Closed tripotents and weak compactness in the dual space of a JB*-triple

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1 Introduction

One of the most celebrated and useful results characterizing weakly compact subsets in the dual space of a C*-algebra is due to Pfitzner, who established that weak compactness in the dual space of a C*-algebra is commutatively determined (see [29]). More concretely, Pfitzner shows, in a "tour de force", that if K is a bounded subset in the dual space of a C*-algebra A, then K is relatively weakly compact if and only if the restriction of K to each maximal abelian subalgebra of A is relatively weakly compact. This result has many important consequences, one of the most interesting is that every C*-algebra satisfies property (V) of Pelczynski.

Pfitzner's result is the latest advance in the study of weak compactness in the dual space of a C*-algebra developed by Takesaki [32], Akemann [1], Akemann, Dodds and Gamlen [3], Saitô [31] and Jarchow [21, 22].

In the more general setting of dual spaces of JB*-triples the study of weak compactness has been developed by Chu and Iochum [12] and Rodríguez-Palacios and the second author of the present paper [28, 27]. However, all the results concerning weak compactness in the dual space of a JB*-triple give characterizations in terms of the abelian subtriples of its bidual instead of the abelian subtriples of the JB*-triple itself. The question clearly is whether a bounded subset in the dual space of a JB*-triple, E, is relatively weakly compact whenever its restriction to any abelian subtriple of E is.

In the main result of this paper we show that weak compactness in the dual space of a JC*-triple is commutatively determined, by showing that a bounded subset K in the dual space of a JC*-triple E is relatively weakly

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compact if and only if the restriction of K to each separable abelian subtriple of E also is relatively weakly compact (see Theorem 3.5).

In order to obtain our characterisation, in Section 2, we introduce the concept of closed tripotents in the bidual of a JB*-triple. This concept generalices the so-called closed projections in the bidual of a C*-algebra introduced and developed by Akemann and Pedersen in [2, 4, 5] and [26, Proposition 3.11.9].

In the general setting of dual Banach spaces we introduce the following concept of "open subspace": Let X be a Banach space and E a weak*-dense subset of X^*. We say that a subset O in X^* is *open relative to* E if $O \cap E$ is weak*-dense in the weak*-closure of O. We say that a tripotent e in the bidual of a JB*-triple E is closed if and only if the orthogonal space of $\{e\}$ in E^{**} is an open subset of E^{**} relative to E.

Closed tripotents also have strong connections with the so-called compact tripotents introduced by Edwards and Ruttimann [16]. We show that every compact tripotent in the bidual of a JB*-triple, say E, is a closed tripotent relative to E, while the reverse implication is not true in general (see Theorem 2.6).

Pfitzner's result is one of the main ingredients in the proof of the non-commutative generalisation of a theorem of Dieudonne obtained by J. Brooks, K. Saitô and J. D. M. Wright in [9]. In the last section of this paper we establish a Dieudonne's type theorem for JC*-triples by a similar approach to the one given by Brooks, Saitô and Wright. More concretely, Theorem 4.2 shows that a sequence (ϕ_n) in the dual of a JC*-triple E, satisfying that $(\phi_n(r))$ converges for each range tripotent $r \in E^{**}$, is weakly convergent.

We recall (c.f. [24]) that a JB*-triple is a complex Banach space together with a continuous triple product $\{.,.,.\} : E \times E \times E \to E$, which is conjugate linear in the middle variable and symmetric bilinear in the outer variables satisfying that,

(a) $L(a,b)L(x,y) = L(x,y)L(a,b) + L(L(a,b)x,y) - L(x,L(b,a)y)$, where $L(a,b)$ is the operator on E given by $L(a,b)x = \{a,b,x\}$;

(b) $L(a,a)$ is an hermitian operator with non-negative spectrum;

(c) $\|L(a,a)\| = \|a\|^2$.

Every C*-algebra is a JB*-triple via the triple product given by

$$2\{x,y,z\} = xy^*z + zy^*x,$$

and every JB*-algebra is a JB*-triple under the triple product

$$\{x, y, z\} = (x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^*.$$

A JC*-triple is a norm-closed subspace of the Banach space, $L(H)$, of all bounded linear operators on a complex Hilbert space H, which is also closed for the law $(x, x, x) \mapsto xx^*x$, equivalently, a closed subtriple of $L(H)$.

A JBW*-triple is a JB*-triple which is also a dual Banach space (with a unique predual [8]). The second dual of a JB*-triple is a JBW*-triple [15]. Elements a, b in a JB*-triple E are *orthogonal* if $L(a, b) = 0$. With each tripotent u (i.e. $u = \{u, u, u\}$) in E is associated the *Peirce decomposition*

$$E = E_2(u) \oplus E_1(u) \oplus E_0(u),$$

where for $i = 0, 1, 2$, $E_i(u)$ is the $\frac{i}{2}$ eigenspace of $L(u, u)$. The Peirce rules are that $\{E_i(u), E_j(u), E_k(u)\}$ is contained in $E_{i-j+k}(u)$ if $i-j+k \in \{0, 1, 2\}$ and is zero otherwise. In addition,

$$\{E_2(u), E_0(u), E\} = \{E_0(u), E_2(u), E\} = 0.$$

The corresponding *Peirce projections*, $P_i(u) : E \to E_i(u)$, $(i = 0, 1, 2)$ are contractive and satisfy

$$P_2(u) = D(2D - I), \; P_1(u) = 4D(I - D), \;\text{ and } P_0(u) = (I - D)(I - 2D),$$

where D is the operator $L(u, u)$ and I is the identity map on E (compare [17]).

Given a JBW*-triple W, a norm-one element φ of W_* and a norm-one element z in W such that $\varphi(z) = 1$, it follows from [7, Proposition 1.2] that the assignment

$$(x, y) \mapsto \varphi\{x, y, z\}$$

defines a positive sesquilinear form on W, the values of which are independent of choice of z, and induces a prehilbert seminorm on W given by

$$\|x\|_\varphi := (\varphi\{x, x, z\})^{\frac{1}{2}}.$$

As φ ranges over the unit sphere of W_* the topology induced by these seminorms is termed the strong*-topology of W. The strong* topology is compatible with the duality (W, W_*) (see [7, Theorem 3.2]). The strong*-topology was introduced in [7], and further developed in [30, 28]. In particular, the triple product is jointly strong*-continuous on bounded sets (see [30]).

Given a Banach space X, we denote by B_X, S_X, X^*, and $L(X)$ the closed unit ball, the unit sphere, the dual space of X, and the Banach space of all bounded linear operators on X, respectively.

2 Open Subspaces

In this section we shall introduce the concept of closed tripotent in the bidual of a JB*-triple. We shall begin by introducing the concept of open subspace in the more general setting of dual Banach spaces.

Definition 2.1. *Let X be a Banach space, E a weak*-dense subset of X^* and S a non-zero subset of X^*. We say that S is* open relative to E *if $S \cap E$ is $\sigma(X^*, X)$-dense in $\overline{S}^{\sigma(X^*,X)}$.*

It is clear that if E is any weak*-dense subset in the dual of a Banach space X, then X^* is open relative to E.

Lemma 2.2. *Let X be a Banach space, $E \subset X^*$ a weak*-dense subset and $\{S_i\}_{i\in I}$ a family of subsets of X^* which are open relative to E. Then $\cup_{i\in I} S_i$ is open relative to E.*

Proof. Let x_0 be a weak* cluster point of $\cup_{i\in I} S_i$ in X^*. Then for each $\varphi_1, \ldots, \varphi_n \in X$ and $\varepsilon > 0$, there exists $x_{i_0} \in S_{i_0}$, such that $|\varphi_j(x_0 - x_{i_0})| < \frac{\varepsilon}{2}$, for all $j \in \{1, \ldots n\}$. Since S_{i_0} is open relative to E, there exists $y_{i_0} \in E \cap S_{i_0} \subseteq E \cap (\cup S_i)$ satisfying $|\varphi_j(y_{i_0} - x_{i_0})| < \frac{\varepsilon}{2}$, for all $j \in \{1, \ldots n\}$. This gives the desired statement. □

Let A be a C*-álgebra. Let p be a projection in A^{**}. Following [2] (see also [4, 5] and [26, Proposition 3.11.9]), we say that p is *open* (*relative to* A) if $pA^{**}p \cap A$ is weak*-dense in $pA^{**}p$, that is, $pA^{**}p$ is an open subset of A^{**} relative to A in the terminology introduced in the previous definition. We shall say that p is *closed* (*relative to* A) if $1 - p$ is open relative to A.

