A Saitô-Tomita-Lusin Theorem for JB*-triples and Applications

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Abstract

A Lusin's theorem is proved in the non-ordered context of JB*-triples. This is applied to obtain versions of a general transitivity theorem and to deduce refinements of facial structure in closed unit balls of JB*-triples and duals.

1 Introduction and Preliminaries

If the open unit ball D of a complex Banach space E is a bounded symmetric domain the holomorphy of D determines the geometry of E and induces a ternary algebraic structure upon it. Banach spaces of this kind are known as JB*-triples [13]. The norm closed subspaces E of C*-algebras for which xx^*x lies in E whenever x does form a large class of examples of JB*-triples that, up to linear isometry, includes all Hilbert spaces, spin factors and many other familiar operator spaces. If $\mathbb{O}$ denotes the complex Cayley numbers then the space of all 1 by 2 matrices over $\mathbb{O}$, $M_{1,2}(\mathbb{O})$, appropriately normed, is an example of a JB*-triple not of this form. Despite a general lack of order and other constraints the ternary structure in JB*-triples, which generalises the binary structure in C*-algebras, has been shown to be a natural medium in diverse disciplines such as complex holomorphy, convexity and quantum mechanics [6, 8, 13].

Non-commutative versions of Egoroff's and Lusin's theorems for C*-algebras [17, 19] are extended in this paper to the non-ordered context of JB*-triples. A transitivity theorem for an arbitrary JB*-triple E, in the sense that the "D - operator" associated with a finite rank tripotent of E^{**}

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has coincident image upon E and E^{**}, is obtained as one application and consequences for facial structure discussed. In particular, it is deduced that the norm exposed faces of E_1^* associated with finite rank tripotents in E^{**} are weak* exposed whenever E is separable and that, in general, the norm semi-exposed faces of E_1 are intersections of maximal norm closed faces.

We recall [13] that a JB*-triple is a complex Banach space together with a continuous triple product $\{.,.,.\} : E^3 \to E$ which is conjugate linear in the middle variable and symmetric bilinear in the outer variables such that, for the operator $D(a,b)$ given by $D(a,b)x = \{a,b,x\}$, we have

$$D(a,b)D(x,y) - D(x,y)D(a,b) = D(D(a,b)x,y) - D(x,D(b,a)y);$$

and that $D(a,a)$ is an hermitian operator with non-negative spectrum and $\|D(a,a)\| = \|a\|^2$.

Every C*-algebra is a JB*-triple via the triple product given by

$$2\{x,y,z\} = xy^*z + zy^*x,$$

and every JB*-algebra is a JB*-triple under the triple product

$$\{x,y,z\} = (x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^*.$$

A JBW*-triple is a JB*-triple with (unique) [3] predual. The second dual of a JB*-triple is a JBW*-triple [5]. Elements a, b in a JB*-triple E are *orthogonal* if $D(a,b) = 0$. With each tripotent u (i.e. $u = \{u,u,u\}$) in E is associated the *Peirce decomposition*

$$E = E_2(u) \oplus E_1(u) \oplus E_0(u),$$

where for $i = 0,1,2$ $E_i(u)$ is the $\frac{i}{2}$ eigenspace of $D(u,u)$. The Peirce rules are that $\{E_i(u), E_j(u), E_k(u)\}$ is contained in $E_{i-j+k}(u)$ if $i-j+k \in \{0,1,2\}$ and is zero otherwise. In addition,

$$\{E_2(u), E_0(u), E\} = \{E_0(u), E_2(u), E\} = 0.$$

The corresponding *Peirce projections*, $P_i(u) : E \to E_i(u)$, $(i = 0,1,2)$ are contractive and satisfy

$$P_2(u) = D(2D - I), \; P_1(u) = 4D(I - D), \; \text{ and } P_0(u) = (I - D)(I - 2D),$$

where D is the operator $D(u,u)$ (compare [9]).

Let E be a JBW*-triple with tripotent u. The Peirce space $E_2(u)$ is a JBW*-algebra with Jordan product and involution given by $a \circ b = \{a, u, b\}$, $a^\sharp = \{u, a, u\}$. Order amongst tripotents in E arises as follows. Tripotents u and v satisfy $v \leq u$ if and only if v is a projection in the JBW*-algebra $E_2(u)$. See [6], [14, §5] for several characterisations. A development [7] (see also [6], [9]) of triple functional calculus [13] is that if x is norm-one element of E there is a least tripotent $r(x)$ of E such that x belongs to the positive part of $E_2(r(x))$; the tripotent that arises as the greatest projection in $E_2(r(x))_+$ majorised by x is denoted by $u(x)$. In particular, in $E_2(r(x))_+$ we have

$$u(x) \leq x \leq r(x).$$

A non-zero tripotent u in E is said to be minimal if $E_2(u) = \mathbb{C}u$, and is said to have finite rank if it is an orthogonal sum of finitely many minimal tripotents. Given a convex set K we denote by $\partial_e(K)$ its set of extreme points. If E is a JB*-triple, for each $\rho \in \partial_e(E_1^*)$ there is a unique minimal tripotent, $u(\rho)$, of E^{**} such that $\rho(u(\rho)) = 1$, and all minimal tripotents arise in this way [9].

