THE ALTERNATIVE DUNFORD-PETTIS PROPERTY FOR SUBSPACES OF THE COMPACT OPERATORS

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Abstract. A Banach space X has the alternative Dunford-Pettis property if for every weakly convergent sequences $(x_n) \to x$ in X and $(x_n^*) \to 0$ in X^* with $\|x_n\| = \|x\| = 1$ we have $(x_n^*(x_n)) \to 0$. We get a characterization of certain operator spaces having the alternative Dunford-Pettis property. As a consequence of this result, if H is a Hilbert space we show that a closed subspace M of the compact operators on H has the alternative Dunford-Pettis property if, and only if, for any $h \in H$, the evaluation operators from M to H given by $S \mapsto Sh$, $S \mapsto S^t h$ are DP1 operators, that is, they apply weakly convergent sequences in the unit sphere whose limits are also in the unit sphere into norm convergent sequences. We also prove a characterization of certain closed subalgebras of $K(H)$ having the alternative Dunford-Pettis property by assuming that the multiplication operators are DP1.

1. Introduction

A Banach space X has the Dunford-Pettis property (DP in the sequel) if for any Banach space Y, every weakly compact operator from X to Y is completely continuous, that is it maps weakly compact subsets of X onto norm compact subsets of Y. The DP was introduced by Grothendieck who also showed that a Banach space X has the DP if, and only if, for every weakly null sequences (x_n) in X and (x_n^*) in X^* we have $x_n^*(x_n) \to 0$. Since its introduction by Grothendieck, the DP has had an important development. We refer to [10] as an excellent survey on the DP and to [2], [17], [4], [7], and [8] for more recent results.

Recently, S. Brown and A. Ülger (see [3, 15]) have studied the DP for subspaces of the compact operators on an arbitrary Hilbert space. Indeed, if M is a closed subspace of the compact operators in a Hilbert space H, then M has the DP if, and only if, for any $h \in H$ the point evaluation $\{S(h) : S \in B_M\}$ and $\{S^*(h) : S \in B_M\}$ are relatively compact in H, where B_M is the closed unit ball in M, equivalently, if and only if, for any $h \in H$ the evaluation operators given by

$$\begin{array}{cc} M \longrightarrow H & M \longrightarrow H \\ S \mapsto Sh & S \mapsto S^t h \end{array}$$

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are completely continuous operators. In [16], E. Saksman and H.O. Tylli have extended this characterization for closed subspaces of the compact operators in ℓ^p $(1 < p < \infty)$. Saksman and Tylli also showed that if M is a closed subspace of the compact operators in a reflexive Banach space X having the DP, then for every $x \in X$ the evaluation operators given above are completely continuous.

In [12] W. Freedman introduces a weaker version of the DP, *the alternative Dunford-Pettis property* (DP1 in the sequel). A Banach space X has the DP1 iff for every weakly convergent sequences $(x_n) \to x$ in X and $(x_n^*) \to 0$ in X^* with $\|x_n\| = \|x\| = 1$ we have $x_n^*(x_n) \to 0$. The DP and the DP1 properties are equivalent for von Neumann algebras and C*-algebras (see [12] and [5]) but the DP1 is strictly weaker than the DP for preduals of von Neumann algebras and general Banach spaces. We refer to [13], [1], [5], and [6] for the most recent results on the DP1.

The DP1 was also characterized by W. Freedman in the following terms. A Banach space X has the DP1 if and only if for any Banach space Y, every weakly compact operator T from X to Y is a DP1 operator, that is, if (x_n) converges weakly to x in X with $\|x_n\| = \|x\| = 1$ we have $(T(x_n))$ converges to $T(x)$ in norm. Therefore, the DP1 operators play with respect to the DP1 the same role that completely continuous operators with respect to the DP.

In this paper we study the DP1 for closed subspaces of the compact operators in a Banach space. In the case that M is a closed subspace of the compact operators on a reflexive Banach space X we prove that a necessary condition on M to have the DP1 is that for every $x \in X$ and $x^* \in X^*$ the evaluation operators given by

$$\begin{array}{cc} M \longrightarrow X & M \longrightarrow X^* \\ S \mapsto Sx & S \mapsto S^* x^* \end{array}$$

are DP1 operators.

