Grothendieck's inequalities revisited

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Abstract

2000 Mathematics Subject Classification: 17C65, 46K70, 46L05, 46L10, and 46L70.

Introduction

Let X be a normed space. We denote by S_X, B_X, and X^* the unit sphere, the closed unit ball, and the dual space, respectively of X. If X is a Banach dual space we write X_* for a predual of X.

1 Little Grothendieck's inequality

We recall that a complex JB*-triple is a complex Banach space $\mathcal{E}$ with a continuous triple product $\{.,.,.\} : \mathcal{E} \times \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ which is bilinear and symmetric in the outer variables and conjugate linear in the middle variable, and satisfies:

1. (Jordan Identity) $L(a,b)\{x,y,z\} = \{L(a,b)x,y,z\} - \{x,L(b,a)y,z\} + \{x,y,L(a,b)z\}$ for all a,b,c,x,y,z in $\mathcal{E}$, where $L(a,b)x := \{a,b,x\}$;

2. The map $L(a,a)$ from $\mathcal{E}$ to $\mathcal{E}$ is an hermitian operator with nonnegative spectrum for all a in $\mathcal{E}$;

*Supported by Programa Nacional F.P.I. Ministry of Education and Science grant, D.G.I.C.Y.T. project no. PB 98-1371, and Junta de Andalucía grant FQM 0199

†Partially supported by Junta de Andalucía grant FQM 0199

3. $\|\{a,a,a\}\| = \|a\|^3$ for all a in $\mathcal{E}$.

Complex JB*-triples have been introduced by W. Kaup in order to provide an algebraic setting for the study of bounded symmetric domains in complex Banach spaces (see [K1], [K2] and [U]).

By a complex JBW*-triple we mean a complex JB*-triple which is a dual Banach space. We recall that the triple product of every complex JBW*-triple is separately weak*-continuous [BT], and that the bidual $\mathcal{E}^{**}$ of a complex JB*-triple $\mathcal{E}$ is a JBW*-triple whose triple product extends the one of $\mathcal{E}$ [Di].

Given a complex JBW*-triple $\mathcal{W}$ and a norm-one element φ in the predual $\mathcal{W}_*$ of $\mathcal{W}$, we can construct a prehilbert seminorn $\|.\|_\varphi$ as follows (see [BF1, Proposition 1.2]). By the Hahn-Banach theorem there exists $z \in \mathcal{W}$ such that $\varphi(z) = \|z\| = 1$. Then $(x,y) \mapsto \varphi\{x,y,z\}$ becomes a positive sesquilinear form on $\mathcal{W}$ which does not depend on the point of support z for φ. The prehilbert seminorm $\|.\|_\varphi$ is then defined by $\|x\|_\varphi^2 := \varphi\{x,x,z\}$ for all $x \in \mathcal{W}$. If $\mathcal{E}$ is a complex JB*-triple and φ is a norm-one element in $\mathcal{E}^*$, then $\|.\|_\varphi$ acts on $\mathcal{E}^{**}$, hence in particular it acts on $\mathcal{E}$.

Following [IKR], we define real JB*-triples as norm-closed real subtriples of complex JB*-triples. In [IKR] it is shown that every real JB*-triple E can be regarded as a real form of a complex JB*-triple. Indeed, given a real JB*-triple E there exists a unique complex JB*-triple structure on the complexification $\widehat{E} = E \oplus i\ E$, and a unique conjugation (i.e., conjugate-linear isometry of period 2) τ on $\widehat{E}$ such that $E = \widehat{E}^\tau := \{x \in \widehat{E} : \tau(x) = x\}$. The class of real JB*-triples includes all JB-algebras [HS], all real C*-algebras [G], and all J*B-algebras [Al].

By a real JBW*-triple we mean a real JB*-triple whose underlying Banach space is a dual Banach space. As in the complex case, the triple product of every real JBW*-triple is separately weak*-continuous [MP], and the bidual E^{**} of a real JB*-triple E is a real JBW*-triple whose triple product extends the one of E [IKR].

If U is a real or complex JB*-triple, and A is a subset of U we denote by

$$A^\perp := \{x \in U : \{x, A, U\}\}$$

the orthogonal complement of A.

In [PR] (see also [P]) the authors proved the following appropriated version of the so called "Little Grothendieck's inequality" for real and complex JBW*-triples, which avoids the gaps contained in [BF1].

