SPIN SPACES AND POSITIVE DECOMPOSITION OF LINEAR MAPS ON ORDERED BANACH SPACES

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ABSTRACT. Spin factors and generalisations are used to revisit positive generation of $B(E,F)$ where E and F are ordered Banach spaces. Interior points of $B(E,F)_+$ are discussed and in many cases it is seen that positive generation of $B(E,F)$ is controlled by spin structure in F when F is a JBW-algebra.

1. INTRODUCTION

Given a real Banach space X there is a natural isometric embedding, $i : X \to V(X)$, where the latter is a certain order unit space derived from X by a construction that mimics the construction of a spin factor (as an order unit space) from a Hilbert space, see §2. Advantages of $V(X)$ are a simple order structure and the property that every bounded linear map from X into another Banach space factors through $i : X \to V(X)$, which we shall exploit to revisit the question of positive decomposition of bounded linear maps between ordered Banach spaces considered, diversely, in [6, 5, 12, 14, 15, 17, 23], for example. We remark that inspection of the particular passages [14, pp. 122-125] (see also [21, pages 96-97] and the proof of 3.10 of [23]) reveals hints of the utility of $V(X)$-spaces developed in the present note.

$V(X)$-spaces are formally introduced in §2 which contains a treatment of interior points of $B(E,F)_+$, where E and F are ordered Banach spaces. When X is a Hilbert space $V(X)$ is order isometric to a spin factor and therefore realisable as a JBW-algebra [13]. If F is a JBW-algebra it transpires that, for a wide range of ordered Banach spaces E, positive generation of $B(E,F)$ is controlled by spin structure in F. In particular, if E is a simplex space and F is a JBW-algebra containing an infinite dimensional spin factor then $B(E,F)$ is positively generated only when E is finite dimensional, a result that remains locally true for certain spaces E such as strongly spectral GM-spaces and their JB-algebra motivating examples.

By an ordered Banach space we shall mean a real Banach space E with closed positive cone E_+; thus $E_+ + E_+ \subseteq E_+$ and $\mathbb{R}_+ \ E_+ \subseteq E_+$. The cone E_+ is said to be *proper* if $E_+ \cap (-E_+) = \{0\}$, to be *generating* if

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$E = E_+ - E_+$ and to be *normal* if there exists $\alpha \geq 1$ such that whenever $a \leq b \leq c$ (in E) we have $\|b\| \leq \alpha \max\{\|a\|, \|c\|\}$, in which case E is said to be α-normal. We refer to [4] and the survey [5, §1] for any undefined terms and details used of ordered Banach spaces and to [1, 4] for the theory of order unit spaces.

The set $\mathrm{int}(E_+)$ of norm interior points of E_+ is empty or dense and coincides with the set $u \in E_+$ (of order units) satisfying

$$E = \cup \{[-\lambda\ u, \lambda\ u] : \lambda \geq 0\}.$$

Given $u \in \mathrm{int}(E_+)$

$$\|x\|_u = \inf\{\lambda > 0 : -\lambda u \leq x \leq \lambda u\}$$

defines a seminorm on E. If, in addition, E_+ is normal then $\|.\|_u$ is a norm on E and E is order isomorphic (not necessarily isometric) to an order unit space. Many examples show that E_+ can have interior points without being normal. One instance of the latter is the ordered Banach space E of all continuously differentiable functions on $[0, 1]$ with the usual ordering and norm

$$\|f\| = \|f\|_\infty + \|f^{'}\|_\infty.$$

We have $u \in \mathrm{int}(E_+)$ where $u(\lambda) = 1$ for all λ. But E_+ is not normal since, in this case, $\|.\|_u$ coincides with the supremum norm and the latter is not equivalent to $\|.\|$, since E is not complete in the supremum norm.

An element x in E is said to have a *positive decomposition* if it is the difference of two positive elements. This is equivalent to the existence of $a \in E$ such that $a \geq 0$ and $a \geq x$. Given $\alpha \geq 1$, E_+ is said to be *α-generated* if each $x \in E$ has a positive decomposition, $x = y - z$, with $\|y\| + \|z\| \leq \alpha$.

