

SEPARATE WEAK*-CONTINUITY OF THE TRIPLE PRODUCT IN DUAL REAL JB*-TRIPLES.

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Abstract

We prove that, if E is a real JB*-triple having a predual E_* , then E_* is the unique predual of E and the triple product on E is separately $\sigma(E, E_*)$ -continuous.

Mathematics Subject Classification: 17C65, 46K70, 46L05, 46L10 and 46L70.

1 Introduction.

In last years, a special category of complex Banach spaces, called JB*-triples, has focused the attention of many researchers. Historically, JB*-triples arose in the study of bounded symmetric domains in complex Banach spaces (see [L] and [K1]) and it has been shown by Kaup [K2], that every such domain is biholomorphic to the open unit ball of a JB*-triple. Every C*-algebra is a JB*-triple in the triple product $\{a, b, c\} = \frac{1}{2}(ab^*c + cb^*a)$ and every JB*-algebra is a JB*-triple in the triple product $\{a, b, c\} = (a \circ b^*) \circ c + (c \circ b^*) \circ a - (a \circ c) \circ b^*$. In the context of Functional Analysis, JB*-triples arise in a natural way in the solution of the contractive projection problem

^{*}Partially supported by DGICYT Grant PB95-1146 and Junta de Andalucía FQM 0199.

[†]Supported by Programa Nacional F.P.I., Ministry of Education and Science grant.

for C^* -algebras, concretely, the range of such a projection is a JB^* -triple for a suitable triple product (see [S], [K3] and [FR1]).

We refer to ([R], [Ru] and [CM]) for recent surveys and to [U] for the general theory of JB^* -triples.

Recently, a theory of real JB^* -triples has been developed (see [D], [CDRV], [DR], [BC], [IKR], [K4] and [CGR]) extending to the real context many results in (complex) JB^* -triples. However, the extension to the real case, of the important result proved by Barton-Timoney [BT] assuring that if E is a JB^* -triple which is a dual Banach space, then E has a unique predual and the triple product on E is separately weak*-continuous, was an open problem which explicitly appears in the papers [IKR] and [CGR]. In this paper we solve this problem, so extending the above mentioned result of Barton-Timoney.

Isidro-Kaup-Rodríguez [IKR] introduce the concept of real JBW^* -triple (as a real form of a complex JBW^* -triple) and they have shown [IKR, Theorem 4.4] that every real JB^* -triple E is a real JBW^* -triple if and only if E has a predual in such a way that the triple product is separately weak*-continuous. From this, using our main result, we conclude that every dual real JB^* -triple is a real JBW^* -triple.

JB^* -algebras and JB -algebras are real JB^* -triples. If they have a unit, then the Jordan product is uniquely determined by the triple product (see [U, Proposition 19.13]) and the unit. Therefore our main result also gives the known separately weak*-continuity of the product in dual JB^* -algebras and JB -algebras.

The proof of our main result (Theorem 2.11) follows several steps.

In a first step we prove that the dual of a real JB^* -triple is well-framed, (Lemma 2.2). As a consequence the predual of every dual real JB^* -triple, say E , is unique and every isometric bijection of E is weak*-continuous.

Once it is proved that the Peirce projections on E and the operators $L(e, e)$ and $Q(e)$ are weak*-continuous for all tripotents e in E (Proposition 2.4), it can be concluded that if E has a distinguished unitary element, then the triple product is separately weak*-continuous (Proposition 2.7).

Finally, starting from the existence of complete tripotents in dual real JB^* -triples and using Peirce decomposition, we conclude the proof.

2 Main result.

We recall that a complex JB*-triple is a complex Banach space B with a continuous triple product $\{., ., .\} : B \times B \times B \rightarrow B$ which is bilinear and symmetric in the outer variables and conjugate linear in the middle variable, satisfying:

1. (Jordan Identity) $L(a, b)\{x, y, z\} = \{L(a, b)x, y, z\} - \{x, L(b, a)y, z\} + \{x, y, L(a, b)z\}$ for all a, b, c, x, y, z in B , where $L(a, b)x := \{a, b, x\}$;
2. The map $L(a, a)$ from B to B is an hermitian operator with spectrum ≥ 0 for all a in B ;
3. $\|\{a, a, a\}\| = \|a\|^3$ for all a in B .