Given a JBW*-triple M, a norm-one element φ of M_* and a norm-one element z in M such that $\varphi(z) = 1$, it follows from [1, Proposition 1.2] that the assignment

$$(x, y) \mapsto \varphi\{x, y, z\}$$

defines a positive sesquilinear form on M, the values of which are independent of choice of z, and induces a prehilbert seminorm on M given by

$$\|x\|_\varphi := (\varphi\{x, x, z\})^{\frac{1}{2}}.$$

As φ ranges over the unit sphere of M_* the topology induced by these seminorms is termed the strong*-topology of M. The strong*-topology was introduced in [2], and further developed in [16, 15]. In particular [16], the triple product is jointly strong*-continuous on bounded sets.

2 Saitô-Tomita-Lusin Theorem

The classical Lusin's theorem states that if μ is a Radon measure on a locally compact Hausdorff space T and if f is a complex-valued measurable function on T such that there exists a Borel set $A \subseteq T$ with $\mu(A) < \infty$ and $f(x) = 0$ for all $x \notin A$, then for each $\varepsilon > 0$ there exists a Borel set $E \subseteq T$ with $\mu(T \backslash E) < \varepsilon$ and a function $g \in C_0(T)$ such that f and g coincide on

E. A non-commutative analogue of Lusin's theorem for general C*-algebras was given in [19] and considerably developed subsequently in [17]. The underlying strategy in our approach to non-ordered JB*-triple extensions is to release and exploit local order structures harboured by Peirce 1- spaces and Peirce-2 spaces. Our initial aim is to derive a novel inequality (see Proposition 2.4) involving D-operators and then to employ it as a controlling device thereafter.

The following result is proved in [9, Lemma 1.5] and remark prior to it.

Lemma 2.1. *Let u be a tripotent in a JB*-triple E and let $x \in E_1(u) \cup E_2(u)$. Then $D(x,x)u$ is a positive element in the JB*-algebra $E_2(u)$.*

Lemma 2.2. *Let e be a projection in a JB*-algebra E and let $a \in E_1(e) \cup E_2(e)$ where $a = a^*$. Then $a^2 \circ e \geq 0$ and $\|a\|^2 = \|D(a,a)e\|$.*

Proof. We have $D(a,a)e = a^2 \circ e$. We may suppose without loss that E has an identity element, 1. If $a \in E_2(e)$, then $a^2 = a^2 \circ e$. Let $a \in E_1(e)$. By [20], the JB-subalgebra of E_{sa} generated by $1, e$ and a can be realised as a JC-subalgebra of the self-adjoint part of a C*-algebra B so that, in B, we have $a = 2D(e,e)a = ea + ae$ and therefore $a^2 = ea^2 + aea$, giving $ea^2 = a(1-e)a = a^2e$. Consequently,

$$\|ea^2\| = \|(1-e)a^2(1-e)\| = \|(1-e)a^2\| = \|(1-e)a\|^2$$

and therefore

$$\|a\|^2 = \max\{\|ea^2\|, \|(1-e)a^2\|\} = \|ea^2\| = \|D(a,a)e\|.$$

□

Lemma 2.3. *Let E be a type I von Neumann factor or a finite dimensional simple JB*-algebra. Let u be a tripotent in E. Then there exists a triple embedding $\pi : E \to E$ such that $\pi(u)$ is a projection.*

Proof. In the first case we may suppose $E = B(H)$ for some complex Hilbert space H [18, V.1.28] and that u is a partial isometry. From [12, Lemma 3.12] it follows that there exists a complete tripotent $c \in B(H)$ such that $c \geq u$. Since c is complete in $B(H)$ we have $(1-cc^*)B(H)(1-c^*c) = 0$ and hence $cc^* = 1$ or $c^*c = 1$. We may assume that $cc^* = 1$. Denoting $q = c^*c$, it follows that $c : q(H) \to H$ is a surjective linear isometry. Thus, the mapping $\pi : B(H) \to B(H, q(H)) \subseteq B(H)$ defined by $\pi(x) = c^*x$ is a surjective linear isometry from $B(H)$ onto $B(H, q(H))$ and hence a triple isomorphism. It is clear that $\pi(c) = q$ is a projection in $B(H)$. Moreover, since π is a triple isomorphism and $c \geq u$ we have $\pi(u) \leq \pi(c) = q$, and hence $\pi(u)$ is a projection in $B(H)$.