Such an $\alpha \geq 1$ always exists if E_+ is generating. We remark that the theory of interior points of E_+ (respectively, E_+^*) is equivalent to the theory of weak* compact bases of E_+^* (respectively, bases of E_+).

The following is a slight variation of parts of [23, Theorems 3.5, 4.4] involving positive decomposition in the ordered Banach space $B(E, F)$ of all bounded linear operators from E to F, where the latter are ordered Banach spaces, and the corresponding space $K(E, F)$ of compact operators. The space of all continuous F-valued functions on a compact Hausdorff space Ω is denoted by $C(\Omega, F)$.

Lemma 1.1. *Let E and F be ordered Banach spaces and Ω a compact stonean space, where E_+ is α-normal, F_+ is β-generated and F has finite dimension n. Let $\varphi \in B(E, C(\Omega, F))$. Then there exists $\psi \in B(E, C(\Omega, F))$ with $\psi \geq 0$, $\psi \geq \varphi$ such that $\|\psi\| \leq 2\ \alpha\ \beta\ n\ \|\varphi\|$.*

If $\varphi \in K(E, C(\Omega, F))$, then ψ can be chosen in $K(E, C(\Omega, F))$ such that $\|\psi\| \leq 3\ \alpha\ \beta\ n\|\varphi\|$.

Proof. A suitable choice of basis $\{x_1, \ldots, x_n\}$ of norm one elements of F induces norm one linear maps, $\pi_j : C(\Omega, F) \to C(\Omega)$, satisfying

$$\pi_j\left(\sum_{i=1}^{n} f_i \otimes x_i\right) = f_j \qquad (j : 1, \ldots, n)$$

(where $(f \otimes x)(y) = f(y)x$, $f \in C(\Omega)$, $x \in F$ and $y \in \Omega$)). Thus,

$$\varphi(.) = \sum_{1}^{n} \varphi_i(.) \otimes x_i,$$

where φ_i denotes $\pi_i \varphi$.

By [23, Theorem 3.5] each $\varphi_i : E \to C(\Omega)$ has a positive decomposition $\varphi_i = S_i - T_i$ such that $\|S_i\| \le \alpha\|\varphi\|$ and $\|T_i\| \le 2\alpha\|\varphi\|$, since $\|\varphi_i\| = \|\varphi\|$. By assumption, each x_i has a positive decomposition $x_i = y_i - z_i$ such that $\|y_i\| + \|z_i\| \le \beta$. The map $\psi : E \to C(\Omega, F)$ defined by

$$\psi(.) = \sum_{1}^{n} S_i(.) \otimes y_i + T_i(.) \otimes z_i$$

is positive, majorises φ and $\|\psi\| \le 2\ \alpha\ \beta\ n\ \|\varphi\|$, as required.

If φ is compact then each φ_i is compact and by [23, Theorem 4.4] the positive S_i, T_i (above) can be chosen to be compact satisfying $\|S_i\| \le \frac{3}{2}\alpha\|\varphi\|$, $\|T_i\| \le 3\alpha\|\varphi\|$ so that ψ is compact and $\|\psi\| \le 3\ \alpha\ \beta\ n\|\varphi\|$. □

Corollary 1.2. *Let $E, F_1, \ldots, F_n$ be ordered Banach spaces such that E_+ is normal and all F_i are finite dimensional and $(F_i)_+$ are generating, and let $\Omega_1, \ldots, \Omega_n$ be compact stonean spaces. Let F be the ℓ_∞-sum of the $C(\Omega_i, F_i)$. Then $B(E, F)_+$ and $K(E, F)_+$ are generating.*

2. The Order Unit Space V(X)

Formally replicating the construction and order of a spin factor in the case of a Hilbert space, for any real Banach space X, let $V(X) = \mathbb{R}1 \oplus_1 X$ (ℓ_1-sum) and define

$$V(X)_+ = \{\alpha 1 + a : \alpha \in \mathbb{R}, a \in X, \alpha \ge \|a\|\}$$

to make $V(X)$ into an ordered Banach space with closed proper cone $V(X)_+$. We assume it is well-known that $V(X)$ is an order unit space. The simple proof is included below nevertheless.

