| = \|x\|^3$ for all $x \in E$.

Every C*-algebra is a JB*-triple with respect to the triple product

$$\{x, y, z\} = 2^{-1}(xy^*z + zy^*x).$$

Every JB*-algebra is a JB*-triple with triple product given by

$$\{a, b, c\} = (a \circ b^*) \circ c + (c \circ b^*) \circ a - (a \circ c) \circ b^*.$$

The (classical) Cartan factors constitute and interesting variety of examples of JB*-triples. Cartan factors are defined as follows (see [13] for more details): Let H and K be complex Hilbert spaces. A type 1 Cartan factor is a JB*-triple of the form $L(H, K)$ with operator norm and triple product defined by

$$\{x, y, z\} = \frac{1}{2}(xy^*z + zy^*x). \tag{2}$$

Let $j : H \to H$ be a conjugation (conjugate linear isometry of period 2) on H. For each $x \in L(H)$ we define $x^t = jx^*j$. The the law $x \mapsto x^t$ defines linear isometry of period 2 on $L(H)$. $S_n := \{x \in L(H) : x^t = -x\}$ with product (2) and operator norm is a Cartan factor of type 2 or of *symplectic type* and $H_n := \{x \in L(H) : x^t = x\}$ with product (2) and operator norm is a Cartan factor of type 3 or of *symplectic type.*

A type-4 Cartan factor, (also called *spin factor*) is a complex Hilbert space provided with a conjugation $x \mapsto \overline{x}$, triple product

$$\{x, y, z\} = (x|y)\, z + (z|y)\, x - (x|\bar{z})\, \bar{y},$$

and norm given by $\|x\|^2 = (x|x) + \sqrt{(x|x)^2 - |\,(x|\overline{x})\,|^2}$.

The type 6 Cartan factor is the space $H_3(\mathbb{O})$ of all 3×3 hermitian matrices over the complex Cayley algebra $\mathbb{O}$ with product

$$\{x, y, z\} = (x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^*, \tag{3}$$

where $a \circ b = \frac{1}{2}(ab + ba)$. The type 5 Cartan factor consists of all 1 by 2 matrices over $\mathbb{O}$ and can be regarded as a JB*-subtriple of the Cartan factor of type 6.

A JBW*-triple is a JB*-triple which is also a dual Banach space. The bidual of a JB*-triple is a JBW*-triple with respect to a triple product extending the one of E [4]. Every JBW*-triple has a unique predual and its triple product is separately weak* continuous [2].

Let E be a JB*-triple. An element $e \in E$ is said to be a tripotent if $\{e, e, e\} = e$. The set of all tripotents of E is denoted by $\mathrm{Tri}(E)$. Given a tripotent $e \in E$ there exist a decomposition of E in terms of the eigenspaces of $L(e, e)$ given by

$$E = E_0(e) \oplus E_1(e) \oplus E_2(e), \tag{4}$$

where $E_k(e) := \{x \in \mathcal{E} : L(e, e)x = \frac{k}{2}x\}$ is a subtriple of E $(k : 0, 1, 2)$. The natural projection of E onto $E_k(e)$ will be denoted by $P_k(e)$. The following rules are also satisfied

$$\{E_k(e), E_l(e), E_m(e)\} \subseteq E_{k-l+m}(e),$$

$$\{E_0(e), E_2(e), E\} = \{E_2(e), E_0(e), E\} = 0,$$

where $E_{k-l+m}(e) = 0$ whenever $k - l + m$ is not in $\{0, 1, 2\}$. It is also known that $E_2(e)$ is a unital JB*-algebra with respect to the product and involution given by $x \circ y = \{x, e, y\}$ and $x^* = \{e, x, e\}$, respectively. When E is a JBW*-triple then $E_2(e)$ is a JBW*-algebra.

For background about JB- and JBW-algebras the reader is referred to [11]. We recall that JB-algebras (respectively, JBW-algebras) are nothing but the self-adjoint parts of JB*-algebras (respectively, JBW*-algebras) [21] (respectively, [5]).

2 Grothendieck's Inequalities

The natural strategy to approach Grothendieck's inequalities in the setting of JB*-triples is based on the study of the so called "Little Grothendieck's Theorem" for JB*-triples. The results in [16] provide a new approach to Grothendieck's inequalities for JB*-triples, which allows us to avoid some difficulties in the proofs of [1, Theorems 1.3 and 1.4] and [3, Proposition 4, Theorem 6]. In [16, Corollary 1] it is proved the following Little Grothendieck's Theorem:

Theorem 2.1. *Let $\mathcal{W}$ be a complex JBW*-triple and T a weak*-continuous linear operator from $\mathcal{W}$ to a complex Hilbert space. Then there exist norm-one functionals $\varphi_1, \varphi_2 \in \mathcal{W}_*$ such that, for every $x \in \mathcal{W}$, we have*

$$\|T(x)\| \leq 2\|T\|\|x\|_{\varphi_1,\varphi_2}.$$

$\square$

The question if in the above Theorem we can replace the seminorm $\|.\|_{\varphi_1,\varphi_2}$ by a seminorm of the form $\|.\|_{\varphi}$ remains open. The aim of this section is to give an affirmative answer to the above question in the case of an atomic JBW*-triple.