The second case follows from [14, Corollary 5.12]. □

Proposition 2.4. *Let u be a tripotent in a JB*-triple E and let $x \in E_1(u) \cup E_2(u)$. Then $\|x\|^2 \leq 4\ \|D(x,x)u\|$.*

Proof. By [10, Corollary 1] we may suppose that E is a JB*-subtriple of an ℓ_∞-sum, $M \oplus N$, where M is a type I von Neumann factor and N is an ℓ_∞-sum of finite dimensional simple JB*-algebras. Letting F denote the JB*-algebra $M \oplus N$, it follows from Lemma 2.3 that there is a triple embedding, $\pi : F \to F$, such that $\pi(u)$ is a projection. Thus, since $E_i(u) \subseteq F_i(u)$ for $i = 1, 2$ and π preserves the triple product we may assume without loss that E is a JB*-algebra and that u is a projection in E.

In which case, we have $x = a + ib$ where a and b are self-adjoint elements of E. Let $x \in E_1(u)$. Then a, b and x^* lie in $E_1(u)$. Via the equalities $2u \circ x = x$ and $2u \circ x^* = x^*$ and Lemma 2.2 we have

$$D(x,x)u = (x \circ x^*) \circ u = (a^2 + b^2) \circ u \geq a^2 \circ u, b^2 \circ u \geq 0$$

so that

$$4\ \|D(x,x)u\| \geq 4\ \max\{\|a\|^2, \|b\|^2\} \geq (\|a\| + \|b\|)^2 \geq \|x\|^2.$$

If x lies in $E_2(u)$ the assertion is verified by a similar (easier) argument. □

The following observations illustrate the geometric nature of the inequality in Proposition 2.4. Given a tripotent u in a JB*-triple E the weak*-closed face of E_1^*,

$$F_u = \{\varphi \in E^* : \varphi(u) = \|\varphi\| = 1\}$$

identifies with the state space of the JB*-algebra $E_2(u)$. Let $x \in E_1(u)$. Since $D(x,x)u \in (E_2(u))_+$ we therefore have that

$$\|D(x,x)u\| = \sup\{\varphi(D(x,x)u) : \varphi \in F_u\}.$$

Thus, letting $\|x\|_u$ denote $\|D(x,x)u\|^{\frac{1}{2}}$ we have that

$$\|x\|_u = \sup\left\{\|x\|_\varphi : \varphi \in F_u\right\},$$

the seminorms $\|x\|_\varphi$ being as defined in the introduction. Further, $\|x\|_u = 0$ implies $x = 0$. Thus, $\|.\|_u$ is a norm on $E_1(u)$ satisfying

$$\|.\|_u \leq \|.\| \leq 2\|.\|_u,$$

the second inequality being given by Proposition 2.4. In particular, we record the following.

Corollary 2.5. *If u is a tripotent in a JB*-triple E then $\|.\|_u$ and $\|.\|$ are equivalent norms on $E_1(u)$.* □

Lemma 2.6. *Let u be a tripotent in a JB*-triple E, let $x \in E$ and let $x_j = P_j(u)(x)$ for $j = 1, 2$. Then $P_2(u)D(x,x)u \geq 0$ in $E_2(u)$ and*

$$\|D(x_j, x_j)u\| \leq \|P_2(u)D(x,x)u\| \quad \textit{for } j = 1, 2.$$

Proof. Using the Peirce rules calculation gives

$$D(x_1, x_1)u + D(x_2, x_2)u = P_2(u)D(x,x)u.$$

Thus, by Lemma 2.1, in $E_2(u)$ we have

$$0 \leq D(x_j, x_j)u \leq P_2(u)D(x,x)u \quad \text{for } j = 1, 2,$$

from which the assertion follows. □

For subsequent purposes we remark that for tripotents u, v in a JB*-triple with $v \leq u$ we have

$$P_i(u)P_j(v) = P_j(v)P_i(u) \quad \text{for } i, j = 1, 2;$$

$$P_1(v)P_0(u) = P_2(v)P_1(u) = 0 \text{ and } P_2(v)P_2(u) = P_2(v).$$

In particular,

$$P_i(v)\left(P_2(u) + P_1(u)\right) = P_i(v) \;\; \text{for } i = 1, 2.$$

In addition,

$$P_2(u) + P_1(u) = 3D(u,u) - 2D(u,u)^2 \;\; \text{and } 2P_2(u) + P_1(u) = 2D(u,u).$$

We employ the geometric inequality obtained in Proposition 2.4 as a key tool in arguments below that culminate with a non-ordered Lusin's theorem for JB*-triples. The general process owes much to the scheme and ideas of Saitô [17]. We begin with a Saitô-Egoroff theorem for JB*-triples.