Let $i_X : X \to V(X)$ and $P_X : V(X) \to V(X)$ denote respectively the natural (isometric) inclusion and projection onto X.

Define $\tau_X : V(X) \to \mathbb{R}$ by $\tau_X(\alpha 1 + a) = \alpha$. We note that P_X is bicontractive and that $\tau_X \in V(X)^*_+$ with $\|\tau_X\| = 1$. For each x in $V(X)$ we have $x = \tau_X(x)1 + P_X(x)$.

Lemma 2.1. *Let X be a real Banach space. Let τ denote τ_X. Then*

(a) *$\|x\| \le 2\tau(x)$ for all $x \in V(X)_+$.*
(b) *If $a \in X$, $x \in V(X)_+$ and $a \le x$ we have $\|a\| \le 2\tau(x)$.*

(c) *$\rho \leq 2\|\rho\|\tau$ for all $\rho \in V(X)^*$.*

Proof.

(a) $x = \tau(x)1 + P_X(x) \geq 0$ implies $\|x\| = \tau(x) + \|P_X(x)\| \leq 2\tau(x)$.
(b) a, x being as stated implies $\tau(x) \geq \|P_X(x)\|$, $\tau(x) \geq \|P_X(x) - a\|$, giving $2\tau(x) \geq \|a\|$.
(c) If $\rho \in V(X)^*$ and $x \in V(X)_+$, then $\rho(x) \leq \|\rho\|\|x\| \leq 2\|\rho\|\tau(x)$, using (a).

□

Lemma 2.2. *Let X be a real Banach space. Then $V(X)$ is an order unit space, $\tau_X \in int(V(X)^*_+)$ and $V(X)^*$ is order isomorphic to an order unit space.*

Proof. Let τ, P denote τ_X, P_X. Given $x \in V(X)$ we have

$$\|x\|1 - x = (|\tau(x)| - \tau(x) + \|P(x)\|)1 - P(x) \geq 0,$$

so that $-\|x\|1 \leq x \leq \|x\|1$. If $\lambda > 0$ such that $-\lambda 1 \leq x \leq \lambda 1$ then

$$\lambda \pm \tau(x) \geq \|P(x)\|, \text{ and } \lambda \geq |\tau(x)| + \|P(x)\| = \|x\|.$$

Hence, $\|x\| = \inf\{\lambda > 0 : -\lambda 1 \leq x \leq \lambda 1\}$; proving the first statement. The second statement is immediate from Lemma 2.1(c) whilst the final statement follows from the first two. □

For Banach spaces X and Y with $\rho \in X^*$ and $y \in Y$, let $\rho \otimes y$ denote the rank one operator in $B(X, Y)$ given by $(\rho \otimes y)(x) = \rho(x)y$. Let X_1 denote the closed unit ball of the Banach space X.

Lemma 2.3. *Let $u \in int(E_+)$ where E is an ordered Banach space. Then $\tau \otimes u$ is an interior point of $B(V(X), E)_+$, where X is a real Banach space and τ denotes τ_X.*

Proof. Let $\varphi \in B(V(X), E)$ with $\|\varphi\| = 1$. We have $\varphi = \tau \otimes \varphi(1) + \varphi P$, where $P = P_X$. By assumption, $E_1 \subseteq [-\alpha u, \alpha u]$ for some $\alpha > 0$. Let $x \in V(X)_+$. Then

$$\varphi(P(x)) \leq \alpha\|\varphi(P(x))\|u \leq \alpha\|P(x)\|u \leq \alpha\tau(x)u = \alpha(\tau \otimes u)(x).$$

Also $\varphi(1) \leq \alpha u$ so that $\tau \otimes \varphi(1) \leq \alpha(\tau \otimes u)$. Hence, $\varphi \leq 2\alpha(\tau \otimes u)$, as required. □