Remark 2.2. Let E be a finite dimensional JB*-triple and let T be a bounded linear operator from E to a complex Hilbert space $\mathcal{H}$. Since T attains its norm we conclude from [16, Lemma 3] that there exists a norm-one functional $\varphi \in E_*$ satisfying

$$\|T(x)\| \leq \sqrt{2}\ \|T\|\ \|x\|_{\varphi},$$

for all $x \in E$.

Let H and K be Hilbert spaces. Let h in H and k in K we denote by $k \otimes h$ the element in $L(H,K)$ given by $k \otimes h(x) := (x|h)k$ $(x \in H)$.

Proposition 2.3. *Let H be a complex Hilbert space and let p be a projection in $L(H)$. Suppose that $p(H) = K$ is infinite dimensional. Let $E = L(H,K)$ be the JBW*-subtriple of $L(H)$ of all bounded linear operators from H to K. Then for every normal state $\phi \in L(H)_*$ there exists a norm-one element $\varphi \in E_*$ satisfying*

$$\|x\|_{\phi} \leq \sqrt{6}\ \|x\|_{\varphi}$$

for all $x \in E$.

Proof. Let ϕ be a normal state of $L(H)$. Write $q = 1 - p$. Let $x \in L(H)$. By the Cauchy-Schwarz inequality we deduce that

$$|\phi(px^*xq)|^2 = |\phi(qx^*xp)|^2 \leq \phi(px^*xp)\ \phi(qx^*xq),$$

which implies that

$$\phi(x^*x) = \phi(px^*xp) + \phi(px^*xq) + \phi(qx^*xp) + \phi(qx^*xq)$$

$$\leq \phi(px^*xp) + \phi(qx^*xq) + 2\ \sqrt{\phi(px^*xp)\ \phi(qx^*xq)} \leq 2(\phi(px^*xp) + \phi(qx^*xq)).$$

Write $\varphi_1(x) := \phi(pxp)$ and $\varphi_2(x) := \phi(qxq)$. Then φ_1 and φ_2 are positive normal functionals of $L(H)$, $\|\varphi_1\| + \|\varphi_2\| = \varphi_1(1) + \varphi_2(1) = \phi(1)$, $\varphi_1(p) = \varphi_1(1)$, $\varphi_2(q) = \varphi_2(1)$, and for every positive element $y \in L(H)$ we have

$$\phi(y) \leq 2(\varphi_1(y) + \varphi_2(y)). \tag{5}$$

Since φ_1 is a positive normal functional of $pL(H)p$, it follows that

$$\varphi_1(x) := \sum_{n\in\mathbb{N}} \lambda_n (x(\eta_n)|\eta_n) \quad (x \in L(H)), \tag{6}$$

where (η_n) is an orthonormal sequence in $p(H) = K$ and (λ_n) is a sequence of non-negative real numbers with $\sum_{n\in\mathbb{N}} \lambda_n = \|\varphi_1\| = \varphi_1(1)$ (compare [20, Corollary 1.15.4]). Analogously we deduce that

$$\varphi_2(x) := \sum_{n\in\mathbb{N}} \mu_n (x(\xi_n)|\xi_n) \quad (x \in L(H)), \tag{7}$$

where (ξ_n) is an orthonormal sequence in $p(H) = K$ and (μ_n) is a sequence of non-negative real numbers with $\sum_{n\in\mathbb{N}} \mu_n = \|\varphi_2\| = \varphi_2(1)$.

If $\|\varphi_2\| = 0$ then $\phi = \varphi_1$ is a norm-one element in $(pL(H)p)_* = (Ep)_* \subseteq E_*$, which gives the desired conclusion for $\varphi = \phi$.

If $\|\varphi_1\| = 0$ then $\phi = \varphi_2$ is a normal state of $qL(H)q$. Since $p(H) = K$ is infinite dimensional we can choose an orthonormal sequence (ν_n) in K. Let ϕ be the norm-one functional in E_* defined by

$$\varphi(x) = \sum_{n\in\mathbb{N}} \mu_n (x(\xi_n)|\nu_n) \quad (x \in E).$$

Let us denote by e the tripotent in E given by $e = \sum_{n\in\mathbb{N}} \nu_n \otimes \xi_n$. Let x be an arbitrary element in E. Since $\varphi(e) = 1$ and $\phi(q) = 1$, we deduce that

$$\|x\|_{\varphi}^2 = \varphi\{x, x, e\} = \frac{1}{2}\varphi(xx^*e + ex^*x) = \sum_{n\in\mathbb{N}} \frac{\mu_n}{2}((xx^*e + ex^*x)(\xi_n)|\nu_n)$$

$$= \sum_{n\in\mathbb{N}} \frac{\mu_n}{2}\ \left(\|x^*(\nu_n)\|^2 + \|x(\xi_n)\|^2\right)$$

and

$$\|x\|_{\phi}^2 = \phi\{x, x, q\} = \frac{1}{2}\phi(xx^*q + qx^*x) = \frac{1}{2}\phi(qx^*x)$$

$$= \sum_{n\in\mathbb{N}} \frac{\mu_n}{2}((qx^*x)(\xi_n)|\xi_n) = \sum_{n\in\mathbb{N}} \frac{\mu_n}{2}\ \left(\|x(\xi_n)\|^2\right).$$

From the above expressions we see that

$$\|x\|_{\phi}^2 \leq \|x\|_{\varphi}^2,$$

for all $x \in E$.