Theorem 2.7. *Let x belong to the strong*-closure of a bounded subset X of E^{**}, where E is a JB*-triple. Let u be a tripotent in E^{**}, let $\varphi \in E^*$ and let $\varepsilon > 0$. Then there exist a sequence (x_n) in X and a tripotent v in E^{**} such that $v \leq u$, $|\varphi(u-v)| < \varepsilon$, and $\|S(x_n - x)\| \to 0$, for $S = P_2(v), P_1(v)$ and $D(v,v)$.*

Proof. Choose a net (x_λ) in X with strong*-limit x. We may suppose that X lies in the closed unit ball of E^{**}, that $\|\varphi\| = 1$ and (by translation) that $x = 0$.

For each λ let y_λ denote $P_2(u)D(x_\lambda, x_\lambda)u$. By the joint strong*-continuity of the triple product on bounded sets (y_λ) is strong*-null. By Lemma 2.6 each y_λ is a positive element of the JBW*-algebra $E_2^{**}(u)$, and $\|y_\lambda\| \leq 1$. In particular, for each λ, the JBW*-subtriple of E^{**} generated by y_λ and u is an abelian W*-subalgebra of the JBW*-algebra $E_2^{**}(u)$. For each λ, let u_λ denote $\chi(y_\lambda)$, where χ is the characteristic function of the interval $(-2^{-4}, 2^{-4})$. Then u_λ is a projection in $E_2^{**}(u)$ satisfying

$$2^4 y_\lambda \geq u - u_\lambda \geq 0,$$

for all λ. Since (y_λ) is strong*-null and hence weak*-null, $u - u_\lambda$ must be weak*-null in (the JBW*-algebra) $E_2^{**}(u)$ and thus weak*-null in E^{**}.

Choose λ_1, such that $|\varphi(u - u_{\lambda_1})| < 2^{-1}\,\varepsilon$. Denote u_{λ_1}, x_{λ_1} and y_{λ_1} by u_1, x_1 and y_1, respectively. Using Proposition 2.4 in the second inequality below and Lemma 2.6 in the third, we have

$$\|P_2(u_1)x_1 + P_1(u_1)x_1\|^2 \leq 2\left(\|P_2(u_1)x_1\|^2 + \|P_1(u_1)x_1\|^2\right)$$

$$\leq 2^3\ \left(\|D(P_2(u_1)x_1, P_2(u_1)x_1)u_1\| + \|D(P_1(u_1)x_1, P_1(u_1)x_1)u_1\|\right)$$

$$\leq 2^4 \|P_2(u_1)D(x_1,x_1)u\| \leq 1.$$

Duplicating the above argument for $\lambda \geq \lambda_1$ where this time χ represents the characteristic function of the interval $(-2^{-6}, 2^{-6})$ we obtain a $\lambda_2 \geq \lambda_1$ and elements u_2, x_2 and y_2, respectively denoting u_{λ_2}, x_{λ_2} and y_{λ_2}, satisfying

$$u_2 \leq u_1, \quad |\varphi(u_1 - u_2)| < 2^{-2}\varepsilon \quad \text{and } \|P_2(u_2)x_2 + P_1(u_2)x_2\| < 2^{-1}.$$

Proceeding inductively we obtain a decreasing sequence (u_n) of projections in the JBW*-algebra $E_2^{**}(u)$ and a sequence (x_n) in X such that, with $u = u_0$,

$$|\varphi(u_{n-1} - u_n)| < 2^{-n}\varepsilon \quad \text{and } \|P_2(u_n)x_n + P_1(u_n)x_n\| \leq n^{-1}, \quad \text{for all } n \geq 1.$$

The sequence (u_n) decreases in the weak*-topology to a projection v of $E_2^{**}(u)$ giving $u - v = \sum_1^{+\infty}(u_{n-1} - u_n)$ and so $|\varphi(u-v)| \leq \varepsilon$. Further, for each n,

$$\|P_i(v)(x_n)\| = \|P_i(v)\left(P_2(u_n)x_n + P_1(u_n)x_n\right)\| \leq n^{-1},$$

for $i = 1, 2$. The remaining assertions follow. $\square$

Corollary 2.8. *Let u be a tripotent in E^{**}, where E is a JB*-triple. Let $x \in E^{**}$ and $\varphi \in E^*$. Let $\varepsilon > 0$ and $\delta > 0$. Then there exists $y \in E$ and a tripotent v in E^{**} such that $v \leq u$, $|\varphi(u-v)| < \varepsilon$, $\|P_i(v)(x-y)\| < \delta$ for $i = 1, 2$ and $\|y\| \leq \|\left(P_2(u) + P_1(u)\right)(x)\|$.*