Theorem 1.1 *[PR, Theorems 2.1 and 2.9]*
Let $K > \sqrt{2}$ (respectively, $K > 1 + 3\sqrt{2}$) and $\varepsilon > 0$. Then, for every complex (respectively, real) JBW-triple $\mathcal{W}$, every complex (respectively, real) Hilbert space $\mathcal{H}$, and every weak*-continuous linear operator $T : \mathcal{W} \to \mathcal{H}$, there exist norm-one functionals $\varphi_1, \varphi_2 \in \mathcal{W}_*$ such that the inequality*

$$\|T(x)\| \leq K \ \|T\| \ \left(\|x\|_{\varphi_2}^2 + \varepsilon^2 \ \|x\|_{\varphi_1}^2\right)^{\frac{1}{2}},$$

holds for all $x \in \mathcal{W}$.

Let T be a bounded linear operator from a real (respectively, complex) JB*-triple E to a real (respectively, complex) Hilbert space H, since E^{**} is a real (respectively, complex) JBW*-triple and T^{**} is a weak*-continuous operator from E^{**} to H, then the following result follows from the previous theorem.

Corollary 1.2 *Let $K > \sqrt{2}$ (respectively, $K > 1 + 3\sqrt{2}$) and $\varepsilon > 0$. Then, for every complex (respectively, real) JB*-triple $\mathcal{E}$, every complex (respectively, real) Hilbert space $\mathcal{H}$, and every bounded linear operator $T : \mathcal{E} \to \mathcal{H}$, there exist norm-one functionals $\varphi_1, \varphi_2 \in \mathcal{E}^*$ such that the inequality*

$$\|T(x)\| \leq K \ \|T\| \ \left(\|x\|_{\varphi_2}^2 + \varepsilon^2 \ \|x\|_{\varphi_1}^2\right)^{\frac{1}{2}},$$

holds for all $x \in \mathcal{E}$.

The question is whether in Corollary 1.2 the value $\varepsilon = 0$ is allowed for some value of the constant K. We are going to give an affirmative answer to this question whenever we replace the prehilbertian seminorm $\|.\|_\varphi$ with another prehilbertian seminorm associated with a "state" of a real or complex JB*-triple.

Given a Banach space X, $BL(X)$, and I_X will denote the normed algebra of all bounded linear operators on X, and the identity operator on X, respectively. If u is a norm-one element in X, the set of **states** of X relative to u, $D(X, u)$, is defined as the non empty, convex, and weak*-compact subset of X^* given by

$$D(X, u) := \{\Phi \in B_{X^*} : \Phi(u) = 1\}.$$

Let $\mathcal{E}$ be a complex JB*-triple and $\Phi \in D(BL(\mathcal{E}), I_{\mathcal{E}})$. Since for every $x \in \mathcal{E}$, the map $L(x, x)$ is an hermitian operator with non-negative spectrum, we can define the prehilbertian seminorm $|||.|||_\Phi$ by $|||x|||_\Phi := \Phi(L(x,x))$ for all $x \in \mathcal{E}$.

Let $\varphi \in S_{\mathcal{E}^*}$ and let $e \in S_{\mathcal{E}^{**}}$ such that $\varphi(e) = 1$. We define $\Phi_{\varphi,e} \in D(BL(\mathcal{E}), I_{\mathcal{E}})$ by $\Phi_{\varphi,e}(T) := \varphi T^{**}(e)$ for all $T \in BL(\mathcal{E})$. We notice that in this case $|||.|||_{\Phi_{\varphi,e}}$ and $\|.\|_\varphi$ coincide on $\mathcal{E}^{**}$ (and hence in $\mathcal{E}$).

Theorem 1.3 *Let $\mathcal{E}$ be a complex (respectively, real) JB*-triple, $\mathcal{H}$ a complex (respectively, real) Hilbert space and $T : \mathcal{E} \to \mathcal{H}$ a bounded linear operator. Then there exists $\Phi \in D(BL(\mathcal{E}), I_{\mathcal{E}})$ such that*

$$\|T(x)\| \leq \sqrt{2}\ \|T\|\ |||x|||_\Phi,$$

(respectively, $\|T(x)\| \leq (1+3\sqrt{2})\ \|T\|\ |||x|||_\Phi$) for all $x \in \mathcal{E}$.