Finally we assume $\|\varphi_1\|, \|\varphi_2\| \neq 0$. Set $\phi_i := \|\varphi_i\|^{-1}\ \varphi_i$ $(1 \leq i \leq 2)$ and $\widetilde{\phi} = 2^{-1}(\varphi_1 + \varphi_2)$. It is clear that $\widetilde{\phi}$ is a normal state of $L(H)$. Since for each $x \in E$, $\{x, x, 1\}$ is a positive element in $L(H)$, we conclude from (5) that the inequality

$$\|x\|_{\phi}^2 = \phi\{x, x, 1\} \leq 2(\phi_1 + \phi_2)\{x, x, 1\} = 4\widetilde{\phi}\{x, x, 1\} = 4\|x\|_{\widetilde{\phi}}^2 \qquad (8)$$

holds for each $x \in E$. From (6) and (7) we see that the expression

$$\|x\|_{\widetilde{\phi}}^2 = \widetilde{\phi}\{x, x, 1\} = \frac{1}{2}\ \widetilde{\phi}(xx^* + x^*x)$$

$$= \sum_{n\in\mathbb{N}} \Big(\frac{\lambda_n}{4\|\varphi_1\|}\left(\|x^*(\eta_n)\|^2 + \|x(\eta_n)\|^2\right) + \frac{\mu_n}{4\|\varphi_2\|}\|x(\xi_n)\|^2\Big), \qquad (9)$$

holds for all $x \in E$.

Let φ be the norm-one functional in E_* given by

$$\varphi(x) = \sum_{n\in\mathbb{N}} \delta_n(x(\xi_n^{'})|\eta_n)\ \ (x \in E),$$

where $(\xi_n^{'})$ is the orthonormal sequence in H defined by $\xi_{2k}^{'} = \eta_k$ and $\xi_{2k-1}^{'} = \xi_k$ $(\forall k \in \mathbb{N})$, and (δ_n) is the sequence in $\mathbb{R}_0^+$ given by $\delta_{2k} = \frac{\lambda_{2k}+\lambda_k}{3\ \|\varphi_1\|}$

and $\delta_{2k-1} = \frac{\mu_k \ \|\varphi_2\|^{-1}+\lambda_{2k-1} \ \|\varphi_1\|^{-1}}{3}$ $(\forall k \in \mathbb{N})$. It is not hard to check from the above definition that $3\delta_{2n} \geq \lambda_n \ \|\varphi_1\|^{-1}$, $3\delta_{2n-1} \geq \mu_n \ \|\varphi_2\|^{-1}$, $3\delta_n \geq \frac{\lambda_n}{\|\varphi_1\|}$, and $\sum_{n\in\mathbb{N}} \delta_n = 1$. Set $e = \sum_{n\in\mathbb{N}} \eta_n \otimes \xi_n^{'} \in E$. It is easy to see that $\varphi(e) = 1$. Thus, for each $x \in E$ we get

$$\|x\|_{\varphi}^2 = \varphi\{x, x, e\} = \sum_{n\in\mathbb{N}} \frac{\delta_n}{2} \|x^*(\eta_n)\|^2 + \sum_{n\in\mathbb{N}} \frac{\delta_{2n-1}}{2} \|x(\xi_n)\|^2 + \sum_{n\in\mathbb{N}} \frac{\delta_{2n}}{2} \|x^*(\eta_n)\|^2.$$

From (8), (9) and the above expression we deduce that

$$\|x\|_{\phi}^2 \leq 4\|x\|_{\widetilde{\phi}}^2 \leq 6 \ \|x\|_{\varphi}^2,$$

for all $x \in E$. □

Corollary 2.4. *Let $E = L(H, K)$ be a type 1 Cartan factor with H and K infinite dimensional and let $\mathcal{H}$ be a complex Hilbert space. Then for every weak*-continuous linear operator T from E to $\mathcal{H}$ there exists a norm-one functional $\varphi \in E_*$ satisfying*

$$\|T(x)\| \leq 2\sqrt{3} \ \|T\| \ \|x\|_{\varphi},$$

for all $x \in E$.

Proof. Since $L(H, K)$ and $L(K, H)$ are triple isomorphic we may assume that K is a Hilbert subspace of H. Let p be a projection in $L(H)$ such that $p(H) = K$. Let $T : E \to \mathcal{H}$ be a weak*-continuous linear operator. The law $z \mapsto T(pz)$ defines a weak*-continuous linear operator $\widetilde{T}$ from $L(H)$ to $\mathcal{H}$ which satisfies $\widetilde{T}(x) = T(x)$ for all $x \in E$. By [10, Proposition 2.3] (see also [16, Remark 1]) there exists a normal state $\phi \in L(H)_*$ satisfying

$$\|\widetilde{T}(z)\| \leq \sqrt{2} \ \|T\| \ \|z\|_{\phi},$$

for all $z \in L(H)$. From Proposition 2.3 it follows that there exists a norm-one functional $\varphi \in E_*$ satisfying

$$\|x\|_{\phi} \leq \sqrt{6} \ \|x\|_{\varphi},$$

for all $x \in E$. Therefore

$$\|T(x)\| \leq 2\sqrt{3} \ \|T\| \ \|x\|_{\varphi},$$

for all $x \in E$. □

The case of a type 1 Cartan factor $E = L(H, K)$ with $\dim(K)$ finite will need an special development.