A real Banach space A together with a trilinear map $\{., ., .\} : A \times A \times A \rightarrow A$ is called (see [IKR]) a real JB*-triple if there is a complex JB*-triple B and an \mathbb{R} -linear isometry λ from A to B such that $\lambda\{x, y, z\} = \{\lambda x, \lambda y, \lambda z\}$ for all x, y, z in A .

Real JB*-triples are essentially the closed real subtriples of complex JB*-triples and, by [IKR, Proposition 2.2], given a real JB*-triple A there exists a unique complex JB*-triple B and a unique conjugation (conjugate linear and isometric mapping of period 2) τ on B such that $A = B^\tau := \{x \in B : \tau(x) = x\}$. In fact, B is the complexification of the vector space A , with triple product extending in a natural way the triple product of A and a suitable norm.

The class of real JB*-triples includes all JB-algebras [H], all real C*-algebras [Go], and all J*B-algebras [A].

Real JB*-triples are Jordan triples. So, given a tripotent e ($\{e, e, e\} = e$) in a real JB*-triple A , there exists a decomposition of A into the eigenspaces of $L(e, e)$, known as the Peirce decomposition;

$$A = A_0(e) \oplus A_1(e) \oplus A_2(e)$$

where $A_k(e) = \{x \in A : L(e, e)x = \frac{k}{2}x\}$ for $k = 0, 1, 2$.

$A_k(e)$ is called the Peirce k -space of e . Peirce k -spaces satisfy the following multiplication rules:

1. $\{A_i(e), A_j(e), A_k(e)\} \subseteq A_{i-j+k}(e)$, where $i, j, k = 0, 1, 2$ and $A_l(e) = 0$ for $l \neq 0, 1, 2$.

$$2. \{A_0(e), A_2(e), A\} = \{A_2(e), A_0(e), A\} = 0.$$

These rules are known as Peirce arithmetic. In particular, Peirce k -spaces are subtriples.

The projection $P_k(e)$ of A onto $A_k(e)$ is called the Peirce k -projection of e . These projections are given by

$$\begin{aligned} P_2(e) &= Q(e)^2; \\ P_1(e) &= 2(L(e, e) - Q(e)^2); \\ P_0(e) &= Id_A - 2L(e, e) + Q(e)^2; \\ \text{where } Q(e)x &= \{e, x, e\}. \end{aligned}$$

If X is a dual Banach space with predual X_* , we will denote by w^* the $\sigma(X, X_*)$ topology in X . The next Lemma summarizes some important results on well-framed Banach spaces, relevant to our purpose. We refer to [G] for a detailed presentation of the well-framed property and the proof of the next Lemma.

Lemma 2.1 [G, Th. 15 and Th. 16]

Let X be a real or complex Banach space, then

1. *If X is well-framed, then X is the unique predual of X^* . Furthermore, every isometric bijection on X^* is w^* -continuous.*
2. *If X is well-framed, then so is any closed linear subspace of X .*

In the proof of [BT, Theorem 2.1] it is shown that the dual B^* of a complex JB*-triple B is well-framed. The next lemma shows that this is still true for real JB*-triples.

