THE ALTERNATIVE DUNFORD-PETTIS PROPERTY, CONJUGATIONS AND REAL FORMS OF C*-ALGEBRAS

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Abstract. Let τ be a conjugation, alias a conjugate linear isometry of order 2, on a complex Banach space X and let X^τ be the real form of X of τ-fixed points. In contrast with the Dunford-Pettis property, the *alternative* Dunford-Pettis property of [10] need not lift from X^τ to X. But if X is a C*-algebra it is shown that X^τ has the alternative Dunford-Pettis property if and only if X does and an analogous result is shown when X is the dual space of a C*-algebra. One consequence is that both Dunford-Pettis properties coincide on all real forms of C*-algebras.

1. Introduction

Investigations of Dunford-Pettis and associated properties of spaces of operators and their dual spaces include [1, 2, 4, 5, 6, 7, 10, 11]. In particular, the Alternative Dunford-Pettis property (definitions are given below) was introduced in [10] and shown to coincide with the usual Dunford-Pettis property on von Neumann algebras. This was extended to all C*-algebras in [4] and the class of those C*-algebras whose dual space has the Alternative Dunford-Pettis property was determined. In this paper we investigate the Alternative Dunford-Pettis property on spaces of fixed points of conjugations on C*-algebras.

A Banach space X is said to have the Dunford-Pettis property (DP) if whenever (x_n) is a weakly null sequence in X and (ρ_n) is a weakly null sequence in the dual space of X, then $\rho_n(x_n) \to 0$. The reader is referred to [8] for several characterisations. On the other hand, if $\rho_n(x_n) \to 0$ whenever $x_n \to x$ weakly in the unit sphere of X and (ρ_n) is weakly null in the dual space of X, then X is said to have the Alternative Dunford-Pettis property (DP1). The DP1 is preserved

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by linear isometries but not by linear isomorphisms [10, Example 1.6] whereas, visibly, the DP is preserved by linear isomorphisms. Both the DP and the DP1 are inherited by complemented subspaces. If X is a complex Banach space we let X^* denote its dual space and X_* a predual, if the latter exists. We use the corresponding notations of $X^{'}$ and $X_,$ if X is a real Banach space. Further, for a complex Banach space X we shall denote by X_r the real Banach space obtained by reduction of scalars.

Let X be a complex Banach space. The map

$$\varphi : (X^*)_r \to (X_r)^{'} \quad (\rho \mapsto \Re e\rho)$$

is a surjective linear isometry with inverse $\gamma \mapsto \gamma(\cdot) - i\gamma(i\ \cdot)$. The bidual surjective linear isometry

$$\psi : (X^{**})_r \to (X_r)^{''}$$

sends the canonical embedding of X in X^{**} onto the canonical embedding of X_r in $(X_r)^{''}$. The maps φ and ψ are homeomorphisms for the respective weak topologies and for the respective weak* topologies. Similarly, the identity map on X is a $\sigma(X, X^*)$-$\sigma(X, (X_r)^{'})$ homeomorphism. The following is evident.

Lemma 1.1. *If X is a complex Banach space, then*

(a) *X has the DP (respectively, the DP1) if and only if X_r has the DP (respectively, the DP1);*

(b) *X^* has the DP (respectively, the DP1) if and only if $(X_r)^{'}$ has the DP (respectively, the DP1).*

Let $\tau : X \to X$ be a *conjugation*, alias a conjugate linear isometry of order 2, on the complex Banach space X. The associated *real form* of X is the set of fixed points $X^\tau := \{x \in X : \tau(x) = x\}$. We note that X^τ is the image of the real contractive projection $\frac{1}{2}(id + \tau)$, and that

$$X = X^\tau \oplus iX^\tau. \qquad (1)$$

In particular, if X has the DP or the DP1, respectively, then so does X^τ. The map $\widetilde{\tau} : X^* \to X^*$ given by $\widetilde{\tau}(\rho) = \overline{\rho \circ \tau}$, is a weak* continuous conjugation and satisfies $\widetilde{\tau} = \varphi^{-1} \circ \tau^{'} \circ \varphi$, where $\tau^{'}$ is the real transpose of τ and φ is as defined above. The restriction map $\alpha : (X^*)^{\widetilde{\tau}} \to (X^\tau)^{'}$ is a surjective linear isometry, the inverse being $\rho \mapsto \widetilde{\rho}$, where $\widetilde{\rho}$ is the unique complex linear extension of $\rho \in (X^\tau)^{'}$. Via the obvious identification we have