Proof. We suppose that $\mathcal{E}$ is a complex JB*-triple. The proof for a real JB*-triple is the same. By Corollary 1.2, for every $n \in \mathbb{N}$ there are norm-one functionals $\varphi_1^n, \varphi_2^n \in \mathcal{E}^*$ such that the inequality

$$\|T(x)\| \leq (\sqrt{2} + \frac{1}{n}\ )\ \|T\|\ \left(\|x\|^2_{\varphi_2^n} + \frac{1}{n}\ \|x\|^2_{\varphi_1^n}\right)^{\frac{1}{2}}$$

$$= (\sqrt{2} + \frac{1}{n}\ )\ \|T\|\ \left(|||x|||^2_{\Phi_{\varphi_2^n, e_2^n}} + \frac{1}{n}\ |||x|||^2_{\Phi_{\varphi_1^n, e_1^n}}\right)^{\frac{1}{2}},$$

holds for all $x \in \mathcal{E}$, where $e_i^n \in S_{\mathcal{E}^{**}}$ with $\varphi_1^n(e_i^n) = 1$ ($i = 1,2,\ n \in \mathbb{N}$). Let $i \in \{1,2\}$, since $D(BL(\mathcal{E}), I_{\mathcal{E}})$ is weak*-compact, we can take a weak* cluster point $\Phi_i \in D(BL(\mathcal{E}), I_{\mathcal{E}})$ of the sequence $\Phi_{\varphi_i^n, e_i^n}$ ($i = 1,2$). Then the inequality

$$\|T(x)\| \leq \sqrt{2}\ \|T\|\ |||x|||_{\Phi_2}$$

holds for all $x \in \mathcal{E}$. □

From the previous Theorem we can now derive a remarkable result of U. Haagerup.

Corollary 1.4 *[H1, Theorem 3.2]*

Let A be a C^*-algebra, H a complex Hilbert space, and $T : A \to H$ a bounded linear operator. There exist two states φ and ψ on A, such that

$$\|T(x)\|^2 \leq \|T\|^2(\varphi(x^*x) + \psi(xx^*)),$$

for all $x \in A$.

Proof. By Theorem 1.3 there exists $\Phi \in D(BL(A), I_A)$ such that

$$\|T(x)\|^2 \leq 2\|T\|^2\Phi(L(x,x))$$

for all $x \in A$. Since for every $x \in A$, $L(x,x) = \frac{1}{2}(L_{xx^*} + L_{x^*x})$ (where L_a and R_a stands for the left and right multiplication by a, respectively), we have

$$\|T(x)\|^2 \leq \|T\|^2\Phi(L_{xx^*} + R_{x^*x})$$

for all $x \in A$.

Now denoting by $\widehat{\varphi}$ and $\widehat{\psi}$ the positive functionals on A given by $\widehat{\varphi}(x) := \Phi(L_x)$, and $\widehat{\psi}(x) := \Phi(R_x)$, respectively, we conclude that $\varphi = \frac{\widehat{\varphi}}{\|\widehat{\varphi}\|}$ and $\psi = \frac{\widehat{\psi}}{\|\widehat{\psi}\|}$ are states on A and

$$\|T(x)\|^2 \leq \|T\|^2(\varphi(x^*x) + \psi(xx^*)),$$

for all $x \in A$. □

The concluding section of the paper [PR] deals with some applications of the Theorem 1.1, including certain results on the strong*-topology, $S^*(W, W_*)$, of a real or complex JBW*-triple W. We recall that if W is a real or complex JBW*-triple then the $S^*(W, W_*)$ topology is defined as the topology on W generated by the family of seminorms $\{\|.\|_\varphi : \varphi \in W_*,\ \|\varphi\| = 1\}$. For every dual Banach space X (with a fixed predual denoted by X_*), we denote by $m(X, X_*)$ the Mackey topology on X relative to its duality with X_*.