Lemma 2.4. *Let $\varphi \in B(E, V(X))$ where E is an ordered Banach space and X is a real Banach space. Let τ denote τ_X.*

(a) *$\varphi \geq 0$ if, and only if, $\varphi \leq 2(\tau\varphi) \otimes 1$.*
(b) *φ has a positive decomposition if, and only if, $\varphi \leq \rho \otimes 1$ for some $\rho \in E^*_+$.*

Proof. (a) We have $\varphi = (\tau\varphi) \otimes 1 + P\varphi$, where $P = P_X$. Since, by definition of $V(X)_+$, we have $\varphi \geq 0$ if, and only if, $(\tau\varphi) \otimes 1 \geq \pm P\varphi$, (a) follows.

(*b*) Let $\psi \in B(E, V(X))$ with $\psi \geq 0$, $\psi \geq \varphi$. Then (*a*) gives

$$\psi \leq 2(\tau\psi) \otimes 1 \text{ and } -(\psi - \varphi) \leq \psi - \varphi \leq 2(\tau(\psi - \varphi)) \otimes 1$$

so that $\varphi \leq \rho \otimes 1$ where $\rho = 2\tau\psi$. The converse is obvious.

□

Let E be an ordered Banach space. We recall (compare [6, page 374]) that $\text{int}(E_+) \neq \emptyset$ if, and only if, there exists $v \in E_+$ such that $E_1 \subseteq [-v, v]$.

Lemma 2.5. *Let E be an ordered Banach space.*

(*a*) *int$(E^*_+) \neq \emptyset$ if, and only if, $i_E : E \to V(E)$ has a positive decomposition in $B(E, V(E))$.*

(*b*) *int$(E_+) \neq \emptyset$ if, and only if, $P_E : V(E) \to E$ has a positive decomposition in $B(V(E), E)$.*

Proof. (*a*) Let $\rho \in E^*_+$. The condition that $E^*_1 \subseteq [-\rho, \rho]$ is equivalent to the condition that

$$|\sigma(a)| \leq \rho(a) \text{ for all } \sigma \in E^*_1 \text{ and all } a \in E_+,$$

which, in turn, is equivalent to the condition that $\|a\| \leq \rho(a)$ for all $a \in E_+$. But the latter holds if, and only if, $i_E \leq \rho \otimes 1$. The result now follows from Lemma 2.4(*b*).

(*b*) Suppose $P_E : V(E) \to E$ has a positive decomposition in $B(V(E), E)$. Choose $\varphi \in B(V(E), E)$ with $\varphi \geq 0$ and $\varphi \geq P_E$. Let $a \in E$. Then $\|a\|1 \pm a \in V(E)_+$. So, $a = P_E(\|a\|1 + a) \leq \varphi(\|a\|1 + a) \leq 2\|a\|\varphi(1)$, implying that $\varphi(1) \in \text{int}(E_+)$.

Conversely, by Lemma 2.3, $B(V(E), E)_+$ has an interior point and so is generating if E_+ has an interior point.

□

If E and F are ordered Banach spaces with $\pi \in \text{int}(B(E, F)_+)$ then, since $B(E, F)_+$ is generating, E^*_+ and F_+ are generating, by [23, Proposition 3.2]. Moreover, if $a \in E_+$ with $a \neq 0$ then $\pi(a) \in \text{int}(F_+)$. The reason being that, since $E^* = E^*_+ - E^*_+$, there exist $\rho \in E^*_+$ such that $\rho(a) > 0$. Thus, if $b \in F$ we have $\rho \otimes b \leq \lambda\pi$ for some $\lambda > 0$ by assumption, giving $b \leq \rho(a)^{-1}\lambda\pi(a)$. Similar argument shows that $\pi^*(\rho) \in \text{int}(E^*_+)$ for all $\rho \in F^*_+ \backslash \{0\}$. A simple special case is that if $\rho \in E^*_+$ and $u \in F_+$ are such that $\rho \otimes u \in \text{int}(B(E, F)_+)$, then $\rho \in \text{int}(E^*_+)$ and $u \in \text{int}(F_+)$. The converse is shown next.