Proof. Since, by [2, Corollary 3.3], the closed unit ball E_1 of E is strong*-dense in the closed unit ball of E^{**}, the assertions follow from replacing x and X in Theorem 2.7 with $\left(P_2(u) + P_1(u)\right)(x)$ and $\|\left(P_2(u) + P_1(u)\right)(x)\|\ E_1$, respectively. $\square$

A Lusin's theorem for JB*-triples is proved next.

Theorem 2.9. *Let E be a JB*-triple, let $\varphi \in E^*$ and let $x \in E^{**}$. Let u be a tripotent in E^{**} and let $\varepsilon > 0$ and $\delta > 0$. Then there is an element $y \in E$ and a tripotent $v \in E^{**}$ such that $v \leq u$, $|\varphi(u-v)| < \varepsilon$, $S(x-y) = 0$ for $S = P_2(v), P_1(v)$ and $D(v,v)$, and $\|y\| \leq (1+\delta)\ \|\left(P_2(u) + P_1(u)\right)(x)\|$.*

Proof. We may assume without loss that $\|\left(P_2(u) + P_1(u)\right)(x)\| = 1$. By Corollary 2.8, there is an element y_1 in E and a tripotent $u_1 \leq u$ in E^{**} satisfying

$$|\varphi(u-v)| < 2^{-1}\ \varepsilon, \quad \|P_i(u_1)(x-y_1)\| < 2^{-2}\ \delta \ \text{ for } i = 1, 2$$

$$\text{and } \|y_1\| \leq \|\left(P_2(u) + P_1(u)\right)(x)\| = 1.$$

Replacing u with u_1 and x with $\left(P_2(u_1) + P_1(u_1)\right)(x - y_1)$ in Corollary 2.8 now gives an element y_2 in E and a tripotent u_2 in E^{**} such that $u_2 \leq u_1$ satisfying

$$|\varphi(u_1 - u_2)| < 2^{-2}\ \varepsilon,$$

$$\|P_i(u_2)(x - y_1 - y_2)\| = \|P_i(u_2)\left(P_2(u_1) + P_1(u_1)\right)(x - y_1 - y_2)\| < 2^{-3}\ \delta$$

for i=1,2 and

$$\|y_2\| \leq \|\left(P_2(u_1) + P_1(u_1)\right)(x - y_1)\| < 2^{-1}\ \delta.$$

Proceeding in this way gives rise to a sequence (y_n) in E and a decreasing sequence (u_n) of tripotents in $E_2^{**}(u)$, which, for $u_0 = u$, and all $n \geq 1$ satisfies

$$|\varphi(u_{n-1} - u_n)| < 2^{-n}\ \varepsilon,\ \|P_i(u_n)(x - \sum_{k=1}^{n} y_k)\| < 2^{-(n+1)}\ \delta \text{ for } i = 1, 2$$

$$\text{and } \|y_{n+1}\| \leq \|\ (P_2(u_n) + P_1(u_n))\ (x - \sum_{k=1}^{n} y_k)\| < 2^{-n}\ \delta.$$

Letting v denote the weak*-limit of (u_n) in $E_2^{**}(u)$, and $y = \sum_{n=1}^{+\infty} y_n$, we have that v is a tripotent in E^{**} with $v \leq u$ and $y \in E$ such that $|\varphi(u - v)| < \varepsilon$ and $\|y\| \leq 1 + \delta$.

Finally, for $i = 1, 2$

$$\|P_i(v)(x - \sum_{k=1}^{n} y_k)\| = \|P_i(v)\ \ (P_2(u_n) + P_1(u_n))\ (x - \sum_{k=1}^{n} y_k)\| < 2^{-n}\ \delta$$

for all $n \geq 1$, so that $P_i(v)(x - y) = 0$. In turn, this implies

$$D(v, v)(x - y) = 0.$$

□

3 Applications

In [6] Edwards and Rüttiman investigated facial structure of unit balls of a JBW*-triple and predual giving a complete description, and made significant inroads into corresponding general JB*-triple theory in the subsequent treatise [7]. In this section we exploit Theorem 2.9 to obtain versions of Kadison transitivity for JB*-triples (c.f. [18, II.4.18]) and use it to contribute observations on facial structure.