In [16, page 167], C. M. Edwards and G. T. Rüttimann introduced the following concept of open tripotent. Let W be a JBW*-triple, E a strong*-dense subtriple of W, and e a tripotent in W, we say that e is *open relative to* E if $W_2(e) \cap E$ is weak*-dense in $W_2(e)$. It is well known that when a C*-algebra A is regarded as a JB*-triple, then tripotents in A coincide with partial isometries of A. Thus, a projection (respectively, a partial isometry) $p \in A^{**}$ is open relative to A if and only if p is an open tripotent relative to A.

Inspired by the previous arguments we introduce the concept of closed tripotent. Let E be a weak*-dense JB*-subtriple of a JBW*-triple W and e a tripotent in W, we say that e is *closed relative to* E if $W_0(e)$ is an open subset of W relative to E. It is not hard to see that when a C*-algebra A is regarded as a JB*-triple, then a projection $p \in A^{**}$ is closed relative to A if and only if p is a closed projection in the sense of [2, 4, 26].

The concept of closed tripotent has a natural strong connection with the notions of compact projections and compact tripotents introduced by C. Akemann and G. K. Pedersen [5] and C. M. Edwards and G. T. Rüttimann [16], respectively. Let A be a C*-algebra and let p be a projection in A^{**}. We say that p is a *compact* if p is closed relative to A and there exists a norm-one element $x \in A^+$ such that $p \leq x$ (compare [5, page 422]).

Let W be a JBW*-triple and let a be a norm-one element in W. The sequence (a^{2n-1}) defined by $a^1 = a$, $a^{2n+1} = \{a, a^{2n-1}, a\}$ $(n \in \mathbb{N})$ converges in the strong*-topology (and hence in the weak*-topology) of W to a tripotent $u(a) \in W$ (compare [16, Lemma 3.3]). This tripotent will be called the *support tripotent* of a. There exists a smallest tripotent $r(a) \in W$ satisfying that a is positive in the JBW*-algebra $W_2(r(a))$, and $u(a) \leq a^{2n-1} \leq a \leq r(a)$. This tripotent $r(a)$ will be called the *range tripotent* of a. (Beware that in [16], $r(a)$ is called the support tripotent of a).

Let W be a JBW*-triple and let E be a weak*-dense subtriple of W. A tripotent u in W is said to be *compact-G_δ relative to E* if u is the support tripotent of a norm one element in E. The tripotent u is said to be *compact relative to E* if $u = 0$ or there exist a decreasing net of compact-G_δ tripotents relative to E, (u_λ), in W converging, in the strong*-topology of W, to the element u (compare [16, §4]). When E is a JB*-triple, the range (respectively, the support) tripotent of every norm-one element in E is always an open (respectively, compact) tripotent in E^{**} relative to E.

The next result shows the connection between closed and compact tripotents.

Proposition 2.3. *Let E be a weak*-dense JB*-subtriple of a JBW*-triple W and let u be a compact tripotent relative to E in W. Then u is closed relative to E.*

For the proof of the above result we shall prepare first some technical lemmas.

Lemma 2.4. *Let A be a unital JB*-algebra, p a projection in A, and x a norm-one element in A satisfying $0 \leq p \leq x$. Then for each y in A, the element $y - 2L(z,z)y + Q(z)^2(y)$ belongs to $A_0(p)$, where $z \in A^+$ satisfies $z^2 = x$.*

Proof. Since $p \leq x$ we have $1 - p \geq 1 - x \geq 0$, and thus $1 - x \in A_0(p)$. Let $y \in A$. From the expressions

$$2L(z,z)(y) := 2\left(z^2 \circ y + (y \circ z) \circ z - (y \circ z) \circ z\right) = 2x \circ y, \text{ and}$$

$$Q(z)^2(y) = U_z^2(y) = U_{z^2}(y) = U_x(y) = Q(x)(y^*), \tag{1}$$

we deduce that $y - 2L(z,z)y + Q(z)^2(y) = Q(1-x)(y^*)$, where in (1) we are applying [19, Lemma 2.4.21] to assure that $U_z^2(y) = U_{z^2}(y)$. Since $1-x$ belongs to $A_0(p)$, it follows, by Peirce arithmetic, that

$$y - 2L(z,z)y + Q(z)^2(y) \in A_0(p).$$

□

Let E be a JB*-triple. It is known that the JB*-subtriple of E generated by a norm-one element x (denoted by E_x) is JB*-triple isomorphic (and hence isometric) to $C_0(\Omega)$ for some locally compact Hausdorff space Ω contained in $(0,1]$, such that $\Omega \cup \{0\}$ is compact and $C_0(\Omega)$ denotes the Banach space of all complex-valued continuous functions vanishing at 0. It is also known that if Ψ denotes the triple isomorphism from E_x onto $C_0(\Omega)$, then $\Psi(x)(t) = t$ $(t \in \Omega)$ (cf. [23, 4.8], [24, 1.15] and [17]). In particular, for each $n \in \mathbb{N}$, we can define the $(2n-1)$-th square root of x denoted by $x^{1/2n-1}$. It is not hard to see that $x^{1/(2n-1)}$ converges to $r(x)$ in the strong*-topology of E^{**}. Therefore, denoting $f(t) := \sqrt{t}$, $(t \in \Omega)$, there exists a unique positive element $z = \Psi^{-1}(f) \in E_x$, such that $\left\{z, z, x^{1/(2n-1)}\right\}$ converges to x in the strong* topology of E^{**}, in particular $\{z, z, r(x)\} = x$. This element z will be called the *positive square root* of x.

Let E be a JB*-triple, e a tripotent in E, and x a norm-one element in E. We shall say that $e \leq x$ (respectively, $e \geq x$) if and only $L(e,e)x = e$ (respectively, x is a positive element in $E_2(e)$). If $e \geq x$ and z denotes the positive square root of x then $\{z, z, e\} = x$, that is, z is a positive square root of x in the JB*-algebra $E_2(e)$.

Lemma 2.5. *Let E be a JB*-triple, let e be a tripotent in E, and let x be a norm-one element in E such that $e \leq x$. Then for each y in E the element $y - 2L(z,z)y + Q(z)^2(y)$ belongs to $E_0(e)$, where z is the positive square root of x in E.*

Proof. By the Gelfand-Naimark Theorem of Friedman and Russo (see [18]), E can be embedded as a subtriple into an ℓ_∞-sum of Cartan factors. Since each Cartan factor can be also embedded as a subtriple of $L(H)$ or $H_3(\mathbb{O})$, then we can assume that E is a JB*-subtriple of the JBW*-álgebra

$$A = L(H) \bigoplus^{\infty} \left(\bigoplus^{\infty} C_\alpha \right),$$

where C_α coincide $H_3(\mathbb{O})$, for all α. We may then assume that

$$e \le x \ (\ \le r(x) \ )$$

in the JBW*-algebra A, where $r(x)$ is the range tripotent of x in A. From [11, Lemma 2.3] and [25, Corollary 5.12] there exists an isometric triple embedding T from A onto A, such that $T(r(x))$ (and hence $T(e)$) is a projection in A. In particular T is a triple isomorphism. By Lemma 2.4, it follows that for every $T(y) \in T(E) \subseteq A$ we have

$$T(y) - 2L(T(z), T(z))T(y) + Q(T(z))^2(T(y)) \in A_0(T(e)),$$

where z is the positive square root of x in E. Therefore,

$$y - 2L(z, z)y + Q(z)^2(y) \in A_0(e) \cap E = E_0(e),$$

for all $y \in E$. □

In the sequel, given x in a JB*-triple E we shall denote by $P_0(x)$, the bounded linear operator from E to E defined by

$$P_0(x)(y) = y - 2L(z, z)y + Q(z)^2(y),$$

where z is the square root of x in E. In the literature this operator is called the Bergman operator associated to z. We should note that this notation is not ambiguous, because when e is a tripotent then $P_0(e)$ is precisely the Peirce projection of E onto $E_0(e)$.