Lemma 2.2 *The dual of a real JB*-triple is well-framed.*

Proof

Let A be a real JB*-triple and suppose that B is a complex JB*-triple such that $A = B^\tau$, where τ is a conjugation on B . Then $\tau^* : B^* \rightarrow B^*$ defined by $(\tau^* f)x := \overline{f\tau(x)}$, for all f in B^* and x in B , is a conjugation on B^* . Furthermore, the map $f \mapsto f|_{B^\tau}$ is an isometric bijection between $(B^*)^{\tau^*}$ and $(B^\tau)^*$, hence $A^* = (B^*)^{\tau^*}$ is a real subspace of B^* . It is known

[IR, Lemma 1.4] that if X is a well-framed complex Banach space, then its underlying real Banach space $X_{\mathbb{R}}$ is well-framed, too. Hence $(B^*)_{\mathbb{R}}$ is well-framed, so A^* is well-framed too by Lemma 2.1, 2. \square

The following proposition is a first application of Godefroy's theory of well-framed Banach spaces to dual real JB*-triples (that is real JB*-triples which are dual Banach spaces).

Proposition 2.3 *Let E be a real JB*-triple with a predual E_* . Then*

1. E_* is the unique predual of E and every isometric bijection on E is w^* -continuous.
2. The operator $L(a, b) - L(b, a)$ on E is w^* -continuous for all a, b in E .

Proof

1. By the above Lemma, E^* is well-framed. Since E_* is a subspace of E^* , Lemma 2.1 gives the first assertion.
2. Let a, b in E . It is known [IKR, proposition 2.5] that $\exp(t(L(a, b) - L(b, a)))$ is an isometric bijection on E , for all t in \mathbb{R} . Hence by the first assertion it is w^* -continuous. Now the operator

$$L(a, b) - L(b, a) = \lim_{t \rightarrow 0} \frac{\exp(t(L(a, b) - L(b, a))) - Id_E}{t}$$

is w^* -continuous, because the set of all w^* -continuous operators on E is norm-closed in the Banach space of all bounded linear operators on E . \square

We recall that if Y is a w^* -closed subspace of a Banach dual space X , then Y is a Banach dual space (with predual X_*/Y_*). Furthermore $\sigma(Y, Y_*)$ and $\sigma(X, X_*)|_Y$ are the same topology on Y . On the other hand if e is a tripotent in a complex JB*-triple B , then e is a tripotent in the subtriple $B_2(e)$ such that $L(e, e)$ is the identity map on $B_2(e)$ (i.e. e is a unitary element in $B_2(e)$). Therefore $B_2(e)$ is a unital JB*-algebra with product $x \circ y = \{x, e, y\}$ and involution $x^* = Q(e)x$ ([BKU, Theorem 2.2] and [KU, Theorem 3.7], see also [U, Proposition 19.13]).

The next proposition shows that the triple product in a dual real JB*-triple is separately w^* -continuous if we fix the same tripotent in two variables.

Proposition 2.4 *Let E be a dual real JB^* -triple and e a tripotent in E . Then the Peirce projections, $L(e, e)$ and $Q(e)$ are w^* -continuous operators on E .*

Proof

Let B a complex JB^* -triple and τ a conjugation on B such that $E = B^\tau$. First we observe that every tripotent in E is a tripotent in B and the restrictions to E of Peirce projections on B are the Peirce projections on E .

For every $\varepsilon \in \mathbb{C}$ let $S_\varepsilon := S_\varepsilon(e) = \sum_{k=0}^2 \varepsilon^k P_k(e)$. Then S_ε is an isometric automorphism of B if $|\varepsilon| = 1$ by [FR2, Lemma 1.1]. Then $S_{\pm 1}$ are isometries of E and hence w^* -continuous. Therefore $P_1(e) = (S_1 - S_{-1})/2$ is w^* -continuous and the subtriple $E_2(e) + E_0(e)$ is w^* -closed in E . But S_i and $P_0(e) - P_2(e)$ have the same restriction to $E_2(e) + E_0(e)$. This implies that $P_0(e), P_2(e)$ and $L(e, e) = P_2(e) + \frac{1}{2}P_1(e)$ are w^* -continuous. The restriction of $Q(e)$ to the w^* -closed subtriple $E_2(e)$ is isometric and hence is w^* -continuous on E . \square

Following [IKR] a real JBW^* -triple is a real JB^* -triple E such that $E = B^\tau$ for a dual complex JB^* -triple (JBW^* -triple) B and a conjugation τ on B .