$$(X^*)^{\widetilde{\tau}} = (X^\tau)^{'} \oplus i(X^\tau)^{'}. \qquad (2)$$

The further conjugation, $\widetilde{\widetilde{\tau}} : X^{**} \to X^{**}$, satisfies $\widetilde{\widetilde{\tau}} = \psi^{-1} \circ \tau'' \circ \psi$. Thus, since $\widetilde{\widetilde{\tau}}$ is weak*-continuous and agrees with τ on X, it is the unique weak*-continuous conjugate linear extension of τ. For this reason, in notation, we shall tend not to distinguish between τ and $\widetilde{\widetilde{\tau}}$ and shall write $\widetilde{\widetilde{\tau}} = \tau$.

Lemma 1.2. *If τ is a conjugation on a complex Banach space X, then X has the DP if and only if X^τ has the DP.*

Proof. This follows from (1) and (2) above. □

The same is not true of the DP1. In contrast to the DP, the DP1 need not lift from a real form to the original space.

Example 1.3. Let H be an infinite dimensional real Hilbert space and let X be the space of all bounded real linear maps $T : H \to \mathbb{C}$. For each T in X let $\overline{T}$ be given by $\overline{T}(h) = \overline{T(h)}$. Now X is a complex Banach space with a conjugation $\tau : X \to X$ given by $\tau(T) = \overline{T}$. The real form X^τ is linearly isometric to $B(H, \mathbb{R})$ and hence to H, and so has the DP1 [10, Corollary 1.5]. However, X is real linearly isometric to $B(H, K)$ where K is a real Hilbert space of dimension two. Therefore, by [1, Proposition 2] (which is independent of the scalar field), X does not have the DP1.

In spite of the general failure embodied in Example 1.3, we shall prove that if A is a C*-algebra with a conjugation τ, then A^τ has the DP1 if and only if A has the DP1 and we shall further prove an analogous result for the dual space of A. We shall conclude that the DP1 and the DP are equivalent for any real form of a C*-algebra.

We remark that if A is a C*-algebra with a $*$-antiautomorphism φ of order 2, and τ is the conjugation given by $\tau(x) = \varphi(x)^*$, then A^τ is a real C*-algebra. All real C*-algebras arise in this way. If φ is relaxed to a Jordan $*$-automorphism of order 2, then A^τ is a real Jordan C*-algebra. In general, a real form of a C*-algebra need not be an algebra, but is invariably a real subspace closed under $x \mapsto xx^*x$. We wish to state that although in the context of C*-algebras this paper is self-contained, it is informed by ideas arising in the theory of JB*-triples [13, 15].

2. Positive Conjugations

Let A be a C*-algebra. We use s to denote the standard conjugation, $x \mapsto x^*$, on A. By a *Jordan $*$-involution* of A we shall mean a Jordan $*$-automorphism of order 2. Given a conjugation τ on A, we let τ

continue to denote its unique weak* continuous extension to A^{**} (see §1). In which case, $s \circ \tau$ is a complex linear isometry on A^{**} and so equals a Jordan $*$-automorphism multiplied by a unitary, by [14, Theorem 7]. In particular, it follows that

$$\tau(xy^*z + zy^*x) = \tau(x)\tau(y)^*\tau(z) + \tau(z)\tau(y)^*\tau(x), \text{ for all } x, y, z \in A^{**}.$$

We shall make extensive use of this property of τ. A conjugation $\tau : A \to A$ is said to be *positive* if $\tau(A_+) \subseteq A_+$. Thus, $\tau(x^*) = \tau(x)^*$ for all x in A if τ is a positive conjugation. The standard conjugation is an example of a positive conjugation. In the following, 1 denotes the identity of A^{**} and $r(x)$ denotes the range projection of x in A^{**}. We do not assume that $1 \in A$.

Lemma 2.1. *Let A be a C^*-algebra with conjugation τ. The following are equivalent.*

(*a*) *τ is positive.*
(*b*) $\tau(1) = 1$.
(*c*) *$s \circ \tau$ is a Jordan $*$-involution.*

Proof. $(a) \Rightarrow (b)$. Assume (a). Then $\tau(1)$ is a self-adjoint unitary and hence a projection of A^{**}. Given any self-adjoint element a in A^{**} we have $\tau(a) = \tau(1a1) = \tau(1)\tau(a)\tau(1)$, so that $\tau(a) = \tau(a)\tau(1)$ and it follows that $\tau(1) = 1$.