Proof. (Proposition 2.3) Suppose first that u is compact-G_δ relative to E, that is, there exists a norm-one element $a \in E$ such that $u = u(a) = \text{Strong}^* - \lim a^{2n-1}$. We note that for each $n \in \mathbb{N}$ $u(a) \le a^{2n-1} \le a$. Let $y_0 \in E_0^{**}(u)$. From [28, Corollary 9] there exists a bounded net $(y_\lambda)_{\lambda \in \Lambda}$ in E such that y_λ converges to y_0 in the strong*-topology of W. By the joint strong*-continuity of the triple product it follows that the net $(z_{\lambda,n})_{\Lambda \times \mathbb{N}}$ defined by

$$z_{\lambda,n} = P_0(a^{2n-1})(y_\lambda)$$

converges, in the strong*-topology of W, to $P_0(u(a))(y_0) = y_0$. Finally, Lemma 2.5 assures that $z_{\lambda,n} \in W_0(u(a)) \cap E$, which shows that $u(a)$ is closed.

If $u \in W$ is a compact tripotent relative to E. Then there exists a decreasing net (u_λ) of tripotents which are compact-G_δ relative to E satisfying that $u = \text{Strong}^* - \lim u_\lambda$. We claim that $W_0(u) = \overline{\cup_\lambda W_0(u_\lambda)}^{w^*}$. Since for each λ, $u \leq u_\lambda$ we have $W_0(u_\lambda) \subseteq W_0(u)$ and hence $W_0(u) \supseteq \overline{\cup_\lambda W_0(u_\lambda)}^{w^*}$. To see the reverse inclusion let $y_0 \in W_0(u)$. The net $y_\lambda := P_0(u_\lambda)(y_0)$ converges in the strong*-topology of W to $P_0(u)(y_0) = y_0$ and $(y_\lambda) \subseteq \cup_\lambda W_0(u_\lambda)$, which gives $y_0 \in \overline{\cup_\lambda W_0(u_\lambda)}^{w^*}$. Therefore $W_0(u) = \overline{\cup_\lambda W_0(u_\lambda)}^{w^*}$. From the first part of the proof we have u_λ is closed relative to E for every λ, and hence $W_0(u_\lambda)$ is an open subset in W relative to E. The statement follows now from Lemma 2.2. □

The reverse of the above result is not true in general. For example, let $E = C_0((0,1])$, and let $e = \chi_{(0,\frac{1}{2}]} \in E^{**}$ (the characteristic function of $(0, \frac{1}{2}]$) is a closed tripotent relative to E in E^{**} but e is not compact relative to E because for each $x \in E$ we have $e \nleq x$ (and hence $e \nleq u(x)$).

Inspired by the above results, we introduce the following definition. Let W be a JBW*-triple, N a subset of W and u a tripotent in W. We say that u is *bounded relative to* N if there exist a norm one element x in N, such that $L(u,u)x = u$, that is, $u \leq x$.

Let X be a Banach space. For each pair of subsets G, F in the unit balls of X and X^*, respectively, let the subsets $G^{'}$ and $F_{,}$ be defined by

$$G^{'} = \{f \in B_{X^*} : \ f(x) = 1 \ \forall x \in G\}, \quad F_{,} = \{x \in B_X : \ f(x) = 1 \ \forall f \in F\}.$$

If C is a convex subset of a complex vector space and F is a convex subset of C, we say that F is a face of C if whenever $tx_1 + (1-t)x_2 \in F$, with $x_1, x_2 \in C$ and $0 < t < 1$, then $x_1, x_2 \in F$.

Theorem 2.6. *Let E be a weak*-dense JB*-subtriple of a JBW*-triple W. Suppose that u is a tripotent in W. The following assertions are equivalent:*

1. *u is a compact tripotent relative to E;*
2. *u is closed and bounded relative to E.*

Proof. 1. ⇒ 2. Let $u \in W$ be a compact tripotent relative to E. Then, by Proposition 2.3, it follows that u closed relative to E. If $u = 0$ then it is clear that u is bounded relative to E. When $u \neq 0$, then there exists a decreasing net, (u_λ), of compact-G$_\delta$ tripotents relative to E converging to u in the strong*-topology of W. In particular, for each λ there exists a norm-one element x_λ in E such that $u \leq u_\lambda \leq x_\lambda$.

2. ⇒ 1. Since u is bounded relative to E there exist a norm one element $x \in E$ such that $L(u,u)x = u$. Clearly $x - u \in W_0(u)$. Since u is closed relative to E, then $W_0(u)$ is an open subset of W relative to E. Therefore, there exists a net $(y_\lambda)_{\lambda\in\Gamma}$ in $W_0(u)\cap E$ converging to $x-u$ in the strong*-topology of W. In particular $x_\lambda := x - y_\lambda$ is a net in E converging to u in the strong-* topology of W. Moreover, since $L(u,u)x_\lambda = u$, it follows that $u \le u(x_\lambda) \le x_\lambda$. Since, for each φ in the unit sphere of W_*, we have

$$\|x_\lambda - u\|_\varphi^2 = \|x_\lambda - u(x_\lambda)\|_\varphi^2 + \|u(x_\lambda) - u\|_\varphi^2 \ge \|u(x_\lambda) - u\|_\varphi^2,$$

then $u(x_\lambda) \to u$ in the strong*-topology of W.

On the other hand, each $u(x_\lambda)$ is a compact-G_δ tripotent relative to E and hence $\{u(x_\lambda)\}_, = \{x_\lambda\}_,$ (compare [16, Lemma 3.1]). We claim that

$$\{u\}_, = \bigcap_\lambda \{u(x_\lambda)\}_, . \tag{2}$$

Indeed, since for each λ, $u \le u(x_\lambda)$, then every $\varphi \in \{u\}_,$ also lies in $\{u(x_\lambda)\}_,$ and hence $\{u\}_, \subseteq \bigcap_\lambda \{u(x_\lambda)\}_,$. To see the equality, take $\varphi \in \bigcap_\lambda \{u(x_\lambda)\}_,$. From [30, Corollary 3], we conclude that φ is strong* continuous. Therefore, since $u(x_\lambda) \to u$ in the strong* topology, we have $1 = \varphi(u(x_\lambda)) \to \varphi(u)$, and hence $\varphi \in \{u\}_,$.

We shall show now that the arguments given in the proof of Theorem 4.2 in [16] can be adapted to show that u is compact relative to E.

Let G denote the convex set $\left(\{u\}_,\right)' \cap E$. The expression (2) shows that G is not empty. It is not hard to see that G is a face of the closed unit ball of E. For each $a \in G$ let face(a) denote the smallest face of B_E containing $\{a\}$ and set $\Lambda = \{\text{face}(a) : a \in G\}$. Since for each $a_1, a_2 \in G$, both face(a_1) and face(a_2) are contained in face$(\frac{1}{2}(a_1+a_2))$, we conclude that Λ is a partially ordered by set inclusion and upwards directed. Moreover, if $a_1 \in \text{face}(a_2) \subset \left(\{a_2\}_,\right)' \cap E = \left(\{u(a_2)\}_,\right)' \cap E$, then we conclude by [16, Lemma 3.1] (see also [17, Lemma 1.6]) that $u(a_1) \ge u(a_2)$. For each $\mu \in \Lambda$ we define $u_\mu = u(a)$, where $a \in G$ is the element satisfying $\mu = \text{face}(a)$. Then, $\{u_\mu\}_{\mu\in\Lambda}$ is a decreasing net of compact-G_δ tripotents relative to E. In particular, the net $\{u_\mu\}_{\mu\in\Lambda}$ converges in the strong*-topology of W to its infimum denoted by v. Clearly, v is a compact tripotent relative to E.