From [IKR, Theorem 4.4] E is a real JBW^* -triple if and only if E has a predual E_* in such a way that the triple product is separately w^* -continuous. In this paper we prove that every dual real JB^* -triple is a real JBW^* -triple. Concretely we will prove that in every dual real JB^* -triple the triple product is separately w^* -continuous.

The following Proposition is a first approach to our purpose.

Proposition 2.5 *Let E be a dual real JB^* -triple. Suppose that for every a in E and for every $\varepsilon > 0$, there exists a family $\{e_1, \dots, e_n\}$ of pairwise orthogonal tripotents and $\lambda_1, \dots, \lambda_n$ in \mathbb{R} , such that $\left\| a - \sum_{i=1}^n \lambda_i e_i \right\| < \varepsilon$. Then the triple product of E is separately w^* -continuous.*

Proof

Let $a \in E$ and $1 > \varepsilon > 0$. Then by hypothesis, there exists a family $\{e_1, \dots, e_n\}$ of pairwise orthogonal tripotents and $\lambda_1, \dots, \lambda_n$ in \mathbb{R} , such that

$$\|a - a_n\| < \frac{\varepsilon}{2(1 + \|a\|)}$$

for $a_n := \sum_{i=1}^n \lambda_i e_i$. By orthogonality $L(a_n, a_n) = \sum_{i=1}^n \lambda_i^2 L(e_i, e_i)$. Hence $L(a_n, a_n)$ is w^* -continuous by Proposition 2.4. Since

$$\|L(a, a) - L(a_n, a)\| = \|L(a - a_n, a)\| \leq \|a - a_n\| \|a\| < \frac{\varepsilon}{2(1 + \|a\|)} (1 + \|a\|)$$

and

$$\begin{aligned} \|L(a_n, a_n) - L(a_n, a)\| &= \|L(a_n, a_n - a)\| \\ &\leq \|a - a_n\| \|a_n\| < \frac{\varepsilon}{2(1 + \|a\|)} (1 + \|a\|) \end{aligned}$$

(where we have used that $\|a - a_n\| < \varepsilon \Rightarrow \|a_n\| < \varepsilon + \|a\| < 1 + \|a\|$). It follows that

$$\|L(a, a) - L(a_n, a_n)\| \leq \|L(a, a) - L(a_n, a)\| + \|L(a_n, a_n) - L(a_n, a)\| < \varepsilon.$$

This implies that $L(a, a)$ is in the norm closure of the set of all w^* -continuous operators and hence is w^* -continuous for all a in E . In particular $L(a, b) + L(b, a) = L(a + b, a + b) - L(a, a) - L(b, b)$ is w^* -continuous. Now by using Proposition 2.3, 2., we have $L(a, b)$ is w^* -continuous for all a, b in E .

It is known [IKR, Lemma 3.6] that e_i, e_j are orthogonal tripotents if and only if $e_i \pm e_j$ are tripotents. Therefore, if e_i, e_j are orthogonal tripotents, then $Q(e_i + e_j)$ is w^* -continuous by Proposition 2.4. Thus

$$Q(a_n, a_n) = \sum_{i,j=1}^n \lambda_i \lambda_j Q(e_i, e_j) = \frac{1}{2} \sum_{i,j=1}^n \lambda_i \lambda_j (Q(e_i + e_j) - Q(e_i) - Q(e_j))$$

is w^* -continuous.