It is worth mentioning that if a JBW*-algebra $\mathcal{A}$ is regarded as a complex JBW*-triple, $S^*(\mathcal{A}, \mathcal{A}_*)$ coincides with the so-called "algebra-strong* topology" of $\mathcal{A}$, namely the topology on $\mathcal{A}$ generated by the family of seminorms of the form $x \mapsto \sqrt{\xi(x \circ x^*)}$ when ξ is any positive functional in $\mathcal{A}_*$ [R1, Proposition 3]. As a consequence, when a von Neumann algebra $\mathcal{M}$ is regarded as a complex JBW*-triple, $S^*(\mathcal{M}, \mathcal{M}_*)$ coincides with the familiar strong*-topology of $\mathcal{M}$ (compare [S, Definition 1.8.7]).

The results of [PR] allow us to avoid the difficulties in [R1] (compare [PR, page 23]), and to extend these results to the real case. We summarize these results in the following theorem.

Theorem 1.5 *[PR, page 23, Corollary 4.2, and Theorem 4.3] (see also [R1, Theorem] and [R2, Theorem D.21])*

1. *Let W be a real or complex JBW*-triple. Then the strong*-topology of W is compatible with the duality (W, W_*).*

2. *Linear mappings between real or complex JBW*-triples are strong*-continuous if and only if they are weak*-continuous.*

3. *If W is a real or complex JBW*-triple, and if V is a weak*-closed subtriple, then the inequality $S^*(W, W_*)|_V \leq S^*(V, V_*)$ holds, and in fact $S^*(W, W_*)|_V$ and $S^*(V, V_*)$ coincide on bounded subsets of V.*

4. *Let W be a real or complex JBW*-triple. Then the triple product of W is jointly $S^*(W, W_*)$-continuous on bounded subsets of W, and the topologies $m(W, W_*)$ and $S^*(W, W_*)$ coincide on bounded subsets of W.*

Remark 1.6 *In a recent work L. J. Bunce has obtained an improvement of the third statement. Concretely in [Bu, Corollary] he proves that if W is a real or complex JBW*-triple, and if V is a weak*-closed subtriple, then*

1. *each element of V_* has a norm preserving extension in W_*;*

2. $S^*(W, W_*)|_V = S^*(V, V_*)$

From the results related with the strong*-topology we derive a Jarchow-type characterization of weakly compact operators from (real or complex) JB*-triples to arbitrary Banach spaces.

Theorem 1.7 *[PR, Theorem 4.6]*

Let E be a real (respectively, complex) JB-triple, X a real (respectively, complex) Banach space, and $T : E \to X$ a bounded linear operator. The following assertions are equivalent:*

1. *T is weakly compact.*

2. *There exist a bounded linear operator G from E to a real (respectively, complex) Hilbert space and a function $N : (0, +\infty) \to (0, +\infty)$ such that*
$$\|T(x)\| \leq N(\varepsilon)\|G(x)\| + \varepsilon\|x\|$$
for all $x \in E$ and $\varepsilon > 0$.

3. *There exist norm one functionals $\varphi_1, \varphi_2 \in E^*$ and a function $N : (0, +\infty) \to (0, +\infty)$ such that*
$$\|T(x)\| \leq N(\varepsilon)\ \|x\|_{\varphi_1,\varphi_2} + \varepsilon\|x\|$$
for all $x \in E$ and $\varepsilon > 0$.

2 Big Grothendieck's inequality

In [PR, Theorems 3.1 and 3.4] we obtained the following result.

Theorem 2.1 *Let $M > 4(1+2\sqrt{3})\ (1+3\sqrt{2})^2$ (respectively, $M > 4(1+2\sqrt{3})$) and $\varepsilon > 0$. For every couple (V, W) of real (respectively, complex) JBW*-triples and every separately weak*-continuous bilinear form U on $V \times W$, there exist norm-one functionals $\varphi_1, \varphi_2 \in V_*$, and $\psi_1, \psi_2 \in W_*$ satisfying*

$$|U(x,y)| \leq M\ \|U\|\ \left(\|x\|_{\varphi_2}^2 + \varepsilon^2\ \|x\|_{\varphi_1}^2\right)^{\frac{1}{2}}\ \left(\|y\|_{\psi_2}^2 + \varepsilon^2\ \|y\|_{\psi_1}^2\right)^{\frac{1}{2}}$$

for all $(x,y) \in V \times W$.

In the case of complex JB*-triples the interval of variation of the constant M can be enlarged with $M > 3+2\sqrt{3}$ (see [PR, Remark 3.6]). Precisely, we have the following theorem.