Since for each $a \in G$ we have $\{u\}_, \subset \{a\}_, = \{u(a)\}_,$ we deduce that $u \le u(a)$, and hence $u \le v$. Since for each $\lambda \in \Gamma$, $x_\lambda \in G$, the expression (2) gives

$$\{u\}_, \supseteq \bigcap_{a\in G} \{u(a)\}_, .$$

And the reverse inclusion clearly follows from the fact that $\{u\}_{\prime} \subset \{a\}_{\prime} = \{u(a)\}_{\prime}$ for all $a \in G$. Therefore, since (u_μ) converges to v in the strong*-topology, proceeding as in the previous paragraph, we have

$$\{u\}_{\prime} = \bigcap_{a \in G} \{u(a)\}_{\prime} = \{v\}_{\prime}.$$

This gives $u = v$ is a compact tripotent relative to E. □

As a consequence of the above result, it can be easily seen that when A is a unital C*-algebra, then every projection in A^{**} closed relative to A is compact relative to A, since every projection is automatically bounded by the unit element. The following example shows that this is not longer true for closed tripotents.

Example 2.7. Let $K = [0,1]$ and let $E = C(K)$ the unital C*-algebra of all complex valued continuous functions on K. Let us denote by u the bounded function defined by

$$u(t) = \begin{cases} 1, & \text{if } t \in [\frac{3}{8}, \frac{1}{2}) \cup (\frac{1}{2}, \frac{5}{8}] \\ \exp i\theta, & \text{if } t = \frac{1}{2} \\ 0, & \text{elsewhere} \end{cases}$$

Clearly, u is a tripotent in E^{**} closed relative to E. However, for each $\theta \notin 2\pi\mathbb{Z}$, u is not compact, since it can not be bounded relative to E.

Corollary 2.8. *Let E be a weak*-dense JB*-subtriple of a JBW*-triple W, u_1, u_2 tripotents in W. Suppose that u_1 is closed in W relative to E and u_2 is closed in $W_0(u_1)$ relative to $W_0(u_1) \cap E$. Then the tripotent $u_1 + u_2$ is a closed tripotent in W relative to E.* □

The following corollary is a generalization of [16, Corollary 4.8]

Corollary 2.9. *Let E be a weak*-dense JB*-subtriple of a JBW*-triple W. Let A be a JB*-subtriple of W which is open relative to E, and let $u \in \overline{A}^{\sigma(W,W_*)}$ be a tripotent. Then u is compact in W relative to E whenever u is compact in $\overline{A}^{\sigma(W,W_*)}$ relative to $A \cap E$.*

Proof. If u is compact-G_δ in $\overline{A}^{\sigma(W,W_*)}$ relative to $A \cap E$, then there exists a norm-one element $x \in A \cap E$ such that $u = u(x)$, in particular u is compact-G_δ in W relative to E. If u is compact in $\overline{A}^{\sigma(W,W_*)}$ relative to $A \cap E$, then there exists a decreasing net of tripotents, $(u_\lambda) \subset \overline{A}^{\sigma(W,W_*)}$, which are compact-$G_\delta$ relative to $A \cap E$, converging to u in the strong*-topology of $\overline{A}^{\sigma(W,W_*)}$. Now, it follows from the previous paragraph and [10, Corollary] that u_λ is a decreasing net of tripotents in W which are compact-G_δ relative to E. $\square$

3 Weak compactness is commutatively determined

Let X be a Banach space. A series $\sum x_i$ in X is said to be *weakly unconditionally convergent* (w.u.c in the sequel) if $\sum |\phi(x_i)| < +\infty$, for every $\phi \in X^*$. The space X satisfies *Pelczynski's property (V)* if a bounded set $K \subset X^*$ is relatively weakly compact whenever $\lim_{n\to+\infty} \sup\{|\phi(x_n)| : \phi \in K\} = 0$, for every w.u.c. series $\sum x_i$ in X. Every C*-álgebra and every JB*-triple satisfies property (V) (compare [29, Corollary 6] and [14, Theorem 3])

Let K be a bounded subset in the dual of a C*-algebra A. Suppose that K is not relatively weakly compact. The classical theorems of Eberlein-Šmul'jan and Rosenthal assure the existence of an ℓ^1-basis (φ_k) in K. Let $r > 0$ such that for each (α_k) in $\mathbb{R}$ we have

$$r \sum_{k\in\mathbb{N}} |\alpha_k| \leq \left\| \sum_{k\in\mathbb{N}} \alpha_k \frac{\varphi_k}{\|\varphi_k\|} \right\| \leq \sum_{k\in\mathbb{N}} |\alpha_k|.$$

The excellent result proved by H. Pfitzner in [29, Theorem 1] and concretely the arguments given in its proof (see page 364 in [29]) assure that taking $1 > \varepsilon > 0$, and $\theta = (1-\varepsilon)\ r\ \inf_{k\in\mathbb{N}} \|\varphi_k\|$, then there exist a sequence (x_n) of pairwise orthogonal selfadjoint elements in the unit ball of A and a subsequence $(\varphi_{\sigma(k)})$ satisfying that $|\varphi_{\sigma(k)}(x_k)| > \theta$, for every $k \in \mathbb{N}$.

Let E be a JC*-triple and let K be a bounded subset in the dual of E. Again the classical theorems of Eberlein-Šmuljan and Rosenthal assure the existence of an ℓ^1-basis (φ_k) in K. Let $r > 0$ such that for each (α_k) in $\mathbb{R}$ we have

$$r \sum_{k\in\mathbb{N}} |\alpha_k| \leq \left\| \sum_{k\in\mathbb{N}} \alpha_k \frac{\varphi_k}{\|\varphi_k\|} \right\| \leq \sum_{k\in\mathbb{N}} |\alpha_k|.$$

From [13] it follows that E^{**} embeds as a subtriple in some von Neumann algebra M and there exists a weak*-continuous contractive projection $P : M \to E^{**}$. Since the mapping $P^*|_{E^*} : E^* \to M_*$ embeds isometrically E^* in M_*, the arguments given in the previous paragraph imply that taking $1 > \varepsilon > 0$, and $\theta = (1-\varepsilon)\ r\ \inf_{k\in\mathbb{N}} \|\varphi_k\|$, then there exists a sequence (x_n) of pairwise orthogonal selfadjoint elements in the unit ball of M and a subsequence, $(\varphi_{\sigma(k)})$, satisfying that $|\varphi_{\sigma(k)}(x_k)| > \theta$, for every $k \in \mathbb{N}$. The same proof given by Ch.-H. Chu and P. Mellon in [14, Theorem 1] can be literally adapted to get a w.u.c. series $\sum z_n$ in E and a subsequence $(\varphi_{\sigma(n)})$ such that $\|z_n\| < 1$, and $|\varphi_{\sigma(n)}(z_n)| > \frac{\theta}{4}$. We have obtain:

Proposition 3.1. *Let E be a JC*-triple, let $r > 0$, and $1 > \varepsilon > 0$. Suppose that (φ_k) is a bounded sequence in the dual of E such that for each (α_k) in $\mathbb{R}$ we have*

$$r \sum_{k\in\mathbb{N}} |\alpha_k| \leq \left\| \sum_{k\in\mathbb{N}} \alpha_k \frac{\varphi_k}{\|\varphi_k\|} \right\| \leq \sum_{k\in\mathbb{N}} |\alpha_k|.$$

Then taking $\theta = (1-\varepsilon)\ r\ \inf_{k\in\mathbb{N}} \|\varphi_k\|$, there exists a w.u.c. series $\sum z_n$ in E and a subsequence $(\varphi_{\sigma(n)})$ such that $|\varphi_{\sigma(n)}(z_n)| > \frac{\theta}{4}$. □