Again

$$\|Q(a) - Q(a_n, a)\| = \|Q(a - a_n, a)\| \leq \|a - a_n\| \|a\| < \frac{\varepsilon}{2(1 + \|a\|)} (1 + \|a\|)$$

and

$$\|Q(a_n) - Q(a_n, a)\| = \|Q(a_n, a_n - a)\| \leq \|a - a_n\| \|a_n\| < \frac{\varepsilon}{2(1 + \|a\|)} (1 + \|a\|),$$

so

$$\|Q(a) - Q(a_n)\| < \varepsilon.$$

Hence $Q(a)$ is w^* -continuous for all a in E . Finally

$$Q(a, b) = \frac{1}{2} (Q(a + b) - Q(a) - Q(b))$$

is w^* -continuous for all a, b in E . \square

If E is a real JB^* -triple there exists a complex JB^* -triple B and a conjugation τ on B such that $E = B^\tau$. Let e be a tripotent in E , as we have commented before, $B_2(e)$ is a JB^* -algebra. Therefore $A(e) := \{x \in E_2(e) : Q(e)x = x\}$ is a JB -algebra as a closed subalgebra of the JB -algebra $\{x \in B_2(e) : Q(e)x = x^* = x\}$.

We are going to show that if E is a dual real JB^* -triple and e is a tripotent in E , then every element in $A(e)$ can be approximated by finite linear combinations of pairwise orthogonal tripotents. An argument similar to that in the proof of Proposition 2.5 then shows that $L(a, b)$ and $Q(a, b)$ are w^* -continuous for all a, b in $A(e)$.

Proposition 2.6 *Let E be a dual real JB^* -triple and e a tripotent in E . Then $L(a, b)$ and $Q(a, b)$ are w^* -continuous for all a, b in $A(e)$.*

Proof

$A(e)$ is a JB -algebra and since $Q(e)$ is w^* -continuous, then $A(e)$ is w^* -closed in E . Therefore $A(e)$ is a JBW -algebra [H, Theorem 4.4.16]. Again by [H, Lemma 4.1.11] if $a \in A(e)$, then the w^* -closure of the subalgebra generated by a , $W(a)$, is isometrically isomorphic to a monotone complete $C(X)$ where X is a compact Hausdorff space and for all $\varepsilon > 0$, there exist pairwise orthogonal idempotents e_1, \dots, e_n in $W(a)$ and $\lambda_1, \dots, \lambda_n$ in \mathbb{R} such that $\left\| a - \sum_{i=1}^n \lambda_i e_i \right\| < \varepsilon$ [H, Proposition 4.2.3]. In fact e_1, \dots, e_n are pairwise orthogonal tripotents in E because $e_i \pm e_j$ are tripotents in E for all $i \neq j$. Finally we proceed as in the proof of Proposition 2.5. \square

The following result is one of the keys in the proof of our main result and gives the separate w^* -continuity of the triple product in the case that the dual real JB^* -triple E has a unitary element u , i.e. $L(u, u) = Id_E$ ($E = E_2(u)$).

Proposition 2.7 *Let E be a dual real JB^* -triple with a unitary element. Then the triple product is separately w^* -continuous.*

Proof

Since E is a real JB*-triple, there exists a complex JB*-triple B and a conjugation τ on B such that $E = B^\tau$. Let u be a unitary element in E . Then u is a unitary element in B . So B is a complex JB*-triple with a unitary element u . By [BKU, Theorem 2.2] and [KU, Theorem 3.7], (see also [U, Proposition 19.13]) it follows that B is a unital JB*-algebra with product $x \circ y := \{x, u, y\}$, involution $x^* := \{u, x, u\} = Q(u)x$ and unit u .

Put $A := \{x \in E : Q(u)x = x\}$ and $D := \{x \in E : Q(u)x = -x\}$. Then $E = A \oplus D$, because $Q(u)$ is an involution on E .