Remark 2.5. Let E be a JBW*-triple. From [16, Remark 3, $(i) \Leftrightarrow (iii)$] it follows that the following assertions are equivalent:

(a) There is a universal constant G such that, for every couple (φ_1, φ_2) of norm-one functionals in $E_* \times E_*$, we can find a norm-one functional $\varphi \in E_*$ satisfying

$$\|x\|_{\varphi_i} \leq G\|x\|_\varphi$$

for every $x \in E$ and $i = 1, 2$.

(b) There is a universal constant $\widetilde{G}$ such that for every weak*-continuous linear operator T from E to a complex Hilbert space, there exists a norm-one functional $\varphi \in E_*$ satisfying

$$\|T(x)\| \leq \widetilde{G}\ \|T\|\ \|x\|_\varphi$$

for all $x \in W$.

Moreover, in the implication $(a) \Rightarrow (b)$ we can take $\widetilde{G} = 2\sqrt{2}G$. and in $(b) \Rightarrow (a)$ we can choose $G = \sqrt{2}\widetilde{G}$ (compare [16, Corollary 1]).

Let V and W be JBW*-triples satisfying one of the above equivalent statements. From [16, Remark 3 $(ii) \Leftrightarrow (i)$] (see also the proof of [16, Theorem 6]), we deduce that there is a universal constant $\widehat{G} = \widetilde{G}^2(1 + 2\sqrt{3})$ such that for every separately weak*-continuous bilinear form U on $V \times W$, there are norm-one functionals $\varphi \in V_*$, and $\psi \in W_*$ satisfying

$$|U(x, y)| \leq \widehat{G}\ \|U\|\ \|x\|_\varphi \|y\|_\psi$$

for all $(x, y) \in V \times W$.

The following result describes the hilbertian semi-norms of the form $\|.\|_\varphi$ in a type 1 Cartan factor.

Let A be a C*-algebra with involution $*$. Let $\circ$ denote the natural Jordan product on A defined by $x \circ y = \frac{1}{2}(xy + yx)$. It is well known that A has a JB*-algebra structure with respect to the product $\circ$, the involution $*$, and the natural norm. The JB*-algebra $(A, \circ, *)$ will be denoted by A^+. Moreover A^+ is a JBW*-algebra whenever A is a von Neumann algebra. It is also known that A^+ and A has the same normal states.

Lemma 2.6. *Let $E = L(H,K)$ be a type 1 Cartan factor. Suppose that there exists a projection $p \in L(H)$ with $p(H) = K$. Let φ be a norm-one functional in E_*. Then there exists a partial isometry $e \in L(H)$ such that $pe = e \in E$, an orthonormal sequence (ξ_n) in $e^*(H)$, and a sequence of non-negative real numbers (λ_n) such that $\sum_{n\in\mathbb{N}} \lambda_n = 1$ and*

$$\varphi(x) = \sum_{n\in\mathbb{N}} \lambda_n \ (x(\xi_n)|e(\xi_n)),$$

for all $x \in E$. As a consequence, for each $x \in E$ we have

$$\|x\|_\varphi^2 = \sum_{n\in\mathbb{N}} \frac{\lambda_n}{2} \ \Big(\|x^*(e(\xi_n))\|^2 + \|x(\xi_n)\|^2\Big).$$

Proof. It is immediate that every tripotent $e \in E$ is also a tripotent in $L(H)$, since $E = pL(H)$ is a JBW*-subtriple of $L(H)$. Thus every tripotent $e \in E$ is a partial isometry $e \in L(H)$ satisfying $pe = e$. Let e be a tripotent in E. Then $ee^* = p_1$ and $e^*e = q_1$ are projections in $L(H)$ with $p_1 \leq p$ and $e|_{e^*(H)} : e^*(H) = p_1(H) \to e(H) = q_1(H)$ is a surjective isometry. It is easy to check that $E_2(e) = p_1 L(H) q_1 = p_1 L(H) q_1$.

Let us denote by $\bullet_e$ and $\sharp_e$ the product and involution on $E_2(e)$ given by

$$x \bullet_e y := xe^*y \quad (x, y \in E_2(e))$$

and

$$x^{\sharp_e} = ex^*e \quad (x \in E_2(e)),$$

respectively. It is clear that $(E_2(e), \bullet_e, \sharp_e)$ is a von Neumann algebra and the mapping

$$\begin{aligned} E_2(e) &\to L(e^*(H)) \\ x &\mapsto e^*x \end{aligned}$$

is a *-isomorphism from $(E_2(e), \bullet_e, \sharp_e)$ to $L(e^*(H))$.