$(b) \Rightarrow (c)$. Denoting $s\circ\tau$ by π, for a in A^{**} we have $\pi(a^2) = \pi(a1a) = \pi(a)1\pi(a) = \pi(a)^2$, that $\pi(a^*) = \pi(a)^*$ and that $\pi^2 = id$, as required.

$(c) \Rightarrow (a)$. This is immediate. □

It follows from Lemma 2.1 that $\tau \mapsto s \circ \tau$ is a bijection from the set of positive conjugations of A onto the set of Jordan $*$-involutions of A.

Lemma 2.2. *Let A be a C^*-algebra with conjugation τ. Suppose there is an element $x \in (A^\tau)_+$ with $r(x) = 1$. Then τ is positive.*

Proof. Let S denote the weak* closure of the real linear space generated by $\{x^{2n+1} : n \geq 0\}$. Since $\tau(x) = x$ we have $S \subseteq (A^{**})^\tau$. But $1 \in S$, by spectral theory. □

In general, a conjugation on a C*-algebra need have no non-zero positive fixed points. However, every conjugation is *locally positive* in the sense described below.

Let A be a weak* dense C*-subalgebra of a W*-algebra W and let $x \in A$. Consider the polar decomposition of x in W

$$x = u|x| \qquad (u^*u = r(x) = r(|x|)).$$

The polar decomposition of x^* in W is then

$$x^* = u^*|x^*| \qquad (uu^* = r(x^*) = r(|x^*|)).$$

We have $u^*x = |x| = x^*u$, $xu^* = |x^*| = ux^*$.

Let $A(x)$ denote the norm closure of xAx. Retaining these notations below, we have:

Lemma 2.3. *uWu is a von Neumann algebra with identity u and with product and standard conjugation given by $a \bullet b = aub$, $a^\sharp = ua^*u$. Moreover, u is the range projection of x in uWu.*

Proof. See [9, Lemmas 3.2, 3.3] □

Lemma 2.4. *$A(x)$ is a C*-subalgebra of the von Neumann algebra uWu. Moreover, $x \in A(x)_+$ and $A(x)$ is weak* dense in uWu.*

Proof. If p and q are polynomials with zero constant term we have $p(xx^*)Aq(x^*x)$ is contained in $A(x)$. Thus if $\alpha, \beta > 0$, functional calculus gives $|x^*|^\alpha A|x|^\beta \subseteq A(x)$ and similarly that $|x^*|^\alpha x$ and $x|x|^\beta$ lie in $A(x)$.

Given $a, b \in A$ we have $(xax) \bullet (xbx) = x(ax|x|b)x \in A(x)$ and $(xax)^\sharp = ux^*a^*x^*u = |x^*|a^*|x| \in A(x)$. It follows that $A(x)$ is closed under the product and standard conjugation on uWu and so is a C*-subalgebra of uWu.

Since $ux^*x = |x^*|x \in A(x)$, we have $up(x^*x) \in A(x)$ for every polynomial p with zero constant term and thus $u|x|^{\frac{1}{2}} \in A(x)$ by functional calculus. Now $(u|x|^{\frac{1}{2}})^\sharp = u|x|^{\frac{1}{2}}u^*u = u|x|^{\frac{1}{2}}$, $u|x|^{\frac{1}{2}} \bullet u|x|^{\frac{1}{2}} = u|x| = x$.

Therefore $x \in A(x)_+$. Since u is the range projection of x in uWu, by Lemma 2.3, we have $A(x)$ is weak* dense in uWu. □

Consider now the weak* continuous extension, $\pi : A^{**} \to W$, of the inclusion $A \hookrightarrow W$. Let v be the range projection of x in A^{**} with u the range projection of x in W as before. Then $\pi(v) = u$ and by restriction, $\pi : vA^{**}v \to uWu$ is a weak* continuous $*$-homomorphism of von Neumann algebras. The set $A(x)$ is a weak* dense C*-subalgebra in both $vA^{**}v$ and $uA^{**}u$, as above. But π acts identically on $A(x)$. It follows that the above C*-structure on $A(x)$ is independent of any faithful representation of A in a von Neumann algebra.

We refer to $A(x)$ as the *C*-homotope* of A with respect to x. If $x \in A_+$, then $A(x)$ is a C*-subalgebra of A.