For $M_a := L(a, u)$ the identities

$$L(a, b) = M_a M_{b^*} - M_{b^*} M_a + M_{a \circ b^*},$$

$$Q(a, b) = (M_a M_b + M_b M_a - M_{a \circ b})Q(u)$$

for all $a, b \in E$ imply that only the w^* -continuity of every M_a , $a \in E$, has to be shown. For $a = a^*$ this follows from Proposition 2.6 and for $a^* = -a$ from the identity

$$2M_a = L(a, u) - L(u, a)$$

and Proposition 2.3, 2.□

Proposition 2.8 *Let E be a dual real JB*-triple and e a tripotent in E . Then $L(a, b)$ and $Q(a, b)$ are w^* -continuous operators on E for all a, b in $E_2(e)$.*

Proof

$E_2(e)$ is a dual real JB*-triple with a unitary element e . Then by Proposition 2.7 the triple product is separately w^* -continuous on $E_2(e)$. Therefore $E_2(e)$ is a real JBW*-triple [IKR, Theorem 4.4].

From [IKR, Proof of Theorem 4.8] it can be concluded that for all $a \in E_2(e)$ and $\varepsilon > 0$, there exist a family of pairwise orthogonal tripotents $\{e_1, \dots, e_n\}$ in $E_2(e)$ and $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ such that $\left\| a - \sum_{i=1}^n \lambda_i e_i \right\| < \varepsilon$. As in the proof of Proposition 2.5, we conclude that $L(a, b)$ and $Q(a, b)$ are w^* -continuous operators on E for all a, b in $E_2(e)$. □

The next two lemmas are needed in the proof of the Main Theorem below. We are inspired in some results of Friedman and Russo [FR2, Propositions 1 and 2] for their proofs.

Lemma 2.9 *Let E be a real JB^* -triple, $f \in E^*$ and e a tripotent in E such that $\|fP_2(e)\| = \|f\|$. Then $f = fP_2(e)$.*

Proof

The same proof as in [FR2, Proposition 1] runs here. \square

Lemma 2.10 *Let E be a dual real JB^* -triple. Then*

1. *If $f \in E_*$, there exists a tripotent e in E such that $f = fP_2(e)$.*
2. *A net $\{x_\alpha\}$ converges to zero in the w^* -topology if and only if $\{P_2(u)x_\alpha\}$ converges to zero in the w^* -topology for every tripotent u in E .*

Proof

1. Let us suppose $\|f\| = 1$, then the set $S = \{x \in E : f(x) = \|x\| = \|f\| = 1\}$ is nonempty convex and w^* -compact. Therefore there exists an extreme point of the closed unit ball of E , e , such that e is in S , too. Hence e is a tripotent [IKR, Lemma 3.3], and $f(e) = \|f\| = 1$. We have $f = fP_2(e)$ by Lemma 2.9.
2. (\Rightarrow) Straightforward because $P_2(u)$ is w^* -continuous (for every tripotent u in E) by Proposition 2.4.
 (\Leftarrow) Suppose that $P_2(u)x_\alpha \xrightarrow{w^*} 0$, for every tripotent u in E . Let $f \in E_*$. From the first assertion, there exists a tripotent e in E such that $f = fP_2(e)$. By hypothesis $P_2(e)x_\alpha \xrightarrow{w^*} 0$. Therefore $fP_2(e)x_\alpha = f(x_\alpha) \rightarrow 0$. \square

A tripotent e in a Jordan triple A is called complete if $A_0(e) = 0$. In a dual real JB^* -triple E we have many complete tripotents because by [IKR, Lemma 3.3], the complete tripotents in E are exactly the extreme points of the closed unit ball of E , B_E . Banach-Alaoglu's and Krein-Millman's theorems give that B_E is the w^* -closed convex hull of its extreme points.

Having disposed of these preliminary steps we can now prove the Main Theorem.

Theorem 2.11 *Let E be a dual real JB^* -triple. Then the triple product is separately w^* -continuous, i.e., E is a real JBW^* -triple.*

Proof

We first prove that $L(a, b)$ is w^* -continuous for all $a, b \in E$.