If X is a reflexive Banach space having a Schauder basis and M is a closed subspace of the compact operators on X having the $\mathcal{P}$-property (defined below) we show that the necessary condition given above if also sufficient. For a closed subspace M of the compact operators on a (non necessarily separable) Hilbert space H, we prove that the above characterization is also valid.

For any subalgebra A of the compact operators on a Hilbert space the multiplication operators by elements of A are DP1 whenever A has DP1. Under a very mild condition on the algebra A we show that the converse is true.

2. The Results

In the following, if X is a Banach space, we will denote by B_X, S_X the closed unit ball and the unit sphere of X, respectively. Along the paper H is a Hilbert space. $L(X, Y)$ will be the Banach space of all bounded linear operators from the Banach space X to a Banach space Y and $K(X,Y)$ will denote the closed subspace of all compact operators.

Definition 2.1. Let X and Y be Banach spaces. A bounded linear operator $T : X \longrightarrow Y$ is said to be DP1 operator if whenever $x_n \to x$ weakly in X with $\|x_n\| = \|x\| = 1$ we have $\|Tx_n - Tx\| \to 0$.

The following proposition gives us a necessary condition, in a closed subspace of the Banach space of all operators on a reflexive Banach space, to have the DP1.

Proposition 2.2. *Let X and Y be reflexive Banach spaces, $M \subseteq L(X,Y)$ be a closed subspace. Suppose that M has the DP1 then for every $x \in X$ and $y^* \in Y^*$ the evaluation operators given by*

$$M \longrightarrow Y \quad M \longrightarrow X^*$$

$$S \mapsto Sx \quad S \mapsto S^*y^*$$

are DP1 operators.

Proof. Assume that M has the DP1, then by using [12, Theorem 1.4], any bounded linear operator from M into any reflexive Banach space is, in fact, DP1. Hence, for any $x \in X$, the operator $T : M \longrightarrow Y$ given by

$$T(S) = Sx \qquad (S \in M),$$

is in fact DP1. The corresponding assertion for the other operator follows from an analogous argument. □

Remark 2.3.

(1) X has the DP1 if and only if for every weakly null sequences x_n in X and x_n^* in X^*, $x \in S_X$ with $\|x_n + x\| = 1$ we have $x_n^*(x_n) \to 0$.

(2) Let X, Y be Banach spaces and $M \subseteq L(X,Y)$ be a closed subspace. Suppose that for every $x \in X$ and $y^* \in Y^*$ the evaluation operators given by

$$M \longrightarrow Y \quad M \longrightarrow X^*$$
$$S \mapsto Sx \quad S \mapsto S^*y^*$$

are DP1 operators. Then if (T_n) converges weakly to T in M with $\|T_n\| = \|T\| = 1$, then for every $x \in X$ we have $\|T_n(x) - T(x)\| \to 0$, that is, (T_n) converges to T strongly. Since the sequence $\{T_n\}$ is bounded, T_n converges to T uniformly on compact subsets. Hence, T_nS converges in norm to TS for every $S \in K(X)$. Similarly, T_n^*K converges in norm to T^*K, for every $K \in K(Y^*)$. Therefore, for every $S \in K(X)$ and $K \in K(Y^*)$, the mappings

$$M \longrightarrow K(X,Y) \quad M \longrightarrow K(Y^*, X^*)$$
$$T \mapsto TS \qquad T \mapsto T^*K$$

are DP1 operators.

W. Freedman characterized Banach spaces having the DP1 as those Banach spaces X for which for any Banach space Y any weakly compact operator $T : X \longrightarrow Y$ is a DP1 operator [12]. Here we prove that if X is a Banach space with a shrinking Schauder basis, then for subspaces M of $K(X)$ satisfying certain isometric assumption, it is enough to assume that the evaluation operators on M are DP1 operators in order to check that M has the DP1.