Proposition 2.6. *Let ρ and u be interior points of E^*_+ and F_+, respectively. Then $\rho \otimes u$ is an interior point of $B(E, F)_+$. Moreover, with $S =$ int$(B(E, F)_+)$ we have*

(*i*) *int$(F_+) = \{\pi(E_+ \backslash \{0\}) : \pi \in S\}$ and*

(*ii*) *int$(E^*_+) = \{\pi^*(F^*_+ \backslash \{0\}) : \pi \in S\}$.*

Proof. Given $\varphi \in B(E, V(E)_+)$ we have $\varphi \leq 2(\tau\varphi) \otimes 1$ by Lemma 2.4(a), where $\tau = \tau_E$. Since $\tau\varphi \leq \lambda\rho$ for some $\lambda > 0$, we have $\varphi \leq 2\lambda(\rho \otimes 1)$ for some $\lambda > 0$.

Now let $\pi \in B(E, F)$. Then $\phi = \psi i$, where $\psi \in B(V(E), F)$ is given by $\psi(\alpha 1 + a) = \phi(a)$ and where i denotes i_E. By Lemma 2.3 and Lemma 2.5, ψ and i have positive decompositions

$$\psi = S - T \text{ and } i = R - U \text{ say.}$$

This gives $\pi = \psi i \leq SR + TU$. By the first part of the proof and Lemma 2.3 we have

$$S, T \leq \alpha(\tau \otimes u) \text{ and } R, U \leq \beta(\rho \otimes 1)$$

for some $\alpha, \beta > 0$, where τ denotes τ_E. It follows that

$$\pi \leq \alpha\beta(\tau \otimes u)(\rho \otimes 1) = 2\alpha\beta(\rho \otimes u)$$

proving the first statement. The equalities (i) and (ii) now follow from this together with the prior remarks. □

We state the following automatic corollaries of Lemma 2.5, Proposition 2.6 and intervening remark. In both corollaries the implication (d) $\Rightarrow$ (b) is effectively proved in [23, Propositions 3.10 and 3.11]. E and F denote ordered Banach spaces throughout.

Corollary 2.7. *The following are equivalent.*

(a) *int*$(B(E, F)_+) \neq \emptyset$ *for some* F *with int*$(F_+) \neq \emptyset$*;*
(b) *int*$(E^*_+) \neq \emptyset$*;*
(c) *int*$(B(E, F)_+) \neq \emptyset$ *whenever int*$(F_+) \neq \emptyset$*;*
(d) $B(E, F)_+$ *is generating whenever* F *is an order unit space.*

Corollary 2.8. *The following are equivalent.*

(a) *int*$(B(E, F)_+) \neq \emptyset$ *for some* E *with int*$(E^*_+) \neq \emptyset$*;*
(b) *int*$(F_+) \neq \emptyset$*;*
(c) *int*$(B(E, F)_+) \neq \emptyset$ *whenever int*$(E^*_+) \neq \emptyset$*;*
(d) $B(E, F)_+$ *is generating whenever* E^* *is an order unit space.*

3. Spin Factors

If X is a real Hilbert space, $V(X)$ is known as an *abstract spin factor* and is realisable as a JW-algebra ([13, §6], [20]) in which guise we refer to it as just a *spin factor*. A general reference for JB-algebras and JBW-algebras is [13], and [19] for von Neumann algebras. An ordered Banach space E with upward filtering open unit ball and satisfying $E_1 = (E_1 + E_+) \cap (E_1 - E_+)$, called a *GM-space*, is order isometric to the space $A_0(K)$ of weak* continuous affine functions on K vanishing at 0, where $K = \{\rho \in E^*_+ : \|\rho\| \leq 1\}$ [8]. Simplex spaces are examples of GM-spaces. *Strongly spectral* GM-spaces, as defined in [8], include all JB-algebras. This follows from [3, Corollary 3.2] and the results of [8, §3].