Let $x \in E$ where E is a JB*-triple. Let E_x and $E(x)$, respectively denote the JB*-subtriple and norm closed inner ideal of E generated by x. We have $E(x)^{**} = E_2^{**}(r(x))$ and, when the latter is realised as a JBW*-algebra, $E(x)$ is a JB*-algebra with $x \in E(x)_+$ and E_x is the abelian C*-algebra (i.e. associative JB*-algebra) of $E(x)$ generated by x, with corresponding spectrum $\sigma(x)$. To avoid posible confusion below, given a continuous real-valued function f on $\sigma(x) \cup \{0\}$ vanishing at 0, $f(x)$ shall have its usual meaning when E_x is regarded as an abelian C*-algebra and $f_t(x)$ shall denote the same element of E_x when the latter is regarded as a JB*-subtriple of E. Thus, for any real odd polynomial, $P(\lambda) = \sum_{k=0}^{n} \alpha_k \lambda^{2k+1}$, we have $P_t(x) = \sum_{k=0}^{n} \alpha_k D(x,x)^k(x)$.

We remark that if $\|x\| = 1$, then the tripotents $u(x)$ and $r(x)$ in E^{**} are projections in the abelian von Neumann algebra $(E_x)^{**}$, $r(x)$ being the identity element.

Lemma 3.1. *Let $x \in E$ and $u \in E^{**}$ where E is a JB*-triple and u is a tripotent such that $D(u,u)x = u$. Then there is an element $a \in E_x$ such that $\|a\| = 1$ and $D(u,u)a = u$. Moreover, $D(u,u)u(a) = D(u,u)r(a) = u$ and $u \leq u(a) \leq a \leq r(a)$ (in $E_2^{**}(r(a))$).*

Proof. We may suppose that u is non-zero and therefore that $\|x\| \geq 1$. Since $P_0(u)(x-u) = 0$, u and $x-u$ are orthogonal which implies

$$D(u,x) = D(u,u) = D(x,u).$$

By Peirce arithmetic $D(x,x)$ and $D(u,u)$ commute and, by induction, we have

$$D(u,u)D(x,x)^n x = u, \quad \text{for all } n \geq 0.$$

Thus,

$$D(u,u)P_t(x) = P(1)u,$$

for all real odd polinomials P. If f is a continuous real valued function on $[0, \|x\|]$ vanishing at 0 it follows that

$$D(u,u)f_t(x) = f(1)u.$$

Putting $a = f_t(x)$, where $f(\lambda) = \min\{\lambda, 1\}$, we have that $D(u,u)a = u$ and $\|a\| = 1$.

Further, since $(f_n)_t(a) \to r(a)$ and $(g_n)_t(a) \to u(a)$ in the weak*-topology, where $f_n(\lambda) = \lambda^{\frac{1}{2n+1}}$ and $g_n(\lambda) = \lambda^{2n+1}$ $(0 \leq \lambda \leq 1)$ we have

$$D(u,u)r(a) = u = D(u,u)u(a).$$

The final assertion follows from this and the above remarks. □

Lemma 3.2. *Let u be a tripotent in E^{**} where E is a JB*-triple. The sets $\{x \in E_1 : D(u,u)x = u\}$, $(u + E_0^{**}(u)) \cap E_1$ and $\{x \in E : u \leq x \leq r(x)\}$ coincide.*

Proof. The coincidence of the first two sets is evident from the fact that, for $x \in E^{**}$, $D(u,u)x = u$ if and only if $x - u \in \ker D(u,u) = E_0^{**}(u)$. The first set is contained in the third by Lemma 3.1. Conversely, given $x \in E$ with $u \leq x \leq r(x)$ we have, since $r(x) - x \geq 0$ in $E_2^{**}(r(x))$ and u is a projection there satisfying $\{u, r(x) - x, u\} = 0$, that u and $r(x) - x$ must be orthogonal so that

$$0 = D(u,u)(r(x) - x) = u - D(u,u)x.$$

□

We shall now prove a transitivity theorem for JB*-triples.