If W is a type I factor then so is uWu. For if z is a central projection of uWu and e is a minimal projection of W we have

$$0 = (u - z) \bullet uWu \bullet z = (u - z)Wz,$$

so that $u - z = 0$ or $z = 0$, if W is a factor, and

$$(ueu) \bullet uWu \bullet (ueu) = ue(u^*Wu^*)eu = u(\mathbb{C}e)u = \mathbb{C}ueu.$$

Proposition 2.5. *Let $\pi : A \to B(H)$ be an irreducible $*$-representation, where A is a C^*-algebra. Let $x \in A$ and let v be the range projection of $\pi(x)$ in $B(H)$. Then by restriction $\pi : A(x) \to vB(H)v$ is a type I factor representation of $A(x)$.*

Proof. Let $\widetilde{\pi} : A^{**} \to B(H)$ be the weak* continuous extension of π. Let u be the range projection of x in A. Then $\widetilde{\pi} : uA^{**}u \to vB(H)v$ is a weak* continuous $*$-homomorphism onto the type I factor $vB(H)v$ and $\pi(A(x))$ is weak* dense in $vB(H)v$. □

Proposition 2.6. *Let A be a C^*-algebra with a conjugation τ. Then τ is a positive conjugation on the C^*-homotope $A(x)$, for all $x \in A^\tau$.*

Proof. Let x be in A^τ. Then $\tau(xa^*x) = x\tau(a)^*x$ for each $a \in A$, implying that τ restricts to a conjugate linear isometry of $A(x)$. By Lemma 2.3 (with $W = A^{**}$) and Lemma 2.4 $x \in A(x)_+$ with range projection the identity element of $A(x)^{**} = uA^{**}u$. □

A more direct reduction to positive conjugations is possible for von Neumann algebras.

Theorem 2.7. *Let W be a von Neumann algebra with conjugation τ. Then there is a positive conjugation σ on W such that W^τ is linearly isometric to W^σ.*

Proof. It follows from [14, Theorem 7] that $\tau(x)^* = u\psi(x)$ for all x in W, where $\psi : W \to W$ is a Jordan $*$-isomorphism and u is a unitary. Since $\tau(1) = u^*$ we have $1 = \tau(u^*) = \psi(u)u^*$ giving $\psi(u) = u$ and hence that $W^*(u) \subset W^\psi$, where $W^*(u)$ is the (abelian) von Neumann subalgebra of W generated by u. By spectral theory $u = e^{ia}$ for some self-adjoint element a in $W^*(u)$. Thus with $v = e^{i\frac{a}{2}}$, we have $v \in W^*(u)$ and $v^2 = u$.

Define $\varphi : W \to W$ by $\varphi(x) = v\psi(x)v^*$. Let $x \in W$. We have, since $\varphi(v) = v$,

$$\tau(x) = \psi(x)^*u^* = v^*\varphi(x)^*vu^* = v^*\varphi(x)^*v^* = \varphi(vxv)^*.$$

In turn,

$$x = \tau(\varphi(vxv)^*) = \varphi\left(v(\varphi(vxv)^*)v\right)^* = \varphi^2(x).$$

Hence, φ is a Jordan $*$-involution. Choose (as before) a unitary $w \in W^*(v)$ such that $w^2 = v$, and consider the positive conjugation, $\sigma = s \circ \varphi$. We claim that $W^\sigma = wW^\tau w$.

If $x \in W^\sigma$ then $\varphi(x) = x^*$ so that, since $w^2 = v$ and $\varphi(w) = w$, we have

$$\tau(w^*xw^*) = \varphi\left(v(w^*xw^*)v\right)^* = \varphi(wxw)^* = w^*xw^*,$$

giving $x \in wW^\tau w$, thereby proving $W^\sigma \subseteq wW^\tau w$. On the other hand, if $x \in W^\tau$ then

$$x^* = \tau(x)^* = \varphi(vxv) = w\varphi(wxw)w,$$

so that $\varphi(wxw) = (wxw)^*$ and thus $x \in wW^\sigma w$. It follows that $W^\sigma = wW^\tau w$ and, since $x \mapsto wxw$ is an isometry, the proof is complete. □

3. Type I Structure

If $\pi : A \to A$ is a Jordan $*$-involution where A is a C*-algebra we shall continue to use π to denote its bitranspose extension on A^{**}.

Let A be a C*-algebra with Jordan $*$-involution π. Then A^π is a JC*-subalgebra of A and is the image of the positive unital bicontractive projection $\frac{1}{2}(\pi + id)$ on A. If the latter is a von Neumann algebra then $\frac{1}{2}(\pi + id)$ is weak* continuous and A^π is a JW*-subalgebra. By these remarks the first result below is immediate from [19, Lemma 7].