Theorem 2.2 *Let $M > 3+2\sqrt{3}$ and $\varepsilon > 0$. Then for every couple $(\mathcal{E}, \mathcal{F})$ of complex JB*-triples and every bounded bilinear form U on $\mathcal{E} \times \mathcal{F}$ there exist norm-one functionals $\varphi_1, \varphi_2 \in \mathcal{E}^*$ and $\psi_1, \psi_2 \in \mathcal{F}^*$ satisfying*

$$|U(x,y)| \leq M\ \|U\|\ \left(\|x\|_{\varphi_2}^2 + \varepsilon^2\ \|x\|_{\varphi_1}^2\right)^{\frac{1}{2}}\ \left(\|y\|_{\psi_2}^2 + \varepsilon^2\ \|y\|_{\psi_1}^2\right)^{\frac{1}{2}}$$

for all $(x,y) \in \mathcal{E} \times \mathcal{F}$.

As in the "Little Grothendieck's inequality", we do not know if the value $\varepsilon = 0$ is allowed in the previous Theorem. However we can take $\varepsilon = 0$ whenever we change norm-one functionals with states. Indeed, when in the proof of Theorem 1.3, Theorem 2.2 and [PR, Corollary 3.5] replace Corollary 1.2, we obtain the following theorem.

Theorem 2.3 *Let $\mathcal{E}$, $\mathcal{F}$ be complex (respectively, real) JB*-triples, $M = 3+2\sqrt{3}$ (respectively, $M = 4(1+2\sqrt{3})\ (1+3\sqrt{2})^2$), and let U be a bounded bilinear form on $\mathcal{E} \times \mathcal{F}$. Then there are $\Phi \in D(BL(\mathcal{E}), I_{\mathcal{E}})$ and $\Psi \in D(BL(\mathcal{F}), I_{\mathcal{F}})$ such that*

$$|U(x,y)| \leq M\|U\| |||x|||_{\Phi} |||y|||_{\Psi}$$

for all $(x,y) \in \mathcal{E} \times \mathcal{F}$.

Another interesting question is whether the interval $M > 3+2\sqrt{3}$, is valid in the complex case of Theorem 2.1. The rest of the paper deals with the affirmative answer of this question. The following proposition gives a first answer in the particular case of biduals of JB*-triples. We recall that if $\mathcal{E}$ and $\mathcal{F}$ are complex JB*-triples, then every bounded bilinear form U on $\mathcal{E}\times\mathcal{F}$ has a (unique) separately weak*-continuous extension, denoted by $\widetilde{U}$, to $\mathcal{E}^{**}\times\mathcal{F}^{**}$ (see [PR, Lemma 1.1]).

Proposition 2.4 *Let $M > 3+2\sqrt{3}$ and $\varepsilon > 0$. Then for every couple $(\mathcal{E},\mathcal{F})$ of complex JB*-triples and every bounded bilinear form U on $\mathcal{E}\times\mathcal{F}$ there exist norm-one functionals $\varphi_1,\varphi_2\in\mathcal{E}^*$ and $\psi_1,\psi_2\in\mathcal{F}^*$ satisfying*

$$|\widetilde{U}(\alpha,\beta)| \leq M\ \|U\|\ \left(\|\alpha\|_{\varphi_2}^2+\varepsilon^2\ \|\alpha\|_{\varphi_1}^2\right)^{\frac{1}{2}}\ \left(\|\beta\|_{\psi_2}^2+\varepsilon^2\ \|\beta\|_{\psi_1}^2\right)^{\frac{1}{2}}$$

for all $(\alpha,\beta)\in\mathcal{E}^{**}\times\mathcal{F}^{**}$.

Proof. By Theorem 2.2, there are norm-one functionals $\varphi_1,\varphi_2\in\mathcal{E}^*$ and $\psi_1,\psi_2\in\mathcal{F}^*$ satisfying

$$|\widetilde{U}(x,y)| \leq M\ \|U\|\ \left(\|x\|_{\varphi_2}^2+\varepsilon^2\ \|x\|_{\varphi_1}^2\right)^{\frac{1}{2}}\ \left(\|y\|_{\psi_2}^2+\varepsilon^2\ \|y\|_{\psi_1}^2\right)^{\frac{1}{2}} \tag{2.1}$$

for all $(x,y)\in\mathcal{E}\times\mathcal{F}$.