Theorem 3.3. *Let E be a JB*-triple and let $u_1, \dots, u_n$ be orthogonal minimal tripotents in E^{**} with sum u. Then*

(*a*) *$D(u,u)E = D(u,u)E^{**}$ and $P_j(u)E = P_j(u)E^{**}$ for $j = 1, 2$.*

(*b*) *There exists a in E such that $\|a\| = 1$ and $D(u,u)a = u$.*

(*c*) *There exist orthogonal elements, $a_1, \dots, a_n$ in E such that $D(u_i, u_i)a_i = u_i$ and $\|a_i\| = 1$ for $i = 1, \dots, n$.*

Proof. (*a*) The JBW*-algebra $E_2^{**}(u)$ is an ℓ_∞-sum of JBW*-algebras $M_1, \dots, M_k$ where each M_i is a type I_{n_i} factor with $n_i < \infty$. For each i, let ψ_i be the (unique) faithful tracial state on M_i, and let φ denote $\psi P_2(u)$ where ψ is the faithful tracial state on $E_2^{**}(u)$ given by $k^{-1} \sum_{i=1}^{k} \psi_i$. By construction, as v ranges over all tripotents in E^{**} such that $v \leq u$ and $v \neq u$, the values of $\varphi(v)$ form a finite set of rational numbers with supremum $\alpha < \varphi(u) = 1$.

Let $x \in E^{**}$. By Theorem 2.9 there exists an element $a \in E$ and a tripotent $v \in E^{**}$ with $v \leq u$ such that $D(v,v)(x - a) = 0$ and $1 - \varphi(v) < 1 - \alpha$. Since $\varphi(v) > \alpha$, we must have $v = u$ and hence $D(u,u)E = D(u,u)E^{**}$. The remaining equalities are immediate from the identities $2P_j(u)D(u,u) = jP_j(u)$ for $j = 1, 2$.

(*b*) By the arguments of the previous paragraph, and Lemma 3.1, there is a norm-one element $a \in E$ such that $u - D(u,u)a = D(u,u)(u - a) = 0$.

(*c*) Let a be as in (*b*) and let $\rho_i \in \partial_e(E_1^*)$ such that $\rho_i(u_i) = 1$ for each i [9, Proposition 4]. By restriction, the ρ_i are pure states of the JB*-algebra $E(a)$ with support projections u_i in $E(a)^{**}$. Hence, by [11, Proposition 2.3], there exist orthogonal norm-one elements $b_1, \dots, b_n \in E(a)_+$ with $\rho_i(b_i) = 1$ for $i = 1, \dots, n$. Since each u_i is now a minimal tripotent of $E(b_i)^{**}$ we can apply (b) to find a norm-one element $a_i \in E(b_i)$ such that $D(u_i, u_i)a_i = u_i$. Since the inner ideals $E(b_1), \dots, E(b_n)$ are mutually orthogonal, so are the elements $a_1, \dots, a_n$. □

Let E be a JB*-triple. In the terminology introduced in [7] the tripotents of E^{**} of the form $u(a)$ where $a \in E$ with $\|a\| = 1$, are referred to as *compact* G_δ*'s* relative to E and each tripotent of E^{**} that is the weak* limit of a decreasing net of compact G_δ's relative to E is called *compact* relative to E. Let $x \in E^{**}$ with $\|x\| = 1$. The norm-exposed face of E_1^*,

$$F_x = \{\varphi \in E_1^* : \varphi(x) = 1\}$$

satisfies $F_x = F_{u(x)}$ [7, Lemma 3.3]. The face F_x is weak*-exposed if $x \in E$. By [7, Corollary 4.4] (c.f. [6, Lemma 3.2, Theorem 4.6]) the assignments

$$u \mapsto F_u \quad \text{and} \quad u \mapsto (u + E_0^{**}(u)) \cap E_1$$

are respectively an order isomorphism, and an anti-order isomorphism, from the non-zero compact tripotents relative to E onto the proper weak*-semi-exposed faces of E_1^*, and onto the proper norm-semi-exposed faces of E_1.

Theorem 3.4. *Let E be a JB*-triple and let u be a finite rank tripotent of E^{**}. Then*

- (a) *u is compact relative to E;*
- (b) *F_u is a weak*-semi-exposed face of E_1^*.*

Moreover, if E is separable, then

- (c) *u is a compact G_δ relative to E;*
- (d) *F_u is a weak*-exposed face of E_1^*;*
- (e) *$\{\rho\}$ is weak*-exposed for all $\rho \in \partial_e(E_1^*)$.*

Proof. (a) Let u be the sum of orthogonal minimal tripotents $u_1, \dots, u_n$ in E^{**}. Via Theorem 3.3 (c), choose orthogonal norm-one elements $a_1, \dots, a_n$ in E such that

$$u_i \leq a_i \leq r(a_i) \quad \text{for } i = 1, \dots, n.$$

Each u_i is a minimal projection of $E(a_i)^{**} = E_2^{**}(r(a_i))$. Thus $r(a_1) - a_1, \dots, r(a_n) - a_n$ are, respectively, weak* limits of increasing nets

$(x_1)_\lambda, \ldots, (x_n)_\lambda$ in $E(a_1), \ldots, E(a_n)$ [4]. Let x be the sum of $a_1, \ldots, a_n$. Via mutual orthogonality of the a_i we have

$$u \leq x \leq r(x) = \sum_{i=1}^{n} r(a_i),$$

and $r(x) - \sum_{i=1}^{n} (x_i)_\lambda$ is a decreasing net in $A(x)^{**}$ with weak* limit u. Thus,

$$\left\{ x, r(x) - \sum_{i=1}^{n} (x_i)_\lambda, x \right\}$$

is a decreasing net in $E(x)$ with weak*-limit $\{x, u, x\} = u$. It follows that u is compact relative to E.