Lemma 3.2. *Let E be a JB*-triple, let c be a tripotent in E, and let φ be a norm-one elements in E^* such that $\|c\|_\varphi^2 < \delta$ for some $\delta > 0$. Then for each $x \in E$ we have*

$$|\varphi P_2(c)(x)| < 3\sqrt{\delta}\ \|P_2(c)(x)\|. \tag{3}$$

and

$$|\varphi P_1(c)(x)| < 6\sqrt{\delta}\ \|P_1(c)(x)\|, \tag{4}$$

Proof. Let $s \in E^{**}$ be a tripotent such that $\varphi(s) = 1$. Arguing as in the proof of Proposition 2.4 in [11], there exists a JB*-algebra, A, and a triple embedding (isometric triple homomorphism) $T : E^{**} \to A$ such that $T(s) = p$ is a projection in A. By the Hahn-Banach theorem there exists a norm-one positive functional ψ in the dual of A extending $\varphi \circ T^{-1}$ from $T(E^{**})$ to A. For each $x \in E^{**}$ we also have

$$\|T(x)\|_\psi^2 = \psi(\{T(x), T(x), T(s)\}) = \psi T\{x, x, s\} = \varphi\{x, x, s\} = \|x\|_\varphi^2. \tag{5}$$

In particular $T(c) = e$ is a tripotent in A and satisfies

$$\delta > \|e\|_\psi^2 = \psi\{e, e, p\} = \psi\{e, e, 1\} = \psi(e \circ e^*).$$

Since $e \circ e^*$ is a positive element with $\|e \circ e^*\| \leq 1$, then $(e \circ e^*)^2 \leq e \circ e^*$, and hence $\psi((e \circ e^*)^2) \leq \psi(e \circ e^*) < \delta$.

Let $z \in A_2(e)$. By the Cauchy-Schwarz inequality we have

$$|\psi(x)| = |\psi L(e, e)x| = |\psi((e \circ e^*) \circ z + (z \circ e^*) \circ e - (e \circ z) \circ e^*)|$$

$$
\begin{aligned}
&\leq |\psi((e\circ e^*)\circ z| + |\psi(z\circ e^*)\circ e| + |\psi(e\circ z)\circ e^*)| \\
&\leq \sqrt{\psi((e\circ e^*)^2)\psi(z\circ z^*)} + \sqrt{\psi((z\circ e^*)\circ(z\circ e^*)^*)\psi((e\circ e^*))} \\
&\quad + \sqrt{\psi((z\circ e)\circ(z\circ e)^*)\psi((e\circ e^*))} \leq 3\sqrt{\delta}\ \|z\|.
\end{aligned}
$$

Since for each $x \in E_2(c)$, we have $T(x) \in A_2(e)$, the above inequality and (5) give the statement in (3). The inequality (4) can be deduced in a similar way. □

Lemma 3.3. *Let φ be a norm-one functional in the predual of a JBW*-triple W. Then for every norm-one element x in W we have that $\|.\|_\varphi$ is an order-preserving map on the set of all positive elements in the abelian von Neumann algebra $\overline{W_x}^{w^*}$, that is, the weak*-closed subtriple generated by x.*

Proof. Let u_1, u_2 be two orthogonal tripotents in W. Then we have

$$\|u_2 + u_1\|_\varphi^2 = \|u_2\|_\varphi^2 + \|u_1\|_\varphi^2.$$

Let x, y two positive elements in $\overline{W_x}^{w^*}$, such that $x \leq y$. Suppose first that x and y are *algebraic elements*, i.e., finite linear combinations of mutually orthogonal projections (p_i) in $\overline{W_x}^{w^*}$. Since $\overline{W_x}^{w^*}$ is an abelian von Neumann algebra, we may assume that $x = \sum_i \lambda_i p_i$ and $y = \sum_i \mu_i p_i$ with $0 \leq \lambda_i \leq \mu_i$. It follows from the above paragraph that

$$\|y\|_\varphi^2 = \sum_i \mu_i^2 \|p_i\|_{\varphi_i}^2 \geq \sum_i \lambda_i^2 \|p_i\|_{\varphi_i}^2 = \|x\|_\varphi^2.$$

Suppose that $0 \leq x \leq y$ in $\overline{W_x}^{w^*}$. From [19, Proposition 4.2.3] there exist two sequences of algebraic elements $(a_n), (b_n) \in W_2(e)$ satisfying that $0 \leq a_n \leq x$, $y \leq b_n$, $\|x - a_n\| \to 0$, and $\|y - b_n\| \to 0$. Since in particular $0 \leq a_n \leq b_n$ it follows, from the above case, that $\|b_n\|_\varphi \geq \|a_n\|_\varphi$. Finally, since $\|.\|_\varphi$ is norm-continuous we have $\|y\|_\varphi \geq \|x\|_\varphi$. □

Lemma 3.4. *Let E be a JB*-triple, let $\theta > 0, \delta > 0$ and let (φ_n) be a sequence in the closed unit ball of E^*. Suppose that x is a norm-one element in E such that $|\varphi_1(x)| > \theta$ and $\|x\|_{\varphi_n} < \delta$, for all $n \geq 2$. Then there exist a norm-one element z in E and a compact tripotent u in E^{**} such that $z \leq u$, $|\varphi_1(z)| > \frac{3\theta}{4}$ and $\|u\|_{\varphi_n} < \frac{8\delta}{\theta}$, for all $n \geq 2$.*

Proof. Let C denote the norm-closed subtriple of E generated by x. From [23, 4.8] and [24, 1.15] (see also [17]) we known that there exists a locally compact subset S_x of $(0, \|x\|]$ such that $S_x \cup \{0\}$ is compact and C is JB*-triple isomorphic to the C*-algebra of all complex-valued continuous functions on S_x vanishing at 0, $C_0(S_x)$, via a triple isomorphism Ψ, which satisfies $\Psi(x)(t) = t$ $(t \in S_x)$. Let $z \in C_0(S_x)$ be the function defined by

$$z(t) := \begin{cases} 0, & \text{if } 0 \leq t \leq \frac{\theta}{8}; \\ \text{linear}, & \text{if } \frac{\theta}{8} \leq t \leq \frac{\theta}{4}; \\ t, & \text{if } \frac{\theta}{4} \leq t \leq 1. \end{cases}$$

Since $\|x - z\| < \frac{\theta}{8}$ and $|\varphi_1(x)| > \theta$ it follows that $|\varphi_1(z)| \geq \frac{7\theta}{8} > \frac{3\theta}{4}$.

On the other hand, identifying $\Psi(x)|_{[0,\frac{\theta}{8})}$ with $x|_{[0,\frac{\theta}{8})}$ (in E^{**}), we have $x \geq x|_{[\frac{\theta}{8},1]}$ (in C), and hence, by Lemma 3.3, we get $\|x|_{[\frac{\theta}{8},1]}\|_{\varphi_n} < \delta$ for all $n \geq 2$. In particular $\|\frac{8}{\theta}x|_{[\frac{\theta}{8},1]}\|_{\varphi_n} < \frac{8}{\theta}\delta$. Again Lemma 3.3 applied to $u = \chi_{[\frac{\theta}{8},1]} \leq \frac{8}{\theta}x|_{[\frac{\theta}{8},1]}$ assures that

$$\|u\|_{\varphi_n} < \frac{8}{\theta}\delta,$$

for all $n \geq 2$. It is clear that $z \leq u$ and $u \in E^{**}$ is a compact tripotent relative to E, because u is compact in $E_2^{**}(r(x))$ and $r(x)$ is an open tripotent (compare Corollary 2.9 or [16, Corollary 4.8]). $\square$

Let φ be a non-zero functional in the dual of a JB*-triple E. Then $\frac{\varphi}{\|\varphi\|}$ is a norm-one functional in E^* and hence the law $(x, y) \mapsto \frac{\varphi}{\|\varphi\|}\{x, y, e\}$ is a positive sexquilinear form on $E \times E$, where e is a tripotent in E^{**} such that $\frac{\varphi}{\|\varphi\|}(e) = 1$. Therefore the mapping $(x, y) \mapsto \varphi\{x, y, e\}$ is also a positive sexquilinear form on $E \times E$ the corresponding prehilbertian seminorm will be also denoted by $\|x\|_\varphi^2 := \varphi\{x, x, e\}$. From the comments preceding [7, Definition 3.1] we have

$$|\varphi(x)| \leq \|\varphi\| \; \|x\|_\varphi \quad (\varphi \in E^*, \; x \in E).$$

We can state now the main result of this section, which assures that weak compactness in the dual of a JC*-triple is commutatively determined.