Lemma 2.4. *Let X be a Banach space with a shrinking (Schauder) basis $\{x_n\}$ and Y be a Banach space with basis $\{y_n\}$. Then for every $S \in K(X,Y)$, the sequence*

$$\|S - Q_n S P_n\| \to 0$$

where Q_n, P_n are the projections onto the subspaces generated by $\{y_1, \cdots, y_n\}$ and $\{x_1, \cdots, x_n\}$, respectively.

Proof. Since $(Q_n(y)) \to y$ for any $y \in Y$ and (Q_n) is bounded, then (Q_n) converges uniformly on compact subsets. This gives that $\Big\|(I - Q_n)S\Big\| \to 0$.

Since the basis of X is shrinking, by using the same argument for S^*, which is also compact, it follows that the sequence

$$\|Q_n S P_n - S\| = \|P_n^* S^* Q_n^* - S^*\| \leq \|P_n^* S^* Q_n^* - P_n^* S^*\| + \|(P_n^* - I)S^*\| \leq$$
$$\leq \|P_n\| \; \|(I - Q_n)S\| + \|(P_n^* - I)S^*\|$$

converges to zero. $\square$

Remark 2.5. By using the same argument, if H is any Hilbert space and $S \in K(H)$, then for every $\varepsilon > 0$ there is a finite dimensional subspace $V \subset H$ such that $\|S - P_V S P_V\| < \varepsilon$, where P_V is the orthogonal projection onto V.

According to [14], given two Banach spaces with Schauder basis X and Y, we say that a closed subspace M of $L(X,Y)$ has the $\mathcal{P}$-property if for all natural numbers m, n and every operators $T, S \in M$

$$\|Q_W T P_V + (I - Q_W) S (I - P_V)\| \leq \max\{\|Q_W T P_V\|, \|(I - Q_W) S (I - P_V)\|\},$$

where V, W are the subspaces of X, Y, respectively, generated by the first m, n vectors of the basis and P_V, Q_W are the canonical projections onto V and W, respectively.

Theorem 2.6. *Let X be a Banach space with a shrinking Schauder basis, Y a Banach space with basis and assume that M is a closed subspace of $K(X,Y)$ satisfying the $\mathcal{P}$-property. If for any $x \in X$, $y^* \in Y^*$, the evaluation operators given by*

$$M \longrightarrow Y \quad M \longrightarrow X^*$$

$$S \mapsto Sx \quad S \mapsto S^* y^*$$

are DP1 operators, then M satisfies the DP1.

Proof. Let us denote by $\{x_n\}$ and $\{y_n\}$ the Schauder bases of X and Y, respectively. We will argue by contradiction, so we assume that M does not satisfy the DP1. Therefore, assume there are $r > 0$, a weakly null sequence T_n in M, an element $T \in S_M$ and a weakly null sequence $m_n^* \in M^*$ satisfying that

$$\|T_n + T\| = 1, \qquad |m_n^*(T_n)| \geq r, \qquad \forall n \in \mathbb{N}. \tag{1}$$

By assumption, the evaluations at elements of X and Y^* are DP1 operators from M into Y and X^*, respectively. For a fixed natural number k, we denote by $V = [x_1, \cdots, x_k] \subset X, W = [y_1, \cdots, y_k] \subset Y$. By Remark 2.3, the operators from M to $K(X, Y)$ given by $T \mapsto TP_V$ and $T \mapsto Q_W T$, respectively, are DP1 operators, where P_V and Q_W denote the projections of X and Y onto V and W, respectively. Hence $\|T_n P_V\|, \|Q_W T_n\| \to 0$. As a consequence,

$$\|T_n - (I - Q_W)T_n(I - P_V)\| = \|T_n P_V + Q_W T_n(I - P_V)\| \leq$$
$$\leq \|T_n P_V\| + (1 + C)\ \|Q_W T_n\|,$$

where C is the basic constant of the basis of X. The above inequality implies that the sequence

$$\|T_n - (I - Q_W)T_n(I - P_V)\| \to 0. \tag{2}$$

Now we will construct a subsequence of m_n^* that does not converge weakly to zero. This is a contradiction since m_n^* is weakly null. In order to do this, let us take