Let φ be a norm-one functional in E_*. By [6, Proposition 2], there exists a tripotent $e \in E$ such that $\varphi = \varphi P_2(e)$ and $\varphi|_{E_2(e)}$ is a positive normal functional on the JBW*-algebra $(E_2(e), \circ, *) = (E_2(e), \bullet_e, \sharp_e)^+$. Therefore, by [20, Corollary 1.15.4], there exists an orthonormal sequence (ξ_n) in $e^*(H)$, and a sequence of non-negative real numbers (λ_n) such that $\sum_{n\in\mathbb{N}} \lambda_n = 1$ and

$$\varphi(x) = \sum_{n\in\mathbb{N}} \lambda_n \ (x(\xi_n)|e(\xi_n)),$$

for all $x \in E_2(e)$. Finally, the above expression remains valid for all $x \in E$, since $\varphi = \varphi P_2(e)$ and $P_2(e)(x) = ee^*xe^*e$ $(\forall x \in E)$. $\square$

Remark 2.7. Let $E = L(H, K)$ be a type 1 Cartan factor with $\dim(H) \geq \dim(K)$. Let φ be a norm-one element in the predual of E and let e be the tripotent given in E given by Lemma 2.6 above. We claim that we can always assume that ee^* coincide with the orthogonal projection of H onto K (i.e., $ee^*(H) = e(H) = K$). Indeed, from the above proposition we deduce that there is an orthonormal sequence (ξ_n) in $e^*(H)$, and a sequence of non-negative real numbers (λ_n) such that $\sum_{n\in\mathbb{N}} \lambda_n = 1$ and

$$\varphi(x) = \sum_{n\in\mathbb{N}} \lambda_n \ (x(\xi_n)|e(\xi_n)),$$

for all $x \in E$. If $e(H) \neq K$ we write $K_1 = (e(H))^{\perp} \cap K$. Since $\dim(H) \geq \dim(K)$, there exists a Hilbert subspace $H_1 \subseteq (e^*(H))^{\perp} \cap H$ and a surjective isometry e_1 mapping H_1 to K_1. Then, when e_1 is regarded as a tripotent in E it follows that $u = e + e_1$ is a tripotent in E satisfying $\varphi(u) = 1$ and $u(H) = K$.

Proposition 2.8. *Let K be a finite dimensional subspace of a Hilbert space H. Let $E = L(H, K)$ be a type 1 Cartan factor. Then for every couple of norm-one functionals $\varphi_1, \varphi_2 \in E_*$ there exists a norm-one functional $\varphi \in E_*$ satisfying*

$$\|x\|_{\varphi_i} \leq 2\sqrt{2}\ \|x\|_{\varphi},$$

for all $x \in E$, $i \in \{1, 2\}$.

Proof. Let p denote the orthogonal projection of H onto K. Let φ_1, φ_2 norm-one functionals in E_*. By Lemma 2.6 there are partial isometries $e_1, e_2 \in L(H)$ such that $pe_i = e_i \in E$ $(i \in \{1, 2\})$, orthonormal sequences $(\xi_n) \subset e_1^*(H)$ and $(\eta_n) \subset e_2^*(H)$, and sequences of non-negative real numbers (λ_n) and (μ_n) such that $\sum_{n\in\mathbb{N}} \lambda_n = \sum_{n\in\mathbb{N}} \mu_n = 1$,

$$\varphi_1(x) = \sum_{n\in\mathbb{N}} \lambda_n \ (x(\xi_n)|e_1(\xi_n)) \quad (\forall x \in E), \tag{10}$$

and

$$\varphi_2(x) = \sum_{n\in\mathbb{N}} \mu_n \ (x(\eta_n)|e_2(\eta_n)) \quad (\forall x \in E), \tag{11}$$

As a consequence, for each $x \in E$ we have

$$\|x\|^2_{\varphi_1} = \sum_{n\in\mathbb{N}} \frac{\lambda_n}{2} \left(\|x^*(e_1(\xi_n))\|^2 + \|x(\xi_n)\|^2\right) \tag{12}$$

and

$$\|x\|_{\varphi_2}^2 = \sum_{n\in\mathbb{N}} \frac{\mu_n}{2} \left(\|x^*(e_2(\eta_n))\|^2 + \|x(\eta_n)\|^2\right). \tag{13}$$

Let H_1 be the subspace of H generated by $e_1^*(H)$ and $e_2^*(H)$ and let p_1 be the orthogonal projection of H onto H_1. Since K is finite dimensional and for each $i \in \{1,2\}$, $e_i|_{e_i^*(H)} : e_i^*(H) \to e_i(H) \subseteq K$ is a surjective isometry, we conclude that H_1 is finite dimensional. Set $F = Ep_1 = pL(H)p_1$. Then F is a finite dimensional JBW*-subtriple of E and $e_1, e_2 \in F$. Since $\|.\|_{\varphi_1,\varphi_2}|_F$ comes from a suitable separately weak*-continuous positive sesquilinear form $(.|.)$ on F given by the equality $\|x\|_{\varphi_1,\varphi_2}^2 := (x|x)$, it follows from the proof of [19, Corollary] that there exists a weak*-continuous linear operator T from F to a Hilbert space satisfying $\|T(x)\| = \|x\|_{\varphi_1,\varphi_2}$ for each $x \in F$. Since F is finite dimensional, it follows from Remark 2.2 that there exists a norm-one functional $\varphi \in F_* \subseteq E_*$ satisfying

$$\|y\|_{\varphi_1,\varphi_2}^2 \leq 2\|y\|_{\varphi}^2, \tag{14}$$

for all $y \in F$. Let e be a tripotent in F such that $\varphi(e) = 1$. We note that $F_2(e) = E_2(e)$ and $ep_1 = e$. We may also assume $ee^* = p$ (see Remark 2.7).