Theorem 3.5. *Let E be a JC*-triple and let K be a bounded subset in E^*. The following are equivalent:*

(a) *K is not relatively weakly compact;*

(b) *There exist a sequence of pairwise orthogonal elements (a_n) in the unit ball of E, a sequence (φ_n) in K and $\theta > 0$ such that $|\varphi_n(a_n)| > \theta$ for all $n \in \mathbb{N}$;*

(c) *There exists a separable abelian subtriple $C \subset E$ such that $K|_C$ is not relatively weakly compact.*

Proof. $(a) \Rightarrow (b)$ Let us assume that $K \subseteq B_{E^*}$. Since K is not relatively weakly compact, there exists (φ_k) in K and $r > 0$ such that for each (α_k) in $\mathbb{R}$ we have

$$r \sum_{k\in\mathbb{N}} |\alpha_k| \leq \left\| \sum_{k\in\mathbb{N}} \alpha_k \frac{\varphi_k}{\|\varphi_k\|} \right\| \leq \sum_{k\in\mathbb{N}} |\alpha_k|. \tag{6}$$

Let $A = \inf_{k\in\mathbb{N}} \|\varphi_k\| > 0$. We claim that there exist a sequence of mutually orthogonal compact tripotents (u_n) in E^{**}, $\mathbb{N} = N_0 \supset N_1 \supset N_2 \supset \ldots \supset N_n \supset N_{n+1} \supset \ldots$ infinite subsets of $\mathbb{N}$, a sequence (a_n) in E, and a subsequence $(\varphi_{\sigma(n)})$ such that $u_1 + \ldots + u_n$ is closed, u_{n+1} is closed in $E_0^{**}(u_1 + \ldots + u_n)$, $a_k \leq u_k$, $\sigma(k) \in N_{k-1}\backslash N_k$,

$$|\varphi_{\sigma(k)}(a_k)| > 3\frac{\theta_k}{4^2} = \frac{3}{4^2}\left(\frac{r}{2}(A-3\varepsilon) - 3\sum_{i=1}^{k-1}\varepsilon_i\right), \quad (k \in \{2,\ldots,n\}), \tag{7}$$

$$(r - \frac{6}{A-3\varepsilon}\sum_{k=1}^{n}\varepsilon_k) \sum_{k\in N_n} |\alpha_k| \leq \left\| \sum_{k\in N_n} \alpha_k \frac{\varphi_k|_{E_n}}{\|\varphi_k|_{E_n}\|} \right\| \leq \sum_{k\in N_n} |\alpha_k|, \tag{8}$$

where $E_n = E \cap E_0^{**}(u_1 + u_2 + \ldots + u_n)$ and $A - 3\sum_{k=1}^{n}\varepsilon_k \leq \inf_{k\in\mathbb{N}} \|\varphi_k|_{E_n}\|$.

Let $\frac{A}{6} > \varepsilon > 0$, $3(r+2)\varepsilon < rA$ and (ε_n) in $\mathbb{R}^+$ such that $\sum_n \varepsilon_n = \varepsilon$.

From Proposition 3.1 taking $\theta_1 = \frac{r}{2}\ A$, then there exists a w.u.c. series $\sum z_n$ in E and a subsequence $(\varphi_{\sigma(n)})$ such that $\|z_n\| < 1$ and $|\varphi_{\sigma(n)}(z_n)| > \frac{\theta_1}{4}$. Since $\sum z_n$ is a w.u.c. series, there exists $C_1 > 0$ such that $\|\sum_{n\in F} \sigma_n z_n\| \leq C_1$ for every finite set $F \subset \mathbb{N}$ and $\sigma_n = \pm 1$.

Let $j_1 \in \mathbb{N}$ such that $\frac{C_1^2}{j_1} < \frac{\varepsilon_1^4\theta_1^4}{4^4\ 3^4\ 8^2}$. Let $m \in \mathbb{N}$. Since every Hilbert space is of cotype 2 we have

$$\sum_{k=1}^{j_1} \frac{1}{j_1}\ \|z_k\|_{\varphi_m}^2 \leq \frac{1}{j_1}\int_D \left\| \sum_{k=1}^{j_1} \sigma_k z_k \right\|_{\varphi_m}^2 d\mu \leq \frac{C_1^2}{j_1} < \frac{\varepsilon_1^4\theta_1^4}{4^4\ 3^4\ 8^2}, \tag{9}$$

where $D = \{-1,1\}^{\mathbb{N}}$ and μ is the uniform probability measure on D. Since (9) is satisfied for every $m \in \mathbb{N}$, then there exist $k_1 \in \{1,\ldots,j_1\}$ and an infinite subset $N_1 \subset \mathbb{N}$ such that for every $m \in N_1$,

$$\|z_{k_1}\|_{\varphi_m}^2 < \frac{\varepsilon_1^4\theta_1^4}{4^4\ 3^4\ 8^2}.$$

Let $\varphi_{\sigma(1)} = \varphi_{k_1}$. Then, since $\|z_{k_1}\| \geq \frac{\theta_1}{4}$, we have

$$\left\| \frac{z_{k_1}}{\|z_{k_1}\|} \right\|_{\varphi_m}^2 < \frac{\varepsilon_1^4 \theta_1^2}{4^2\ 3^4\ 8^2},\ (m \in N_1)$$

and

$$|\varphi_{\sigma(1)}(\frac{z_{k_1}}{\|z_{k_1}\|})| \geq |\varphi_{\sigma(1)}(z_{k_1})| > \frac{\theta_1}{4}.$$

From Lemma 3.4 there exists a norm-one element $a_1 \in E$ and a compact tripotent $u_1 \in E^{**}$ such that

$$a_1 \leq u_1,$$

$$|\varphi_{\sigma(1)}(a_1)| > \frac{3\theta_1}{4^2}$$

and

$$\|u_1\|_{\varphi_m} < \frac{\varepsilon_1^2}{3^2},$$

for all $m \in N_1$. Now Lemma 3.2 assures that for each $m \in N_1$, $x \in E^{**}$ we have

$$|\varphi_m P_1(u_1)(x)| < 2\varepsilon_1\ \|x\|,$$

and

$$|\varphi_m P_2(u_1)(x)| < \varepsilon_1\ \|x\|.$$

In particular

$$\|\varphi_m(I - P_0(u_1))\| \leq 3\varepsilon_1,\ \forall m \in N_1. \tag{10}$$

The inequalities (6) and (10) show that, denoting $E_1 = E_0^{**}(u_1) \cap E$, we have

$$\|\varphi_k - \varphi_k|_{E_1}\| \leq 3\varepsilon_1,$$

$$\left\| \frac{\varphi_k}{\|\varphi_k\|} - \frac{\varphi_k|_{E_1}}{\|\varphi_k|_{E_1}\|} \right\| \leq \frac{1}{\|\varphi_k\|} \|\varphi_k - \varphi_k|_{E_1}\| + \left| \frac{1}{\|\varphi_k\|} - \frac{1}{\|\varphi_k|_{E_1}\|} \right| \|\varphi_k|_{E_1}\|$$

$$\leq \frac{2}{\|\varphi_k\|} \|\varphi_k - \varphi_k|_{E_1}\| \leq \frac{6\ \varepsilon_1}{\|\varphi_k\|} \leq \frac{6\ \varepsilon_1}{\inf_{k\in N_1} \|\varphi_k\|} = \frac{6\ \varepsilon_1}{A}$$

(we also have $A - 3\varepsilon < A - 3\varepsilon_1 = \inf_{k\in\mathbb{N}} \|\varphi_k\| - 3\varepsilon_1 \leq \inf_{k\in\mathbb{N}} \|\varphi_k|_{E_1}\|$), and hence

$$(r - \frac{6}{A - 3\varepsilon}\ \varepsilon_1) \sum_{k\in N_1} |\alpha_k| \leq (r - \frac{6}{A}\ \varepsilon_1) \sum_{k\in N_1} |\alpha_k|$$

$$\leq \left\| \sum_{k\in N_1} \alpha_k \frac{\varphi_k|_{E_1}}{\|\varphi_k|_{E_1}\|} \right\| \leq \sum_{k\in N_1} |\alpha_k|.$$

We note that, since u_1 is a compact tripotent relative to E, then u_1 is closed relative to E. Thus E_1 is a strong*-dense subtriple of $E_0^{**}(u_1)$. We repeat the same argument above in the JB*-triple E_1 with the family $(\varphi_k|_{E_1})_{k\in N_1}$ and $\theta_2 = \frac{r}{2}(A-3\varepsilon) - 3\varepsilon_1$.