$$m_{\sigma(1)}^* = m_1^*, \quad T_{\sigma(1)} = T_1.$$

Since $T_{\sigma(1)} \in K(X, Y)$, by Lemma 2.4 there exists $p_1 \in \mathbb{N}$ such that

$$\|T_{\sigma(1)} - Q_{W_1} T_{\sigma(1)} P_{V_1}\| < \frac{1}{2},$$

where V_1 and W_1 are the closed subspaces generated by $\{x_1, \ldots, x_{p_1}\}$ and $\{y_1, \ldots, y_{p_1}\}$, respectively. By using (2) and the fact that m_n^* and T_n are weakly null sequences of M^* and $K(X, Y)$, respectively, for n large enough it is satisfied

$$\|T_n - (I - Q_{W_1})T_n(I - P_{V_1})\| < \frac{1}{2^3}, \quad |m_{\sigma(1)}^*(T_n)| < \frac{r}{2^3}, \quad |m_n^*(T_{\sigma(1)})| < \frac{r}{2^3}.$$

We choose a natural number $\sigma(2) > \sigma(1)$ satisfying the previous conditions. By the choice of $\sigma(2)$ it is satisfied

$$\|T_{\sigma(2)} - (I - Q_{W_1})T_{\sigma(2)}(I - P_{V_1})\| < \frac{1}{2^3} \qquad (3).$$

Since $(I - Q_{W_1})T_{\sigma(2)}(I - P_{V_1})$ is a compact operator, by Lemma 2.4, there exists $p_2 > p_1$ such that

$$\|(I - Q_{W_1})T_{\sigma(2)}(I - P_{V_1}) - Q_{W_2}(I - Q_{W_1})T_{\sigma(2)}(I - P_{V_1})P_{V_2}\| < \frac{1}{2^3},$$

where V_2 and W_2 are the finite dimensional subspaces generated by $\{x_1, \ldots, x_{p_2}\}$ and $\{y_1, \ldots, y_{p_2}\}$, respectively.

In view of (3) we get

$$\|T_{\sigma(2)} - (Q_{W_2} - Q_{W_1})T_{\sigma(2)}(P_{V_2} - P_{V_1})\| < \frac{1}{2^2}.$$

Of course, we have chosen $T_{\sigma(2)}, m^*_{\sigma(2)}$ such that they satisfy

$$|m^*_{\sigma(1)}(T_{\sigma(2)})| < \frac{r}{2^3}, \qquad |m^*_{\sigma(2)}(T_{\sigma(1)})| < \frac{r}{2^3}, \qquad |m^*_{\sigma(2)}(T_{\sigma(2)})| \geq r.$$

Assume that we have chosen $p_1 < p_2 < \ldots < p_n \in \mathbb{N}$ and $\sigma(1) < \sigma(2) < \ldots < \sigma(n) \in \mathbb{N}$, satisfying

$$\|T_{\sigma(i)} - (Q_{W_i} - Q_{W_{i-1}})T_{\sigma(i)}(P_{V_i} - P_{V_{i-1}})\| < \frac{1}{2^{i+1}}, \qquad |m^*_{\sigma(i)}(T_{\sigma(i)})| \geq r, \quad (i : 1, \ldots, n)$$

and also

$$|m^*_{\sigma(i)}(T_{\sigma(n)})| < \frac{r}{2^{n+1}}, \qquad |m^*_{\sigma(n)}(T_{\sigma(i)})| < \frac{r}{2^{n+1}} \qquad (i < n), \qquad (4)$$

where V_i and W_i are the closed subspace of X and Y generated by $\{x_1, \ldots, x_{p_i}\}$ and $\{y_1, \ldots, y_{p_i}\}$, respectively.