Lemma 3.1. *Let W be a von Neumann algebra with Jordan $*$-involution π. Suppose that z is a minimal central projection of W.*

(*a*) *If $\pi(z) = z$ then z is either minimal central in W^π or is the sum of two minimal central projections of W^π.*
(*b*) *If $\pi(z) \neq z$ then $z + \pi(z)$ is either a minimal central projection of W^π or is the sum of two minimal central projections of W^π.* □

Lemma 3.2. *Let W be a von Neumann algebra with Jordan $*$-involution π such that W^π is a type I_n factor where $n < \infty$. Then W is $*$-isomorphic to $M_n(\mathbb{C})$, $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ or $M_{2n}(\mathbb{C})$.*

Proof. When $n = 1$ this is [19, Lemma 1] (see also [17, Proposition 2.6]). In general we have $e_1 + \ldots + e_n = 1$ where the e_i are minimal projections of W^π. For each i we have $e_1 \sim e_i$ and so e_1We_1 is $*$-isomorphic to e_iWe_i. Since $\pi : e_1We_1 \to e_1We_1$ and $(e_1We_1)^\pi = e_1W^\pi e_1$, by the case for $n = 1$ there are these possibilities; $e_1We_1 \cong \mathbb{C}$, $\mathbb{C} \oplus \mathbb{C}$ or $M_2(\mathbb{C})$.

In the first of these cases the e_i are minimal in W giving $W \cong M_n(\mathbb{C})$. If $e_1We_1 \cong \mathbb{C} \oplus \mathbb{C}$ then, since $Z(e_1We_1) = e_1Z(W)e_1$, there exist non-zero central projections z_1 and z_2 in W such that $e_1 = (z_1 + z_2)e_1$. Thus for each i, $e_i = (z_1 + z_2)e_i$ since $e_i \sim e_1$. It follows that $z_1 + z_2 = 1$, that $Wz_1 \cong Wz_2 \cong M_n(\mathbb{C})$ and hence that $W \cong M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$.

In the final case each $e_i = p_i + q_i$, where the p_i and q_i are minimal projections in e_iWe_i and hence minimal in W, giving $W \cong M_{2n}(\mathbb{C})$. $\square$

We shall make repeated use of the evident fact that every JW*-subalgebra of a type I finite JW*-algebra is again type I finite.

Proposition 3.3. *Let A be a C*-algebra with a Jordan $*$-involution π. Then $(A^{**})^\pi$ is type I finite if and only if A^{**} is type I finite.*

Proof. If A^{**} is type I finite then so is $(A^{**})^\pi$ by the above remark.

Suppose that $(A^{**})^\pi$ $(= (A^\pi)^{**})$ is type I finite. Let z be the central projection of A^{**} for which $A^{**}z$ is the atomic part of A^{**}. Being a Jordan $*$-automorphism of A^{**}, π must preserve the atomic part and so $\pi(z) = z$. Denoting $A^{**}z$ and $(A^{**})^\pi z$ by W and N, respectively, we have that π is a Jordan $*$-involution on W with $W^\pi = N$. By assumption, N is a direct sum of type I_n factors where $n < \infty$. Therefore, by Lemma 3.2, if e and f are minimal central projections of N then eWe and fWf are finite dimensional (in particular, by Lemma 3.2, any spin factor summand of N is contained in a copy of $M_4(\mathbb{C})$, a fact also seen from [19, Lemma 5]) and so $(e+f)W(e+f)$ is finite dimensional. It now follows from Lemma 3.1 that W must be a sum of finite dimensional type I factors.This implies that all irreducible $*$-representations of A are finite dimensional and therefore that A^{**} is type I finite. $\square$

Proposition 3.4. *Let W be a von Neumann algebra with a Jordan $*$-involution π. Then W^π is type I if and only if W is type I.*

Proof. Let W be type I. Suppose W^π has a non-zero central projection z such that $W^\pi z$ is continuous. Being type I, zWz contains a non-zero abelian projection e. Let f be the projection $e \vee \pi(e)$. Then fWf is type I finite and $f = \pi(f) \leq z$. Thus, $fW^\pi f$ is type I finite contained in $W^\pi z$, a contradiction. Therefore, W^π is type I.