Suppose, by mathematical induction, that we have found $\mathbb{N} = N_0 \supset N_1 \supset N_2 \supset \ldots \supset N_n$ infinite subsets of $\mathbb{N}$, $u_1, \ldots, u_n$ mutually orthogonal compact (and hence closed) tripotents in E^{**}, $a_1, \ldots, a_n \in E$, and $\sigma(1) < \ldots < \sigma(n)$ in $\mathbb{N}$ such that $u_1 + \ldots + u_n$ is closed, $a_i \leq u_i$, $\sigma(k) \in N_{k-1}$,

$$|\varphi_{\sigma(k)}(a_k)| > \frac{3\theta_k}{4^2} = \frac{3}{4^2}\left(\frac{r}{2}(A-3\varepsilon) - 3\sum_{i=1}^{k-1}\varepsilon_i\right), \quad (k \in \{2, \ldots, n\}),$$

$$(r - \frac{6}{A-3\varepsilon}\sum_{k=1}^{n}\varepsilon_k) \sum_{k\in N_n} |\alpha_k| \leq \left\| \sum_{k\in N_n} \alpha_k \frac{\varphi_k|_{E_n}}{\|\varphi_k|_{E_n}\|} \right\| \leq \sum_{k\in N_n} |\alpha_k|, \qquad (11)$$

where $E_n = E \cap E_0^{**}(u_1 + u_2 + \ldots + u_n)$ and $A - 3\sum_{k=1}^{n}\varepsilon_k \leq \inf_{k\in\mathbb{N}} \|\varphi_k|_{E_n}\|$.

Let $u = u_1 + \ldots + u_n$. From the inequality (11) and Proposition 3.1 we conclude that taking $\theta_{n+1} = \frac{r}{2}(A-3\varepsilon) - 3\sum_{i=1}^{n}\varepsilon_i$, then there exists a w.u.c. series $\sum z_n$ in E_n and a subsequence $(\varphi_{\tau(n)})$ such that $\|z_n\| < 1$ and $|\varphi_{\tau(n)}(z_n)| > \frac{\theta_{n+1}}{4}$. Since $\sum z_n$ is a w.u.c. series, then there exists $C_{n+1} > 0$, such that $\|\sum_{n\in F}\sigma_n z_n\| \leq C_{n+1}$ for every finite set $F \subset \mathbb{N}$ and $\sigma_n = \pm 1$.

Let $j_{n+1} \in N_n$ such that $\frac{C_{n+1}^2}{j_{n+1}} < \frac{\varepsilon_{n+1}^4\ \theta_{n+1}^4}{4^4\ 3^4\ 8^2}$. Let $m \in N_n$. Since every Hilbert space is of cotype 2 we have

$$\sum_{k=1}^{j_{n+1}} \frac{1}{j_{n+1}} \|z_k\|_{\varphi_m}^2 \leq \frac{1}{j_{n+1}} \int_D \left\| \sum_{k=1}^{j_{n+1}} \varepsilon_k z_k \right\|_{\varphi_m}^2 d\mu \leq \frac{C^2}{j_{n+1}} < \frac{\varepsilon_{n+1}^4\ \theta_{n+1}^4}{4^4\ 3^4\ 8^2}, \quad (12)$$

where $D = \{-1, 1\}^{\mathbb{N}}$ and μ is the uniform probability measure on D. Since (12) is satisfied for every $m \in N_n$, then there exist $k_{n+1} \in N_n$, $k_{n+1} \leq j_{n+1}$ and an infinite subset $N_{n+1} \subset N_n$ such that for every $m \in N_{n+1}$,

$$\|z_{k_{n+1}}\|_{\varphi_m}^2 < \frac{\varepsilon_{n+1}^4\ \theta_{n+1}^4}{4^4\ 3^4\ 8^2}.$$

Let $\varphi_{\sigma(n+1)} = \varphi_{k_{n+1}}$. Then, since $\|z_{k_{n+1}}\| \geq \frac{\theta_{n+1}}{4}$, we have

$$\left\| \frac{z_{k_{n+1}}}{\|z_{k_{n+1}}\|} \right\|^2_{\varphi_m} < \frac{\varepsilon^4_{n+1}\theta^2_{n+1}}{4^2\ 3^4\ 8^2} \quad (m \in N_{n+1})$$

and

$$\left| \varphi_{\sigma(n+1)} \left(\frac{z_{k_{n+1}}}{\|z_{k_{n+1}}\|} \right) \right| \geq |\varphi_{\sigma(n+1)}(z_{k_1})| > \frac{\theta_{n+1}}{4}.$$

From Lemma 3.4 and Corollary 2.9, there exists a norm-one element $a_{n+1} \in E_n$ and a compact tripotent $u_{n+1} \in E_0^{**}(u)$ such that

$$a_{n+1} \leq u_{n+1},$$

$$|\varphi_{\sigma(n+1)}(a_{n+1})| > \frac{3\theta_{n+1}}{4^2}$$

and

$$\|u_{n+1}\|_{\varphi_m} < \frac{\varepsilon^2_{n+1}}{3^2},$$

for all $m \in N_{n+1}$. From Corollary 2.8 we have $u + u_{n+1}$ closed in E^{**}. Now, Lemma 3.2 assures that for each $m \in N_{n+1}$, $x \in E^{**}$ we have

$$|\varphi_m P_1(u_{n+1})(x)| < 2\varepsilon_{n+1}\ \|x\|,$$

and

$$|\varphi_m P_2(u_{n+1})(x)| < \varepsilon_{n+1}\ \|x\|.$$

In particular

$$\|\varphi_m(I - P_0(u_{n+1}))\| \leq 3\varepsilon_{n+1},\ \forall m \in N_{n+1}. \tag{13}$$

Therefore,

$$\left\| \frac{\varphi_m}{\|\varphi_m\|} - \frac{\varphi_m|_{E_{n+1}}}{\|\varphi_m|_{E_{n+1}}\|} \right\| \leq \frac{6\ \varepsilon_{n+1}}{\|\varphi_m\|} \leq \frac{6\ \varepsilon_{n+1}}{\inf_{m \in N_{n+1}} \|\varphi_m\|} = \frac{6\ \varepsilon_{n+1}}{A}$$

(we also have $A - 3\varepsilon < A - 3\sum_{k=1}^{n+1} \varepsilon_k = \inf_{k \in \mathbb{N}} \|\varphi_k\| - 3\varepsilon_{n+1} \leq \inf_{k \in \mathbb{N}} \|\varphi_k|_{E_{n+1}}\|$), and hence

$$(r - \frac{6}{A - 3\varepsilon} \sum_{k=1}^{n+1} \varepsilon_k) \sum_{k \in N_1} |\alpha_k| \leq (r - \frac{6}{A} \sum_{k=1}^{n+1} \varepsilon_1) \sum_{k \in N_1} |\alpha_k|$$

$$\leq \left\| \sum_{k \in N_1} \alpha_k \frac{\varphi_k|_{E_1}}{\|\varphi_k|_{E_1}\|} \right\| \leq \sum_{k \in N_1} |\alpha_k|.$$

Finally, we can take θ any strictly positive real number smaller or equal than $\frac{3}{4^2}\left(\frac{r}{2}(A-3\varepsilon)-3\varepsilon\right)>0$.