By (2) we know that for p large enough, we have

$$\|T_p - (I - Q_{W_n})T_p(I - P_{V_n})\| < \frac{1}{2^{n+2}},$$

and also, because T_p, m^*_p are weakly null

$$|m^*_p(T_{\sigma(i)})| < \frac{r}{2^{n+2}}, \qquad |m^*_{\sigma(i)}(T_p)| < \frac{r}{2^{n+2}} \qquad (i \leq n).$$

We choose a natural number $\sigma(n+1) > \sigma(n)$ for which the previous conditions hold.

By Lemma 2.4 we can approximate the compact operator

$$(I - Q_{W_n})T_{\sigma(n+1)}(I - P_{V_n}),$$

so there exists $p_{n+1} > p_n$ such that

$$\|(I-Q_{W_n})T_{\sigma(n+1)}(I-P_{V_n}) - Q_{W_{n+1}}(I-Q_{W_n})T_{\sigma(n+1)}(I-P_{V_n})P_{V_{n+1}}\| < \frac{1}{2^{n+2}},$$

where V_{n+1} and W_{n+1} are the finite dimensional subspaces generated by $\{x_1, \ldots, x_{p_{n+1}}\}$ and $\{y_1, \ldots, y_{p_{n+1}}\}$, respectively. Thus

$$\|T_{\sigma(n+1)} - Q_{W_{n+1}}T_{\sigma(n+1)}P_{V_{n+1}}\| < \frac{1}{2^{n+1}}, \qquad |m^*_{\sigma(n+1)}(T_{\sigma(n+1)})| \geq r,$$

$$|m^*_{\sigma(n+1)}(T_{\sigma(i)})| < \frac{r}{2^{n+2}}, \quad |m^*_{\sigma(i)}(T_{\sigma(n+1)})| < \frac{r}{2^{n+2}} \quad (i < n+1),$$

where V_{n+1} and W_{n+1} are the closed subspace of X and Y generated by $\{x_1, \ldots, x_{p_{n+1}}\}$ and $\{y_1, \ldots, y_{p_{n+1}}\}$, respectively.

Now we will check that $(m^*_{\sigma(n)})$, the subsequence of (m^*_n) we defined, does not converge weakly to zero. First we observe that the subsequence $\sum_{k=1}^n T_{\sigma(k)}$ is bounded, since in view of (4)

$$\left\|\sum_{k=1}^n T_{\sigma(k)}\right\| \leq \left\|\sum_{k=1}^n (Q_{W_k} - Q_{W_{k-1}})T_{\sigma(k)}(P_{V_k} - P_{V_{k-1}})\right\| +$$

$$+\sum_{k=1}^{n}\|(Q_{W_k}-Q_{W_{k-1}})T_{\sigma(k)}(P_{V_k}-P_{V_{k-1}})\| \leq$$

$$\leq \left\|\sum_{k=1}^{n}(Q_{W_k}-Q_{W_{k-1}})T_{\sigma(k)}(P_{V_k}-P_{V_{k-1}})\right\| + \sum_{k=1}^{\infty}\frac{1}{2^k},$$

where $P_{V_0}=0, Q_{W_0}=0$.

Since M has the $\mathcal{P}$-property, then it holds that

$$\left\|\sum_{k=1}^{n}(Q_{W_k}-Q_{W_{k-1}})T_{\sigma(k)}(P_{V_k}-P_{V_{k-1}})\right\| \leq \max_{1\leq k\leq n}\|Q_{W_k}T_{\sigma(k)}P_{V_k}\| \leq$$

$$\leq 4CC' \max_{1\leq k\leq n}\|T_{\sigma(k)}\|,$$

where C' is the basic constant associated to the basis of Y. Therefore, the previous sequence is bounded since $T_{\sigma(n)}$ is a weakly null sequence.

Let $x^{**}\in M^{**}$ be a cluster point of the sequence $\sum_{k=1}^{n}T_{\sigma(k)}$. We will observe that $\big(x^{**}(m^*_{\sigma(n)})\big)$ does not converge to zero.