Since the first assertion of Theorem 1.5 assures that $\mathcal{E}$ and $\mathcal{F}$ are strong*-dense in $\mathcal{E}^{**}$ and $\mathcal{F}^{**}$, respectively, for every $(\alpha,\beta)\in\mathcal{E}^{**}\times\mathcal{F}^{**}$ we have nets $(x_\lambda)\subseteq\mathcal{E}$ and $(y_\mu)\subseteq\mathcal{F}$ converging to α and β, respectively, in the strong* topology (hence they converge also in the weak* topology of $\mathcal{E}^{**}$ and $\mathcal{F}^{**}$, respectively. Let now $x\in\mathcal{E}$, since for $i\in\{1,2\}$, the seminorm $\|.\|_{\psi_i}$ is strong*-continuous, by (2.1) and the separately weak*-continuity of $\widetilde{U}$ we have

$$|\widetilde{U}(x,\beta)| \leq M\ \|U\|\ \left(\|x\|_{\varphi_2}^2+\varepsilon^2\ \|x\|_{\varphi_1}^2\right)^{\frac{1}{2}}\ \left(\|\beta\|_{\psi_2}^2+\varepsilon^2\ \|\beta\|_{\psi_1}^2\right)^{\frac{1}{2}}$$

for all $(x,\beta)\in\mathcal{E}\times\mathcal{F}^{**}$. Using the same argument, but fixing $\beta\in\mathcal{F}^{**}$ instead of $x\in\mathcal{E}$, we finish the proof. □

By [BDH, Proposition 6] every JBW*-triple is (isometrically) isomorphic to a weak*-closed ideal of its bidual. Indeed, given a JBW*-triple $\mathcal{V}$, then there exists a weak*-closed ideal P of $\mathcal{V}^{**}$ such that $\Psi := \Pi_P J_{\mathcal{V}}$ is a triple isomorphism (and hence weak*-continuous) from $\mathcal{V}$ onto P, where Π_P denotes the natural projection from $\mathcal{V}^{**}$ onto P, and $J_{\mathcal{V}}$ denotes the natural embedding of $\mathcal{V}$ onto $\mathcal{V}^{**}$. It is also known that $J_{\mathcal{V}_*}^*|_P = \Psi^{-1}$.

We can know state the complex case of Theorem 2.1 with constant $M > 3+2\sqrt{3}$.

Theorem 2.5 *Let $M > 3+2\sqrt{3}$ and $\varepsilon > 0$. For every couple $(\mathcal{V},\mathcal{W})$ of complex JBW*-triples and every separately weak*-continuous bilinear form U on $\mathcal{V}\times\mathcal{W}$, there exist norm-one functionals $\varphi_1,\varphi_2\in\mathcal{V}_*$, and $\psi_1,\psi_2\in\mathcal{W}_*$ satisfying*

$$|U(x,y)| \le M\ \|U\|\ \left(\|x\|_{\varphi_2}^2+\varepsilon^2\ \|x\|_{\varphi_1}^2\right)^{\frac{1}{2}}\ \left(\|y\|_{\psi_2}^2+\varepsilon^2\ \|y\|_{\psi_1}^2\right)^{\frac{1}{2}}$$

for all $(x,y)\in\mathcal{V}\times\mathcal{W}$.

Proof. Let $\widetilde{U}$ the unique separately weak*-continuous extension of U to $\mathcal{V}^{**}\times\mathcal{W}^{**}$. By Proposition 2.4 there exist norm-one functionals $\varphi_1,\varphi_2\in\mathcal{V}^*$ and $\psi_1,\psi_2\in\mathcal{W}^*$ satisfying

$$|\widetilde{U}(\alpha,\beta)| \le M\ \|U\|\ \left(\|\alpha\|_{\varphi_2}^2+\varepsilon^2\ \|\alpha\|_{\varphi_1}^2\right)^{\frac{1}{2}}\ \left(\|\beta\|_{\psi_2}^2+\varepsilon^2\ \|\beta\|_{\psi_1}^2\right)^{\frac{1}{2}} \qquad (2.2)$$

for all $(\alpha,\beta)\in\mathcal{V}^{**}\times\mathcal{W}^{**}$.