In order to show the converse, suppose first that W^π is abelian. Then $(W^\pi)^{**} = (W^{**})^\pi$ is abelian and so of type I finite, in particular. Therefore W^{**} is type I finite, by Lemma 3.2, and hence W is type I finite being a quotient of W^{**}.

Now suppose that W^π is type I. The required remaining argument is now virtually that of [18, Theorem 5.5]. Thus, if Wz is the continuous part of W we have $\pi(z) = z$. Therefore, $(Wz)^\pi = W^\pi z$ is a summand of W^π and so is type I and therefore contains a non-zero abelian projection e, if $z \neq 0$, implying that eWe is type I by the first part of the proof. Therefore, $z = 0$. Hence, W is type I. □

We note the following in passing.

Corollary 3.5. *Let W be a von Neumann algebra with a Jordan $*$-involution π. Then W^π is type I finite if and only if W is type I finite.*

Proof. Let W^π be type I finite and denote it by M. We may suppose $M = (\sum Mz_n)_\infty$ where the z_n are orthogonal central projections in M such that Mz_n is type I_n for each n. Let B denote the c_0-sum $(\sum Mz_n)_0$ and let b denote the element $\sum 2^{-n} z_n \in B$. Then $r(b) = 1$ and so $W(b)$ is weak* dense in W. In addition, $\pi : W(b) \to W(b)$ and $W(b)^\pi \subset B$, the latter because $bMb \subset B$.

Each $(Mz_n)^{**}$ is type I_n and so $B^{**} = (\sum (Mz_n)^{**})_\infty$ is type I finite. Therefore, $(W(b)^{**})^\pi = (W(b)^\pi)^{**}$ is type I finite implying that $W(b)^{**}$ is type I finite by Proposition 3.3. Hence, since it contains $W(b)$ as a weak* dense subalgebra, W is type I finite. The converse is clear. □

In essence, the following is [21, Theorem 1.6].

Proposition 3.6. *Let M be a JW-subalgebra of $B(H)_{sa}$ without type I finite part and let R be the weakly closed real $*$-algebra generated by M in $B(H)$. Let W be the formal complexification of R (not necessarily the von Neumann algebra generated by R in $B(H)$). Then there exists a von Neumann algebra N and a weak* continuous positive unital bicontractive projection, $P : M \to M$, such that $P(M)$ is a JW-subalgebra of M Jordan isomorphic to N_{sa} and such that W is $*$-isomorphic to $M_2(N)$.*

Moreover, N has no type I finite part. If M has no type I part then N has no type I part.

Proof. Since M has no finite type I part we have $M = R_{sa}$ [12, Proposition 7.3.3]. Thus, (see [21, Theorem 1.6]) there is a real *-isomorphism $\varphi : R \to M_2(S)$ for some real von Neumann algebra S. Put $N = S \oplus iS$. Then $W = R \oplus iR \cong M_2(N)$. Now embed N as a real von Neumann subalgebra of $M_2(S)$ via $x + iy \mapsto \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$.

Consider the real *-automorphism π of order 2 of $M_2(S)$ given by $\pi(x) = uxu^*$, where $u = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Then $M_2(S)^\pi$ coincides with the image of N in $M_2(S)$, and the corresponding self-adjoint parts also coincide. Thus $P(M_2(S)^\pi_{sa}) = N_{sa}$, where P denotes $\frac{1}{2}(\pi + id)$. By pulling back P to R via φ the desired result obtains. The final statement follows from type theory. □

4. DP1

If A is a C*-algebra, $\widetilde{A}$ denotes the C*-subalgebra of A^{**} generated by A and 1.

Proposition 4.1. *Let A be a C*-algebra with a Jordan $*$-involution π such that A^π_{sa} has the DP. Then A has the DP.*

Proof. Since A^π_{sa} is the fixed point set of the standard conjugation on A^π the latter has the DP by Lemma 1.2. If $1 \notin A$ and $(a_n + \lambda_n 1)$ and ρ_n are weakly null sequences in $(\widetilde{A})^\pi$ and $((\widetilde{A})^\pi)^*$, respectively, where each $a_n \in A^\pi$, then $\lambda_n \to 0$ and (a_n) is weakly null implying that $\rho_n(a_n + \lambda_n 1) \to 0$ and in turn that $(\widetilde{A})^\pi_{sa}$ has the DP. Thus we may suppose that $1 \in A$. Let $\psi : A^\pi \to B(H)$ be an irreducible Jordan $*$-representation. By [7, Proposition 8] and [3, Theorem 5.5] we have $\psi(A^\pi) \subseteq K(H)$ and so H is finite dimensional since $1 \in A^\pi$. It follows that all irreducible $*$-representations of the universal enveloping C*-algebra [12, §7] $C^*(A^\pi)$ of A^π are finite dimensional and thus that $(C^*(A^\pi))^{**}$ is type I finite. Hence, $(A^\pi)^{**}$ is type I finite, as therefore is A^{**} by Proposition 3.3, so that A has the DP by [11, Theorem 1]. □