For $p>n$ we have, in view of (1) and (4), we have

$$\Big|m^*_{\sigma(n)}\big(\sum_{k=1}^{p}T_{\sigma(k)}\big)\Big| \geq |m^*_{\sigma(n)}(T_{\sigma(n)})| - \sum_{\substack{k=1\\k\neq n}}^{p}|m^*_{\sigma(n)}(T_{\sigma(k)})| \geq$$

$$\geq r - \sum_{k=1}^{n-1}|m^*_{\sigma(n)}(T_{\sigma(k)})| - \sum_{k=n+1}^{p}|m^*_{\sigma(n)}(T_{\sigma(k)})| \geq$$

$$\geq r - r\sum_{k=1}^{n-1}\frac{1}{2^{n+1}} - r\sum_{k=n+1}^{p}\frac{1}{2^{k+1}} \geq$$

$$\geq r - r\sum_{k=1}^{\infty}\frac{1}{2^{k+1}} = \frac{r}{2}.$$

Since x^{**} is a weak* cluster point of $\big(\sum_{k=1}^{m}T_{\sigma(k)}\big)$ we get $|x^{**}(m^*_{\sigma(n)})|\geq \frac{r}{2}$, for every natural number n, that is, $\big(m^*_{\sigma(n)}\big)$ is not weakly null, but this is impossible and hence M has the DP1. □

Remark 2.7. There are many pairs of Banach spaces (X,Y) for which $K(X,Y)$ satisfies the $\mathcal{P}$-property. For instance, $K(\ell_1,Y)$, for any Banach space with basis Y. Also $K(\ell_p,\ell_q)$ $(1<p<\infty, q\leq p)$ has the the $\mathcal{P}$-property. Clearly for any Hilbert space H, it is satisfied that

$$\|Q_W T P_V + (I-Q_W)S(I-P_V)\| = \max\{\|T\|,\|S\|\},$$

for any subspaces V,W of H and arbitrary operators $T,S\in L(H)$, where P_V denotes the orthogonal projection onto V. The above equality is the appropriate version of the $\mathcal{P}$-property.

It is well known that every Schauder basis in a reflexive Banach space is shrinking. Actually, M. Zippin proved that a Banach space X with basis is reflexive if, and only if, any basis in X is shrinking [18]. Therefore, our principal result follows now from Proposition 2.2 and Theorem 2.6.

Corollary 2.8. *Let X, Y be reflexive Banach spaces with Schauder basis and M a closed subspace of $K(X,Y)$ satisfying the $\mathcal{P}$-property. Then M has the DP1 if, and only if, for every $x \in X$ and $y^* \in Y^*$ the evaluation operators given by*

$$\begin{array}{cc} M \longrightarrow Y & M \longrightarrow X^* \\ \\ S \mapsto Sx & S \mapsto S^* y^* \end{array}$$

are DP1 operators.

The following result characterizes those closed subspaces of the compact operators on an arbitrary Hilbert space H (non necessarily separable) satisfying the DP1. The proof can be obtained by adapting the proof of Theorem 2.6. We include here and sketch of the proof for completeness.

Theorem 2.9. *Let M be a closed subspace of $K(H)$. Then M has the DP1 if, and only if, for any $h \in H$ the evaluation operators given by*

$$\begin{array}{cc} M \longrightarrow H & M \longrightarrow H \\ \\ S \mapsto Sh & S \mapsto S^t h \end{array}$$

are DP1 operators.

Proof. The 'only if' part follows by Proposition 2.2.

In order to prove the converse it is enough to observe that Remark 2.5 allows us to approximate compact operators by "finite matrices" and also that in this case, the space $K(H)$ satisfies the corresponding version of the $\mathcal{P}$-property (Remark 2.7). With these two ideas in mind, the proof of Theorem 2.6 can be easily adapted to this case. □

Remark 2.10. In order to show that the assumptions of Corollary 2.8 and Theorem 2.9 are sharp, let us observe several examples.