Write $q_1 = 1 - p_1$. Then $E = F \oplus Eq_1$. Let $z_3 \in Eq_1$. Since $\{z_3, z_3, e\}$ is a positive element in the von Neumann algebra $(E_2(e), \bullet_e, \sharp_e)$ (the latter is defined in the proof of 2.6), then there exists $y \in E_2(e)$ satisfying $y^{\sharp_e} = y$ and $\{z_3, z_3, e\} = y \bullet_e y$. From the equality $y^{\sharp_e} = ey^*e = y$ we deduce that $ye^* = ey^*$, $e^*y = y^*e$ and hence $\{y, y, e\} = yy^*e = \{z_3, z_3, e\} = yy^*e = \frac{1}{2}z_3z_3^*e$. As a consequence we get $yy^* = yy^*ee^* = \frac{1}{2}z_3z_3^*ee^* = \frac{1}{2}z_3z_3^*$, and

$$\|y\|_{\varphi} = \|z_3\|_{\varphi}.$$

It follows from (14) that

$$2\|z_3\|_{\varphi}^2 = 2\|y\|_{\varphi}^2 \geq \|y\|_{\varphi_i}^2 \quad (1 \leq i \leq 2).$$

We compute now the right hand side of the above inequality. From (10) and (11) it is easily seen that $\varphi_i(e_iy^*y) \geq 0$ for all $i \in \{1,2\}$. Thus

$$2\|z_3\|_{\varphi}^2 = 2\|y\|_{\varphi}^2 \geq \|y\|_{\varphi_i}^2 = \varphi_i\{y, y, e_i\} = \frac{1}{2}\varphi_i(yy^*e_i + e_iy^*y)$$

$$\geq \frac{1}{2}\varphi_i(yy^*e_i) = \frac{1}{4}\varphi_i(z_3z_3^*e_i) = \frac{1}{2}\varphi_i\{z_3, z_3, e_i\} = \frac{1}{2}\|z_3\|_{\varphi_i}^2.$$

Therefore, for each $z_3 \in Eq_1$ and $i \in \{1,2\}$ we get

$$\|z_3\|^2_{\varphi_i} \leq 4\|z_3\|^2_{\varphi} \tag{15}$$

Finally, let $x \in E$. Then $x = y + z_3$ for suitable $y \in F$ and $z_3 \in Eq_1$. From (14) and (15) we obtain:

$$\|x\|_{\varphi_i} \leq \|y\|_{\varphi_i} + \|z_3\|_{\varphi_i} \leq 2\,\left(\|y\|_{\varphi} + \|z_3\|_{\varphi}\right) \leq 2\sqrt{2}\,\sqrt{\|y\|^2_{\varphi} + \|z_3\|^2_{\varphi}},$$

for all $i \in \{1,2\}$. Since $ez_3^* = ep_1q_1z_3^* = 0 = yz_3^*$ we deduce that

$$\varphi\{z_3, y, e\} = \overline{\varphi\{y, z_3, e\}} = \varphi(0) = 0$$

and hence

$$\|x\|^2_{\varphi} = \|y\|^2_{\varphi} + \|z_3\|^2_{\varphi},$$

which implies

$$\|x\|_{\varphi_i} \leq 2\sqrt{2}\,\|x\|_{\varphi},$$

for all $i \in \{1,2\}$. □

The following corollary shows that every rectangular type 1 Cartan factor satisfies the Little Grothendieck's inequality.

Corollary 2.9. *Let $E = L(H,K)$ be a type 1 Cartan factor with $dim(H) \geq dim(K)$. Then for every complex Hilbert space $\mathcal{H}$ and every weak*-continuous linear operator $T : E \to \mathcal{H}$ there exists a norm-one functional $\varphi \in E_*$ satisfying*

$$\|T(x)\| \leq 8\,\|T\|\,\|x\|_{\varphi},$$

for all $x \in E$

Proof. When H and K are finite dimensional then E is finite dimensional and hence Remark 2.2 gives the desired conclusion. If H and K are infinite dimensional then the statement follows from Proposition 2.3. Finally, if H is infinite dimensional and K is finite dimensional the conclusion follows from Remark 2.5 and Proposition 2.8. □

We have already proved the Little Grothendieck's inequality in the particular case of a finite dimensional Cartan factor (see Remark 2.2) and in the case of a rectangular Cartan factor (Corollary 2.9). We shall discuss now the remaining Cartan factors.

Let J be a JB*-triple. We recall that a tripotent $u \in J$ is said to be *unitary* if $L(u,u)$ coincides with the identity operator on J. In this case

$J = J_2(u)$ and hence J is a JB*-algebra with product and involution given by $x \circ y = \{x, u, y\}$ and $x^* = \{u, x, u\}$, respectively. When E is a JBW*-triple with a unitary element u then E is a JBW*-algebra with respect to the product and involution given above. We can now rephrase [16, Theorem 4] as follows.

Proposition 2.10. *Let $M > 2$ and let E be a JBW*-triple with a unitary element u. Then for every complex Hilbert space and every weak*-continuous linear operator $T : E \to H$ there exists a norm-one functional $\varphi \in E_*$ such that*

$$\|T(x)\| \leq M\ \|T\|\ \|x\|_\varphi,$$

for all $x \in E$.