Lemma 4.2. *Let A be a C*-algebra with a Jordan $*$-involution such that A^π_{sa} has the DP1. Let x and y be non-zero positive elements in $(\widetilde{A})^\pi$ such that $xy = 0$. Then $A(x)$ and $A(y)$ have the DP.*

Proof. Let (a_n) and (ρ_n) be weakly null self-adjoint sequences in $A(x)^\pi$ and $(A(x)^\pi)^*$, respectively. We have $(A(x)^{**})^\pi = e(A^{**})^\pi e$, where e is the range projection of x in A^{**}. For each n define $\varphi_n \in (A^\pi_{sa})'$ by $\varphi_n(a) = \rho_n(eae)$.

Choose a non-zero positive element $a \in A^\pi$ such that $ay \neq 0$ and put $z = \|yay\|^{-1} yay$. Suppose, as we may, that $\|a_n\| \leq 1$ for all n. We have $a_n \leq e$ for all n, and since $zx = 0$ we have $z \leq 1 - e$. Therefore

$$\|z + a_n\| = \max\{\|z\|, \|a_n\|\} = 1, \text{ for all } n.$$

Now $z + a_n \to z$ weakly in A^π_{sa} and (φ_n) is weakly null. Therefore, by hypothesis $\rho_n(a_n) = \varphi_n(z + a_n) \to 0$. This proves that $A(x)^\pi_{sa}$ has the DP and hence that $A(x)$ has the DP by Proposition 4.1. □

Corollary 4.3. *Let A be a C*-algebra with positive conjugation τ such that A^τ has the DP1. Then A has the DP.*

Proof. Using Lemma 2.1, let π denote the Jordan $*$-involution $s \circ \tau$. Then $A^\pi_{sa} = A^\tau_{sa}$ is complemented in A^τ and so has the DP1. It is sufficient to show that there exists a positive element a in $\widetilde{A}$ such that $A(a)$ and $A(1-a)$ have the DP since, by Lemma 4.2, this will imply that all irreducible $*$-representations of A are finite dimensional and consequently that A has the DP [6] and [11].

Since otherwise trivial, suppose that $(\widetilde{A})^\pi$ contains a non-zero non-invertible element x with $0 \leq x \leq 1$. If $\sigma(x)$ is not connected then $(\widetilde{A})^\pi$ contains a non-trivial projection e so that $A(e)$ and $A(1-e)$ have the DP by Lemma 4.2. If $\sigma(x)$ is connected then $\sigma(x) = [0,1]$ and we can choose continuous functions $f, g, h : [0,1] \to [0,1]$ such that $f \leq g \leq h$, $fg = f$, $gh = h$ with $f \neq 0$ and $h \neq 1$. Letting a, b and c be $f(x), g(x)$ and $h(x)$, respectively, we have $a(1-b) = b(1-c) = 0$. Now Lemma 4.2 implies $A(1-b)$ and $A(b)$ have the DP, as required. □

Let τ be a conjugation on a C*-algebra A and, in the notation of §1, let $\widetilde{\tau}$ be the associated conjugation on A^*. When each $\rho \in (A^\tau)'$ is identified with its unique complex linear extension in $(A^*)^{\widetilde{\tau}}$ we have the identification $(A^\tau)' = (A^*)^{\widetilde{\tau}}$ and correspondingly, $(A^\tau)'' = ((A^*)^{\widetilde{\tau}})' = (A^{**})^{\widetilde{\widetilde{\tau}}} = (A^{**})^\tau$.

Lemma 4.4. *Let A be a C*-algebra with conjugation τ such that A^τ has the DP1. Let $x \in A^\tau$. Then $A(x)^\tau$ has the DP1.*

Proof. By Lemma 2.4, $A(x)^{**} = uA^{**}u$ where u is the partial isometry arising in the polar decomposition of x in A^{**}. Let $a_n \to a$ weakly in $A(x)^\tau$ where $\|a_n\| = \|a\| = 1$ for all n, and let (ρ_n) be a weakly null sequence in $(A^*)^{\widetilde{\tau}}$. For each n define $\varphi_n \in A^*$ by $\varphi_n(a) = \rho_n(uu^*au^*u)$. Then (φ_n) is weakly null in A^*.