By the previous comments there are weak*-closed ideals P and Q, of $\mathcal{V}^{**}$ and $\mathcal{W}^{**}$, respectively, such that

$$\Psi_{\mathcal{V}} := \Pi_P J_{\mathcal{V}} : \mathcal{V}\to P$$

and

$$\Psi_{\mathcal{W}} := \Pi_Q J_{\mathcal{W}} : \mathcal{W}\to Q$$

are triple isomorphisms. Let us now define another bilinear form, $\widehat{U}$, on $\mathcal{V}^{**}\times\mathcal{W}^{**}$ by $\widehat{U}(\alpha,\beta) := U(J_{\mathcal{V}_*}^*(\alpha), J_{\mathcal{W}_*}^*(\beta))$. Then $\widehat{U}$ is separately weak*-continuous and extends U to $\mathcal{V}^{**}\times\mathcal{W}^{**}$, so $\widehat{U}=\widetilde{U}$. In particular

$$U(x,y)=\widetilde{U}(\Pi_P J_{\mathcal{V}}(x),\Pi_Q J_{\mathcal{W}}(y))$$

for all $(x,y)\in\mathcal{V}\times\mathcal{W}$.

It is well known that $\mathcal{V}^{**}=P\oplus^{\ell_\infty}P^\perp$ and $\mathcal{W}^{**}=Q\oplus^{\ell_\infty}Q^\perp$, so the norm-one functionals given in (2.2) decompose $\varphi_i=\varphi_i^1+\varphi_i^2$ and $\psi_i=\psi_i^1+\psi_i^2$ $(i\in\{1,2\})$, where

$$\varphi_i^1\in P_*,\ \varphi_i^2\in (P^\perp)_*,\ \|\varphi_i^1\|+\|\varphi_i^2\|=1,$$

and

$$\psi_i^1 \in Q_*, \ \psi_i^2 \in (Q^\perp)_*, \ \|\psi_i^1\| + \|\psi_i^2\| = 1,$$

for $i \in \{1,2\}$. Now taking $x \in \mathcal{V}$ and norm one elements $e_i^1 \in P$ and $e_i^2 \in P^\perp$ such that $\varphi_i^j(e_i^j) = \|\varphi_i^j$ $(i,j \in \{1,2\})$, applying the orthogonality of P and $P^\perp$, we get

$$\|\Phi_{\mathcal{V}}(x)\|_{\varphi_i}^2 = \varphi_i^1 \left\{\Phi_{\mathcal{V}}(x), \Phi_{\mathcal{V}}(x), e_i^1\right\} + \varphi_i^2 \left\{\Phi_{\mathcal{V}}(x), \Phi_{\mathcal{V}}(x), e_i^2\right\}$$

$$= \varphi_i^1 \Phi_{\mathcal{V}} \left\{x, x, \Phi_{\mathcal{V}}^{-1}(e_i^1)\right\} \leq \|x\|_{\widetilde{\varphi}_i}^2,$$

where $\widetilde{\varphi}_i := \dfrac{\varphi_i^1 \Phi_{\mathcal{V}}}{\|\varphi_i^1 \Phi_{\mathcal{V}}\|} \in \mathcal{V}_*$ if $\varphi_i^1 \Phi_{\mathcal{V}} \neq 0$, if $\varphi_i^1 \Phi_{\mathcal{V}} = 0$ we can take as $\widetilde{\varphi}_i$ any other norm-one functional in $\mathcal{V}_*$. Similarly we get norm-one functionals $\widetilde{\psi}_i$ in $\mathcal{W}_*$ such that

$$\|\Phi_{\mathcal{W}}(y)\|_{\psi_i}^2 \leq \|y\|_{\widetilde{\psi}_i}^2$$

for all $y \in \mathcal{W}$, $i \in \{1,2\}$.

Finally applying (2.2) we get

$$|U(x,y)| \leq M \ \|U\| \ \left(\|x\|_{\widetilde{\varphi}_2}^2 + \varepsilon^2 \ \|x\|_{\widetilde{\varphi}_1}^2\right)^{\frac{1}{2}} \ \left(\|y\|_{\widetilde{\psi}_2}^2 + \varepsilon^2 \ \|y\|_{\widetilde{\psi}_1}^2\right)^{\frac{1}{2}}$$

for all $(x,y) \in \mathcal{V} \times \mathcal{W}$.□

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