Lemma 3.1. *Let E and F be ordered Banach spaces. Let $\varphi \in B(E^{**}, F^*)$ be weak* continuous and let ψ denote its restriction in $B(E, F^*)$. Suppose that ψ has a positive decomposition in $B(E, F^*)$. Then φ has a positive decomposition in $B(E^{**}, F^*)$.*

Proof. The canonical inclusion $j : F \to F^{**}$ induces the positive surjective projection $j^* : F^{***} \to F^*$. The composition $j^*\psi^{**} : E^{**} \to F^{***} \to F^*$ is the unique weak* continuous extension of $\psi : E \to F^*$, so that $j^*\psi^{**} = \varphi$. Thus if ψ has a positive decomposition $\psi = S - T$, $\varphi = j^*S^{**} - j^*T^{**}$ is a positive decomposition of φ. □

Lemma 3.2. *Let E be an ordered Banach space containing an infinite dimensional simplex space S as an ordered Banach subspace. Then $B(E,F)_+ - B(E,F)_+$ does not contain $K(E,F)$ in each of the following cases:*

(a) *$F = V(X)$ where X is an infinite dimensional Hilbert space;*
(b) *$F = (\sum_1^\infty V(X_n))_\infty$ (ℓ_∞-sum) where X_n is an n-dimensional Hilbert space for each $n \in \mathbb{N}$.*

Proof. Since S^* is an AL-space, S^{**} is a dual AM-space and therefore order isometric to the self-adjoint part of a commutative von Neumann algebra. It follows that there exists $e \in S_+$ and infinite sequences (x_n) in S_+ and (ρ_n) in E^* such that

$$\|e\| = \|x_n\| = \|\rho_n\| = 1, \ \sum_1^n x_i \leq e \text{ for all n;}$$

$$\rho_m(x_n) = \delta_{m,n} \text{ for all m,n.}$$

This is seen by first choosing (σ_n) in S^* satisfying the relevant conditions (above) and then taking $\rho_n \in E^*$ to be a Hahn-Banach extension of σ_n for each n (We do not assume the ρ_n positive).

Let (h_n) be an infinite orthogonal sequence in an infinite dimensional Hilbert space X. Note that $V(X) = V(X)^{**}$.

(a) Define $\varphi : E^{**} \to V(X)$ by $\varphi = \sum_1^\infty \frac{1}{n}\rho_n \otimes h_n$, the ρ_n being regarded as weak* continuous linear functionals on E^{**}. Let $\psi : E \to V(X)$ denote the restriction of φ to E. For each a in E^{**} with $\|a\| \leq 1$ and each $n \in \mathbb{N}$ we have

$$\left\| \sum_{k=n}^\infty \frac{1}{k}\rho_k(a)h_k \right\|^2 = \sum_{k=n}^\infty \frac{1}{k^2}\rho_k(a)^2 \leq \sum_{k=n}^\infty \frac{1}{k^2},$$

so that $\|\varphi - \varphi_n\| \to 0$, where φ_n is the finite rank operator $\sum_1^n \frac{1}{k}\rho_k \otimes h_k$ for each n. Therefore, φ is compact and weak* continuous since each φ_n has these properties. Suppose φ has a positive decomposition. By Lemma 2.4(b) there exists $\rho \in E_+^*$ such that $\varphi \leq \rho \otimes 1$. Since $\varphi(E^{**}) \subseteq X$ this gives for all $x \in E_+^{**}$ and all $n \in \mathbb{N}$,

$$\frac{1}{n}|\rho_n(x)| \le \left(\sum_1^\infty \frac{1}{k^2}\rho_k(x)^2\right)^{\frac{1}{2}} = \|\varphi(x)\| \le \rho(x).$$

In particular, for all n,

$$\sum_1^n \frac{1}{k} \le \sum_1^n \rho(x_k) = \rho\left(\sum_1^n x_k\right) \le \rho(e),$$

a contradiction. Therefore φ does not have a positive decomposition and hence neither does ψ, by Lemma 3.1.