Since $\tau(u) = 1$ we have $\tau(uu^*au^*u) = \tau(a)$ for each $a \in A^{**}$. Hence $\widetilde{\tau}(\varphi_n) = \varphi_n$ so that $\varphi_n \in (A^*)^{\widetilde{\tau}}$ for each n. Therefore, $\rho_n(a_n) = \varphi_n(a_n) \to 0$ by hypothesis. Hence, $A(x)^\tau$ has the DP1. □

We are now ready to prove our main results.

Theorem 4.5. *Let A be a C*-algebra with a conjugation τ. Then the following are equivalent:*

(*a*) *A^τ has the DP1.*
(*b*) *A has the DP.*
(*c*) *A^τ has the DP.*

(*d*) $(A^\tau)'$ *has the DP.*

Proof. $(a) \Rightarrow (b)$. Let A^τ have the DP1 and let $\pi : A \to B(H)$ be an irreducible $*$-representation. It is enough to show that π is finite dimensional. Let $x \in A^\tau$. By Lemma 4.4, $A(x)^\tau$ has the DP1. Therefore $A(x)$ has the DP by Proposition 2.5 together with Corollary 4.3. Hence, $\pi(x)$ has finite rank in the induced type I factor representation on the homotope $A(x)$ described in Proposition 2.6 and by construction $\pi(x)$ has finite rank in $B(H)$. It follows that $\pi(a)$ has finite rank in $B(H)$ for all $a \in A = A^\tau + iA^\tau$ so that π is finite dimensional, as required.

$(b) \Rightarrow (c)$. A^τ is complemented in A.

$(c) \Rightarrow (a)$. This is immediate.

$(b) \Rightarrow (d)$. If A has the DP then so does A^* by [6]. Thus, since $(A^\tau)' = (A^*)^{\tilde{\tau}}$ is complemented in A^*, $(A^\tau)'$ has the DP.

$(d) \Rightarrow (c)$. This is clear by [8, Corollary 2]. □

If W is a von Neumann algebra with conjugation τ then W^τ has unique predual $(W^\tau)_{,} \cong (W_*)^{\tilde{\tau}}$ (see [13, §4] and [16]).

Theorem 4.6. *Let W be a von Neumann algebra with involution τ. Then the following are equivalent*

(*a*) $(W^\tau)_{,}$ *has the DP1.*
(*b*) W_* *has the DP1.*
(*c*) W *is type I.*

Proof. The equivalence of (b) and (c) was shown in [4, Theorem 6]. The implication $(b) \Rightarrow (c)$ is immediate from the fact that $(W^\tau)_{,}$ is complemented in W_*.

$(a) \Rightarrow (c)$. Let $(W^\tau)_{,}$ have the DP1. By Lemma 2.1 we may suppose that τ is positive and thus that $W^\tau_{sa} = W^\pi_{sa}$, where π is the Jordan $*$-involution associated to τ via Lemma 2.1. Since complemented in $(W^\tau)_{,}$, $(W^\pi_{sa})_{,}$ has the DP1. In order to derive a contradiction, suppose that W^π_{sa} contains a non-zero continuous direct summand M. By Proposition 3.6 there is a weak* continuous contractive projection P on M such that $P(M)$ is isometric to N_{sa} for some continuous von Neumann algebra N. The predual of N_{sa} has the DP1 since it is complemented in the predual of M which is complemented in $(W^\pi_{sa})_{,}$. By [10, Proposition 2.1(b)], in order to determine the DP1 status of N_* it is sufficient to consider weak convergence of sequences of normal states (which lie in $(N_{sa})_{,}$). It follows that N_* has the DP1 and therefore that N is type I, by $(b) \Rightarrow (c)$. This contradiction completes the proof. □

Theorem 4.7. *Let A be a C*-algebra with conjugation τ. Then the following are equivalent*

(*a*) $(A^{\tau})'$ *has the DP1.*
(*b*) A^* *has the DP1.*
(*c*) A *is postliminal.*

Proof. Since $(A^{\tau})'$ is the predual of $(A^{**})^{\tau}$, the equivalence of (a) and (b) is immediate from Theorem 4.6 $(a) \Leftrightarrow (b)$. The equivalence of (b) and (c) was proved in [4, Corollary 7]. □

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