(b) This is immediate from (a) and [7, Corollary 4.4].

Suppose now that E is separable.

(c) Let the minimal tripotents $u_1, \ldots, u_n$ be as in (a) and let $\rho_1, \ldots, \rho_n$ in $\partial_e(E_1^*)$ such that $\{\rho_i\} = F_{u_i}$ for each i [6, Proposition 4]. As in part (a) we can choose x in E such that $u \leq x \leq r(x)$. Passing to the separable JB*-algebra $E(x)$, it follows from [11, Propositions 2.3 and 3.1] that there exist orthogonal norm-one elements $a_1, \ldots, a_n$ in $E(x)_+$ such that for each i,

$$\{\rho_i\} = F_{a_i} = F_{u(a_i)},$$

the second equality coming from [6, Lemma 3.3], so that $u_i = u(a_i)$ by [6, Theorem 4.4]. Thus, letting $a = \sum_{i=1}^{n} a_i$, in the JB*-algebra $E(x)$ we have that

$$a^{2n+1} = \sum_{i=1}^{n} a_i^{2n+1}$$

is a decreasing sequence with weak* limit $\sum_{i=1}^{n} u_i = u$ in $E(x)^{**}$. Therefore, $u = u(a)$.

(d) With a in E as in the proof of (c), we have $F_u = F_a$, as required.

(e) This is contained in the proof of (c). □

We conclude with an observation on the facial structure of the closed unit ball E_1 of a JB*-triple E. If $\mathcal{G}$ is a norm-semi-exposed face of E_1 then (see [7])

$$\mathcal{G}' = \{\rho \in E_1^* : \rho(x) = 1 \text{ for all } x \in \mathcal{G}\}$$

is a weak*-semi-exposed face of E_1^* and

$$\mathcal{G} = \{x \in E_1 : \rho(x) = 1 \text{ for all } \rho \in \mathcal{G}'\}.$$

Let $u(a)$ be a compact G_δ in E^{**} relative to E (where a lies in E with $\|a\| = 1$) and let $\rho \in \partial_e(F_a)$. Then $u(a)$ majorises the support tripotent v of ρ in E^{**}, and we note that v is a minimal tripotent since $\rho \in \partial_e(E_1^*)$. It follows by definition that each element in the set, S, of all non-zero compact tripotents in E^{**} relative to E, majorises a minimal tripotent of E^{**}. Since, by Theorem 3.4 (a), all minimal tripotents of E^{**} are compact relative to E we deduce that the minimal elements of the set S (see [7, Theorem 4.5]) are, precisely, the minimal tripotents of E^{**}. By Theorem 3.3 (b) each $\rho \in \partial_e(E_1^*)$ attains its norm on E_1 so that

$$E_\rho = \{x \in E_1 : \rho(x) = 1\}$$

is a non-empty (norm-exposed) face of E_1.

Corollary 3.5. *Let E be a JB*-triple.*

- (a) *The E_ρ are the maximal proper norm-closed faces of E_1 as ρ ranges over $\partial_e(E_1^*)$.*
- (b) *Each norm semi-exposed face of E_1 is an intersection of maximal norm closed faces of E_1.*

Proof. (a) Each maximal proper norm closed face of E_1 is norm exposed by [7, Lemma 2.1]. Given $\rho \in \partial_e(E_1^*)$ with minimal support tripotent u in E^{**} we clearly have that E_ρ contains $(u + E_0^{**}(u)) \cap E_1$. By the above remarks together with [7, Corollary 4.4 (ii)] the assertion now follows.

(b) Let $\mathcal{G}$ be a norm semi-exposed face of E_1. By the Krein-Milman theorem

$$\mathcal{G} = \{x \in E_1 : \rho(x) = 1 \text{ for all } \rho \in \partial_e(\mathcal{G}')\}.$$

Further, $\partial_e(\mathcal{G}') = \partial_e(E_1^*) \cap \mathcal{G}'$, since $\mathcal{G}'$ is a face. Hence,

$$\mathcal{G} = \bigcap \{E_\rho : \rho \in \partial_e(E_1^*) \cap \mathcal{G}'\}.$$

□

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