(b) We adapt a construction that first appeared in [14, Lemma 2.1, Theorem 2.2]. For each n let V_n denote $V(X_n)$ where X_n is the subspace of X generated by $\{h_1, \ldots, h_n\}$. Fix α satisfying $1 < 2\alpha < 2$. For each n, let $\varphi_n : E^{**} \to V_n$ be the operator $\frac{1}{n^\alpha}\sum_{k=1}^n \rho_k \otimes h_k$. Consider the ℓ_1-sum, $L = \sum_1^\infty V_n^*$.

Define $\varphi : E^{**} \to L^* = \left(\sum_1^\infty V_n\right)_\infty$ by $\varphi = (\varphi_n)$ and let $\psi : E \to L^*$ be its restriction. For each n we have

$$\|\varphi_n\|^2 \le \frac{1}{n^{2\alpha}}n = \frac{1}{n^{2\alpha-1}},$$

so that φ is the norm limit of the weak* continuous finite rank operators $(\varphi_1, \ldots, \varphi_n, 0, 0, 0, \ldots)$ and hence is itself weak* continuous and compact. By Lemma 3.1, since ψ is compact, it is enough to show that φ has no positive decomposition.

In order to derive a contradiction suppose there exists $\pi \in B(E^{**}, L^*)$ such that $\pi \ge 0$ and $\pi \ge \varphi$. Then $\pi = (\pi_n)$ where each $\pi_n : E^{**} \to V_n$ satisfies $\pi_n \ge 0$ and $\pi_n \ge \varphi_n$. For each $k : 1, \ldots, n$ we have

$$\pi_n(x_k) \ge 0 \text{ and } \pi_n(x_k) \ge \varphi_n(x_k) = \frac{1}{n^\alpha}h_k$$

which, since $\varphi_n(x_k) \in X_n$, implies $2\tau_n\pi_n(x_k) \ge \|\varphi_n(x_k)\| = \frac{1}{n}\alpha$ by Lemma 2.1(b), where $\tau_n = \tau_{V_n}$. Hence, for all n,

$$2\|\pi(e)\| \ge 2\tau_n\pi(e) \ge 2\tau_n\pi\left(\sum_1^n x_k\right) \ge n\frac{1}{n^\alpha} = n^{1-\alpha},$$

a contradiction that completes the proof. □

Definition 3.3. We say that a JBW-algebra F is *spin bounded* if there exists $n \in \mathbb{N}$ such that $\dim(V) \le n$ for every spin factor $V \subseteq M$.

Recall the Jordan algebra inclusions

$$V_n \subseteq M_{2^n}(\mathbb{C})_{sa} \subseteq M_{2^{n+1}}(\mathbb{R})_{sa}, M_{2^n}(\mathbb{H})_{sa}$$

where V_n is an $n+1$ dimensional spin factor (and $\mathbb{H}$ represents the quaternions.)

Lemma 3.4. *A JBW-algebra F is spin bounded if, and only if, $F = (\sum_1^n C(\Omega_i, F_i))_\infty$, where the Ω_i are compact hyperstonean spaces and the F_i are finite dimensional type I JBW-algebra factors.*

Proof. If V is a sub-spin factor of F and F has the ℓ_∞-decomposition stated then representation theory implies that V can be faithfully realised as a sub-spin factor of one of the F_i's implying F is spin bounded. If F is not of the stated form but is still type I finite, structure theory [18] implies F contains a spin factor of infinite dimension or a sequence of finite dimensional type I factors of strictly increasing dimension and so is not spin bounded, by prior remarks in the latter case.

Otherwise, passing to a summand, we may suppose that F has no type I finite part and is therefore the self-adjoint part of a real W*-algebra R acting on a complex Hilbert space [13, 7.3.3]. But then (eg., the argument of [22, Theorem 1.6]) there exists a von Neumann algebra W with no type I finite part which is real *-isomorphic to a real *-subalgebra of R. Since W must contain a type II_1 hyperfinite factor, W_{sa} must contain an infinite dimensional spin factor [20], as therefore must F, since W_{sa} is Jordan isomorphic to a Jordan subalgebra of $R_{sa} = F$. □

The following is immediate from Corollary 1.2 and Lemma 3.4.