The implications $(b)\Rightarrow(c)$ and $(c)\Rightarrow(a)$ are obvious. □

4 Applications: A Theorem of Dieudonne for JC*-triples

Let A be a C*-algebra and let (ϕ_n) be a sequence in A^*. It is known that ϕ_n needs not be weakly convergent in A^* even under the hypothesis that, for each $a \in A$, $(\phi_n(a))$ is a convergent sequence. In a recent paper, J. K. Brooks, K. Saitô and J. D. M. Wright have obtained the following generalisation of a classical theorem of Dieudonne: if $\phi_n(p)$ converges whenever p is a range projection in A^{**}, then ϕ_n is weakly convergent in A^*. Their proof is strongly based on the characterisation of weak compactness in the dual of a C*-algebra obtained by Pfitzner in [29] and the Saitô-Tomita-Lusin Theorem for C*-algebras [26, 2.7.3].

This section is devoted to obtain a generalisation of this Theorem of Dieudonne to the more general setting of JC*-triples, in which the characterisation of weak compactness developed in Theorem 3.5 and the corresponding Lusin's Theorem for JB*-triples (c.f. [11]), will play an important role.

The following proposition generalizes [9, Proposition 3.1] to the setting of JB*-triples. We recall first some necessary results. Let W be a JBW*-triple with predual W_*. From [8, Proposition 3.4] it follows that W_* is an L-summand in its bidual W^*, that is, there exists a linear projection π on W^* satisfying $\|x\| = \|\pi(x)\| + \|x-\pi(x)\|$. It follows from [20, Theorem IV.2.2] that W_* is weakly sequentially complete.

Proposition 4.1. *Let W be a JBW*-triple, E a weak*-dense JB*-subtriple of W and (ϕ_n) a sequence in W_* such that, for each a in E, $(\phi_n(a))$ is convergent sequence. Then the following assertions are equivalent:*

(a) *The set $\{\phi_n : n \in \mathbb{N}\}$ is relatively weakly compact in W_*.*

(b) *For each $\alpha \in W$, $\lim \phi_n(\alpha)$ exists.*

(c) *There exists $\phi \in W_*$ such that $\lim \phi_n(\alpha) = \phi(\alpha)$, for each $\alpha \in W$.*

Proof. (a)⇒(b) Let us assume that the set $\{\phi_n : n \in \mathbb{N}\}$ is relatively weakly compact in W_*. Let $\alpha \in W$. We shall show that $(\phi_n(\alpha))$ is a Cauchy sequence. To this end, let us consider $\epsilon > 0$. We may assume, without loosing generality, that $\|\alpha\| < 1/3$.

By [27, Theorem 1.1], there exist norm-one elements $\varphi_1, \varphi_2 \in W_*$ with the following property: Given $\varepsilon/3 = \eta > 0$, there exists $\delta > 0$ such that for every $z \in W$ with $\|z\| \leq 1$ and $\|z\|_{\varphi_1,\varphi_2} < \delta$, we have

$$|\phi_n(z)| < \eta = \varepsilon/3 \tag{14}$$

for each $n \in \mathbb{N}$. Let $N := \{z \in W : \|z\|_{\varphi_1,\varphi_2} = 0\}$. The completion, H, of $(W/N, \|\cdot\|_{\varphi_1,\varphi_2})$ is a Hilbert space, and the natural projection of W onto H is a weak*-continuous linear operator, which will be denoted by $J : W \to H$. Moreover, $\|J\| \leq \sqrt{2}$. By [28, Theorem 2], there exists a norm-one functional $\psi \in W_*$, such that

$$\|z\|_{\varphi_1,\varphi_2} \leq 2\ \|z\|_\psi + \delta/2\ \|z\|, \tag{15}$$

for all $z \in W$.

The result in [11, Theorem 2.9] remains valid when the bidual of E is replaced with a JBW*-triple W such that E is weak*-dense subtriple of W. Thus, denoting u for the support tripotent of ψ in W, then by [11, Theorem 2.9], there exist a tripotent $e \leq u$ in W and $a \in E$, satisfying that $\|a\| < 3/2\ \|\alpha\| < 1/2$ and

$$P_i(e)(a - \alpha) = 0,\ \ (i \in \{1, 2\})$$

$$|\psi(u - e)| < \delta^2/8 \tag{16}$$

Since $e \leq u$ and $a - \alpha = P_0(e)(a - \alpha)$, we deduce, by Peirce arithmetic and [17, Lemma 1.5], that $\{a - \alpha, a - \alpha, u\} = \{a - \alpha, a - \alpha, u - e\}$ is a positive element in the JBW*-algebra $W_2(u - e)$. Moreover, having in mind that ψ is a positive functional on $W_2(u - e)$, $|\psi(u - e)| < \delta^2/8$, and $\|\{a - \alpha, a - \alpha, u - e\}\| \leq \|a - \alpha\|^2\ \|u - e\| < 1$, we get

$$\|a - \alpha\|_\psi^2 = \psi\{a - \alpha, a - \alpha, u - e\} \leq \psi(u - e) < \delta^2/8.$$

The above inequality together with (15) give that $\|a - \alpha\|_{\varphi_1,\varphi_2} < \delta$, and hence by (14) we deduce that

$$|\phi_n(a - \alpha)| < \varepsilon/3,\ \ \ (n \in \mathbb{N}). \tag{17}$$

Now, since by hypothesis, $(\phi_n(a))$ is a Cauchy sequence, there exists $m_0 \in \mathbb{N}$ such that for each $n, m \geq m_0$ we have

$$|(\phi_n - \phi_m)(a)| < \varepsilon/3. \tag{18}$$

Finally, from (17) and (18) it follows that for each $n, m \geq m_0$ we have

$$|\phi_n(\alpha) - \phi_m(\alpha)| \leq |\phi_n(\alpha - a)| + |(\phi_n - \phi_m)(a)| + |\phi_m(\alpha - a)| < \varepsilon.$$

As we have commented in the introduction of this section, the predual of every JBW*-triple is weakly sequentially complete. Therefore, the implication (b)⇒(c) follows straightforwardly.

Finally, the implication (c)⇒(a) follows from the Eberlein-Šmul'jan Theorem.

□

We can establish now a Dieudonne type Theorem in the setting of JC*-triples.

Theorem 4.2. *Let (ϕ_n) be a sequence in the dual of a JC*-triple E such that, for every range tripotent r in E^{**} (i.e. $r = r(a)$, for some a in E with $\|a\| = 1$), we have $\lim \phi_n(r)$ exits. Then there exits ϕ in E^* satisfying that (ϕ_n) converges weakly to ϕ.*

Proof. Let C_0 be a separable abelian JB*-subtriple of E satisfying that C_0 is JB*-triple isomorphic to an abelian C*-álgebra. Let x be a positive norm-one element in C_0. When C_0 is regarded as a C*-álgebra the range projection of x, $RP(x) \in C_0^{**}$, coincide with the $\sigma(C_0^{**}, C_0^*)$-limit of the sequence $x^{1/n}$. When we consider the JB*-triple structure in C_0, then the range tripotent of x, $r(x) \in C_0^{**}$, coincide with the $\sigma(C_0^{**}, C_0^*)$-limit of the sequence $x^{1/3^n}$. In particular $RP(x) = r(x)$ in C_0^{**}. This gives that $RP(x) = r(x)$ is a range tripotent in E^{**} (compare [6, Theorem 4]), and hence $\phi_n(RP(x))$ converges by hypothesis.

We have actually proved that, whenever C_0 is a separable abelian JB*-subtriple of E satisfying that C_0 is JB*-triple isomorphic to an abelian C*-álgebra, then $(\phi_n(p))$ converges for every range projection $p \in C_0^{**}$. Now, Theorem 3.2 in [9] assures that $\phi_n|C_0$ is weakly convergent in C_0^* (in particular, $\{\phi_n|_{C_0} : n \in \mathbb{N}\}$ is a relatively weakly compact subset in C_0^*). Now, Theorem 3.5 gives that $\{\phi_n : n \in \mathbb{N}\}$ is a relatively weakly compact subset in E^*.

Finally, Proposition 4.1 will give the desired statement provided we can assure that $\phi_n(a)$ converges for every $a \in E$. Since for every norm-one element $b \in E$, the (closed) JB*-subtriple of E generated by b is isometrically isomorphic to a separable abelian C*-álgebra, we conclude from the preceding paragraphs that $\phi_n(b)$ converges. □

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