Let e be a complete tripotent in E . If we fix $a \in E_1(e)$ and $b \in E_2(e)$, using Peirce arithmetic, it is easy to check that

$$\begin{aligned} L(a, b) &= L(a, b) P_2(e) \text{ and} \\ L(b, a) &= L(b, a) P_1(e). \end{aligned}$$

From Proposition 2.3, 2., $L(a, b) - L(b, a)$ is w^* -continuous and by Proposition 2.4, $P_2(e)$ and $P_1(e)$ are w^* -continuous. Therefore

$$L(a, b) = L(a, b) P_2(e) - L(b, a) P_1(e) P_2(e) = (L(a, b) - L(b, a)) P_2(e)$$

is w^* -continuous. (3.1)

In a similar way $L(b, a) = -(L(a, b) - L(b, a)) P_1(e)$ is w^* -continuous. (3.2)

Now if $a \in E$ and $b \in E_2(e)$, then $a = a_1 + a_2$ where $a_i \in E_i(e)$ for $i = 1, 2$. Since

$$\begin{aligned} L(a, b) &= L(a_1, b) + L(a_2, b) \\ L(b, a) &= L(b, a_1) + L(b, a_2) \end{aligned}$$

we can conclude by (3.1), (3.2) and Proposition 2.8 that $L(a, b)$ and $L(b, a)$ are w^* -continuous for all $a \in E$ and $b \in E_2(e)$. (3.3)

Let $a \in E$, by applying Jordan identity, we have

$$L(a, L(e, e)a) = -L(e, a)L(a, e) + L(\{e, a, a\}, e) + L(a, e)L(e, a).$$

Thus by (3.3), $L(a, L(e, e)a)$ is w^* -continuous because $e \in E_2(e)$. (3.4)

From

$$\begin{aligned} L(a, L(e, e)a) &= L(a_1 + a_2, L(e, e)a_1 + a_2) = \\ &= L\left(a_1 + a_2, \frac{1}{2}a_1 + a_2\right) = \\ &= \frac{1}{2}L(a_1, a_1) + \frac{1}{2}L(a_2, a_1) + L(a_1, a_2) + L(a_2, a_2) \end{aligned}$$

we deduce that

$$L(a_1, a_1) = 2\left(L(a, L(e, e)a) - \frac{1}{2}L(a_2, a_1) - L(a_1, a_2) - L(a_2, a_2)\right)$$

is w^* -continuous which follows from (3.3) and (3.4). We have proved that $L(a_1, a_1)$ is w^* -continuous for all $a_1 \in E_1(e)$. (3.5)

Finally since $E = E_1(e) \oplus E_2(e)$, and $L(., .)$ is bilinear, by (3.3) and (3.5) we can conclude that $L(a, b)$ is w^* -continuous for all $a, b \in E$. (3.6)

The last part of the proof is devoted to prove that $Q(a, b)$ is w^* -continuous for all $a, b \in E$.

We fix $a, b \in E$. It is easy to check that $Q(b)Q(a) = 2L(b, a)L(b, a) - L(\{b, a, b\}, a)$. Thus $Q(b)Q(a)$ is w^* -continuous for all $a, b \in E$ by (3.6). In particular $Q(u)Q(a)$ is w^* -continuous for every tripotent u in E . So (using Proposition 2.4) $P_2(u)Q(a) = Q(u)Q(u)Q(a)$ is w^* -continuous for every tripotent u in E . (4.1)

Now by Lemma 2.10, 2., $Q(a)$ is w^* -continuous if and only if $P_2(u)Q(a)$ is w^* -continuous for every tripotent u in E . Hence, using (4.1), we conclude the proof. \square

Edwards [E, Theorems 3.2 and 3.4] has shown that the complexification of a JB-algebra, J , is a JBW*-algebra if and only if J is a JBW-algebra. This result now is a consequence of our main result and [IKR, Theorem 4.4].

Corollary 2.12 *Let J be a JB-algebra. Then J is a JBW-algebra if and only if its complexification is a JBW*-algebra.*

Proof

Consider $B = J \oplus iJ$ (the complexification of J) as JB*-triple, τ the natural involution on B and $J = B^\tau$. \square

Acknowledgments

The authors wish to thank W. Kaup for his suggestions.

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