Corollary 3.5. *If E is an ordered Banach space such that E_+ is normal and F is a spin bounded JBW-algebra, then $K(E,F)_+$ and $B(E,F)_+$ are generating.*

Theorem 3.6. *Let E be an ordered Banach space and F be a JBW-algebra which is not spin bounded. Suppose that $K(E,F) \subseteq B(E,F)_+ - B(E,F)_+$. Then every ordered Banach simplex subspace of E is finite dimensional. If E is a strongly spectral GM-space, then every element of E has finite spectrum.*

Proof. Suppose E contains an infinite dimensional simplex space as an ordered Banach subspace. Since F is not spin bounded previous arguments imply that F contains a JW-subalgebra M where M is either an infinite dimensional spin factor, or is a sum of an infinite sequence of mutually orthogonal finite dimensional spin factors of strictly increasing dimension. It follows from [10, Lemma 2.3] in the first case, and by elementary application of that result in the second, that M is the image of a positive norm one projection $P : F \to F$ (vanishing on any exceptional part). By Lemma 3.2 there is a compact operator $\psi : E \to M$ with no positive decomposition in $B(E,M)$. Therefore the compact operator $\psi_1 : E \to F$, given by $\psi_1(a) = \psi(a)$, has no positive decomposition in $B(E,F)$, else $\psi = P\psi_1 \in B(E,M)_+ - B(E,M)_+$ in contradiction.

Suppose E is a strongly spectral GM-space and that $a \in E$ with spectrum $\sigma(a)$. By [2, Proposition 9.10, Theorem 10.6] and [8, Theorem 3.3, Theorem 3.6] a lies in a subspace $M_0(a)$ of E order isometric to the space of real valued continuous functions vanishing at 0 on the compact Hausdorff space

$\sigma(a) \cup \{0\}$ which, by the first part of the proof, must be finite dimensional. □

We note that if E and F are any ordered Banach spaces such that $B(E,F)_+$ has an interior point then $K(E,F)_+$ is generating since then (see § 2) there exist $\rho \in E_+^*$ and $u \in F_+$ such that the finite rank operator $\rho \otimes u$ is an interior point of $B(E,F)_+$ and hence is an interior point of $K(E,F)_+$.

Corollary 3.7. *Let E be a JB-algebra and F a JBW-algebra which is not spin bounded. The following are equivalent:*

(*a*) *$K(E,F)_+$ is generating;*
(*b*) *$B(E,F)_+$ is generating;*
(*c*) *$B(E,F)$ is order isomorphic to an order unit space;*
(*d*) *$E = A \oplus B$ where A is finite dimensional and B is an ℓ_∞-sum (possibly empty) of finitely many infinite dimensional spin factors.*

Proof. $(a) \Rightarrow (d), (b) \Rightarrow (d)$. Given (a) or (b), the spectrum of every element of E must be finite by Theorem 3.6. In particular, E is a dual JB-algebra [7]. Further, being the self-adjoint part of a commutative C*-algebra, every associative JB-subalgebra of E must be finite dimensional. Thus, (d) now follows from [7, Theorem 4.3].

$(d) \Rightarrow (c)$. Suppose $E = A \oplus B$, where A is finite dimensional and B is the orthogonal sum of infinite dimensional spin factors $B_1, \ldots, B_n$. Then A_+^* and each $(B_i^*)_+$ has non-empty interior, by Lemma 2.2 in the latter case, and so E_+^* has non-empty interior. Hence, $B(E,F)_+$ has non-empty interior, by Proposition 2.6, and since also normal [23, Theorem 3.1], (c) results.

The implication $(c) \Rightarrow (b)$ is obvious, and $(c) \Rightarrow (a)$ is immediate from the remark prior to the statement. This completes the proof. □

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