Proof. Let T be a weak*-continuous linear operator from E to a complex Hilbert space. Since E contains a unitary element u, then E is a JBW*-algebra with product and involution given by $x \circ y = \{x, u, y\}$ and $x^* = \{u, x, u\}$, respectively. By [16, Theorem 4], there exists a norm-one positive linear functional $\varphi \in E_*$ such that

$$\|T(x)\| \leq M\ \|T\|\ \left(\varphi(x \circ x^*)\right)^{\frac{1}{2}},$$

for all $x \in E$. Since φ is norm-one and positive then $\varphi(u) = 1 = \|\varphi\|$, and hence for each $x \in E$ we have $\|x\|_\varphi^2 = \varphi\{x, x, u\} = \varphi(x \circ x^*)$, which completes the proof. $\square$

Let S be a spin factor and let u be a norm-one element in S satisfying $u = \overline{u}$. It is easily seen that $L(u, u)$ coincides with the identity operator on S and hence u is a unitary element in S. It is also known that every Cartan factor of type 1 with $\dim(H)$ $=\dim(K)$, every Cartan factor of type 2 with $\dim(H)$ even , or infinite, every Cartan factors of type 3 and every type 6 Cartan factor contains a unitary element (see for instance [12, Proposition 2]). As a consequence, we can assure that when C is one of the above Cartan factors and Ω is a hyperstonean compact Hausdorff space then $C(\Omega, C)$ is a JBW*-triple containing a unitary element.

Corollary 2.11. *Let $E = C(\Omega, C)$, where Ω is a hyperstonean Hausdorff space and C is a Cartan factor of type 1 with with dim(H) =dim(K), or a Cartan factor of type 2 with dim(H) even , or infinite, or a Cartan factors of type 3, or a spin factor, or a type 6 Cartan factor. Let $M > 2$. Then for every complex Hilbert space and every weak*-continuous linear operator $T : E \to H$ there exists a norm-one functional $\varphi \in E_*$ such that*

$$\|T(x)\| \leq M\ \|T\|\ \|x\|_\varphi,$$

for all $x \in E$. $\square$

The next theorem shows that the family of all JBW*-triples satisfying the Little Grothendieck's inequality is stable by ℓ_∞-sums.

Theorem 2.12. *Let $M > 0$. Let $\{E_\alpha\}_{\alpha\in\Lambda}$ be a family of JBW*-triples such that for every $\alpha \in \Lambda$ and every weak*-continuous linear operator T from E_α to a complex Hilbert space H there exists a norm-one functional $\varphi_\alpha \in (E_\alpha)_*$ satisfying that*

$$\|T(x)\| \leq M \ \|T\| \ \|x\|_{\varphi_\alpha}, \tag{16}$$

for all $x \in E_\alpha$. Let $E = \bigoplus_{\alpha\in\Lambda}^{\ell_\infty} E_\alpha$. Then for every complex Hilbert space $\mathcal{H}$ and every weak-continuous linear operator $T : E \to \mathcal{H}$ there exists a norm-one functional $\varphi \in E_*$ such that*

$$\|T(x)\| \leq 4\sqrt{2} \ M \ \|T\| \ \|x\|_{\varphi},$$

for all $x \in E$.

Proof. By [16, Remark 3] (see also Remark 2.5 above) it suffices to prove that for every pair (φ_1, φ_2) of norm-one functionals in $E_* \times E_*$ there exists a norm-one functional $\varphi \in E_*$ satisfying

$$\|x\|_{\varphi_1,\varphi_2} \leq 2M\|x\|_{\varphi},$$

for all $x \in E$. Let φ_1, φ_2 norm-one functionals in $E_* = \bigoplus_{\alpha\in\Lambda}^{\ell_1} (E_\alpha)_*$. Then there are countably subsets $\Lambda_1, \Lambda_2 \subseteq \Lambda$ such that

$$\varphi_1 = \sum_{\alpha\in\Lambda_1} \mu_\alpha \phi_\alpha^1, \text{ and } \varphi_2 = \sum_{\alpha\in\Lambda_2} \nu_\alpha \phi_\alpha^2,$$

where $(\mu_\alpha) \in \ell_1(\Lambda_1)$, $(\nu_\alpha) \in \ell_1(\Lambda_2)$, $\mu_\alpha, \nu_\alpha \geq 0$, ϕ_α^j are norm-one elements in $(E_\alpha)_*$ $\forall j : 1, 2, \forall \alpha$ and $\|\varphi_1\| = \sum_{\alpha\in\Lambda_1} \mu_\alpha$ and $\|\varphi_2\| = \sum_{\alpha\in\Lambda_2} \nu_\alpha$.

Let $I = \Lambda_1 \cap \Lambda_2$, $I_1 = (\Lambda\backslash\Lambda_2) \cap \Lambda_1$ and $I_2 = (\Lambda\backslash\Lambda_1) \cap \Lambda_2$. By hypothesis and [16, Remark 3] it follows that for each $\alpha \in I$ there exists a norm-one functional $\psi_\alpha \in (E_\alpha)_*$ such that

$$\|x\|_{\varphi_1,\varphi_2} \leq \sqrt{2}M \ \|x\|_{\psi_\alpha},$$

for all $x \in E_\alpha$. Let φ be the norm-one functional in E_* defined by

$$\varphi := \sum_{\alpha\in I} \frac{\mu_\alpha + \nu_\alpha}{2} \ \psi_\alpha + \sum_{\alpha\in I_1} \frac{\mu_\alpha}{2} \ \phi_\alpha^1 + \sum_{\alpha\in I_2} \frac{\nu_\alpha}{2} \ \phi_\alpha^2.$$

It is not hard to see that in this case

$$\|x\|_{\varphi_1,\varphi_2} \leq 2M \ \|x\|_\varphi,$$

for all $x \in E$, which proves the theorem. $\square$

Let e be a tripotent in a JB*-triple J. When $J_2(e) = \mathbb{C}e$ we say that e is a minimal tripotent. A JBW*-triple E is called *atomic* if E coincides with the weak*-cosed ideal generated by all its minimal tripotents. From [7, Proposition 2] it follows that every atomic JBW*-triple coincides with an ℓ_∞-sum of Cartan factors.

Theorem 2.13. *Let E be an atomic JBW*-triple. Then for every weak*-continuous linear operator T from E to a complex Hilbert space there exists a norm-one functional $\varphi \in E_*$ satisfying*

$$\|T(x)\| \leq 32\sqrt{2} \ \ \|T\| \ \|x\|_\varphi,$$

for all $x \in E$.

Proof. Let E be an atomic JBW*-triple. We have already commented that E admits a decomposition in the form $\bigoplus^{\ell_\infty} C_\alpha$, where each C_α is a Cartan factor. If we prove that each factor C_α satisfies the hypothesis of Theorem 2.12 for $M = 8$, then the assertion will follow from the just quoted theorem.

Let $T : C_\alpha \to \mathcal{H}$ be a weak*-continuous linear operator from C_α to a complex Hilbert space. If C_α is a type 1 Cartan factor with $\dim(H) > \dim(K)$, then Corollary 2.9 assures the existence of a norm-one functional $\varphi_\alpha \in (C_\alpha)_*$ satisfying inequality (16) for $M = 8$. If C_α is a Cartan factor of type 1 with with $\dim(H) = \dim(K)$, or a Cartan factor of type 2 with $\dim(H)$ even, or infinite, or a Cartan factors of type 3, or a type 6 Cartan factor, then 2.11 gives the existence of a norm-one functional $\varphi_\alpha \in (C_\alpha)_*$ satisfying (16) for $M > 2$. Finally, if C_α is finite dimensional, then it follows from Remark 2.2 that there exists a norm-one functional $\varphi_\alpha \in (C_\alpha)_*$ satisfying (16) for $M = \sqrt{2}$. $\square$

Let E be a JB*-triple. We have already mentioned that E^{**} is a JBW*-triple. From [6, Theorems 1 and 2] it follows that E^{**} and E^* admit the following decompositions:

$$E^{**} = A \oplus^\infty N$$

and

$$E^* = A_* \oplus^{\ell_1} N_*,$$

where A is the weak*-closed ideal of E^{**} generated by all minimal tripotents of E^{**}, N contains no minimal tripotents, A_* is the predual of A and coincides with the norm closure of the linear span of the extreme points of the closed unit ball of E^*, and the closed unit ball of N_* contains no extreme points. A is called the atomic part of E^{**}. Moreover, by [7, Proposition 2] we conclude that A is an ℓ_∞-sum of Cartan factors.

Corollary 2.14. *Let E be a JB*-triple and let A denote the atomic part of E^{**}. Then for every pair (φ_1, φ_2) of norm-one functionals in $A_* \subseteq E^*$, there exists a norm-one functional $\varphi \in A_* \subseteq E^*$ satisfying*

$$\|x\|_{\varphi_1,\varphi_2} \leq 32\sqrt{2}\ \ \|x\|_\varphi,$$

for all $x \in E$.

Proof. Let (φ_1, φ_2) be a couple of norm-one functionals in $A_* \subseteq E^*$. By Theorem 2.13 there exists a norm-one functional $\varphi \in A_*$ satisfying

$$\|z\|_{\varphi_1,\varphi_2} \leq 32\sqrt{2}\ \|z\|_\varphi, \tag{17}$$

for all $z \in A$.

Let i denote the canonical embedding of E in its bidual and let π be the natural projection of E^{**} onto A. From the proof of [7, Proposition 1] we deduce that $\pi \circ i$ is a triple embedding of E into A. Let ϕ be a norm-one functional in A_*. ϕ can be also regarded as a norm-one element in E^*. Since ϕ attains its norm at a tripotent in A, it is not hard to see that for each $x \in E$ we have

$$\|\pi \circ i(x)\|_\phi = \|x\|_\phi.$$

The conclusion of the corollary follows now from the above expression and inequality (17). □

From the last part of Remark 2.5 and Theorem 2.13 we derive the following Grothendieck's Theorem for atomic JBW*-triples.

Theorem 2.15. *Let V and W be atomic JBW*-triples. Then for every separately weak*-continuous bilinear form U on $V \times W$, there are norm-one functionals $\varphi \in V_*$, and $\psi \in W_*$ satisfying*

$$|U(x,y)| \leq 2^{11}\ (1 + 2\sqrt{3})\ \|U\|\ \|x\|_\varphi \|y\|_\psi$$

for all $(x,y) \in V \times W$. □

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