(1) If the space X is not reflexive, there can be subspaces of $K(X)$ satisfying the DP1 such that the evaluation operators are not DP1. Indeed, this is the case of an example appearing in [16]. Take $X = c_0$, and $M = K(c_0)$. Since $M \equiv c_0(\ell_1)$, it follows that M has the DP and hence M also has the DP1. Let $T = e_1^* \otimes e_1$ and $T_n = e_2^* \otimes e_n$ $(n \geq 2)$, where e_i is the natural coordinate basis in c_0 and $e_i^* \otimes e_j((\lambda_n)) := \lambda_i e_j$. Then $\|T\| = \|T + T_n\| = 1$ for every $n \geq 2$, $T_n \to 0$ weakly and $\|T_n(e_2)\| = 1$ $(n \geq 2)$. Therefore, the evaluation operator at e_2 is not DP1.

(2) Finally, for any (infinite-dimensional) Hilbert space H, one we can find a subspace $M \subset L(H)$ which does not satisfy the DP1 although the evaluations at elements of H are DP1 operators from M to H.

Clearly it is enough to check the statement for ℓ_2. Let $H = \ell_2$ and V_n, W_n be pairwise orthogonal subspaces of H such that H can be decomposed into the (orthogonal) sum

$$H = \left(\oplus^{\ell_2} V_n\right) \oplus^{\ell_2} \left(\oplus^{\ell_2} W_n\right)$$

and $\dim(V_n) = \dim(W_n) = n \ \ (n \in \mathbb{N})$. Let M the C*-algebra given by

$$M = \Big(\bigoplus_{n=1}^{\infty} L(V_n)\Big)_{\infty} \oplus_{\infty} \Big(\bigoplus_{n=1}^{\infty} L(W_n)\Big)_{\infty}.$$

By [12, Examples 3.3 (ii)] we know that M does not satisfy the DP1. Given $h \in H$, we denote by $E_h : M \to H$ the evaluation operator at h. To see that E_h is completely continuous, it is enough to check that E_h maps weakly convergent sequences to norm-convergent. Let us fix $h \in H$ and a weakly null sequence (m_k) in M. For every $\varepsilon > 0$ there exist $p \in \mathbb{N}$, $h_i \in V_i$ and $k_i \in W_i$ $(i \leq p)$ such that $\|h - \sum_{i=1}^{p}(h_i + k_i)\| \leq \varepsilon$, that is

$$\|E_h - \sum_{i=1}^{p}(E_{h_i} + E_{k_i})\| \leq \varepsilon. \qquad (*)$$

For any k, $m_k \in M$, and then $m_k(h_i) = P_{V_i} m_k P_{V_i}(h_i)$. Now, since $(P_{V_i} m_k P_{V_i})_k$ is a weakly null sequence in the finite dimensional space $L(V_i)$, it follows that $\lim_{k\to\infty} \|P_{V_i} m_k P_{V_i}\| = 0$. Therefore $E_{h_i} : M \to H$ is a completely continuous operator. The analogous argument shows that E_{k_i} is a completely continuous operator for every $k_i \in W_i$. Finally, since, by $(*)$, E_h is in the norm closure of all completely continuous operators from M to H, we conclude that E_h is completely continuous and hence DP1.

The last example actually provides us an example of a subspace of $L(H)$ which does not satisfy the DP although the evaluations at elements of H are completely continuous operators from M to H. This example shows that the characterization of the DP for subspaces of $K(H)$ obtained by Brown and Ülger [3, 15] or Saksman-Tylli [16] is not valid for subspaces of $L(H)$.

For subspaces of $K(H)$ which are C^*-subalgebras the DP1 can be characterized in the following way:

Proposition 2.11. *Let $A \subset K(H)$ be a C^*-subalgebra. The following conditions are equivalent:*

i) *A has the DP1.*

ii) *For every sequence $(T_n) \subset A$ satisfying that $(T_n) \xrightarrow{w} 0$ and such that for some $T \in A$, it holds $\|T + T_n\| = 1 = \|T\|, \forall n$, then $\|T_n h\| \to 0$ for any $h \in H$.*

iii) *A has the DP.*

iv) *For every sequence $(T_n) \subset A$, which is w-null, then $\|T_n h\| \to 0$ for any $h \in H$.*

Proof. It is known that for C^*-algebras the DP and the DP1 are equivalent (see [5]).

Now we will check that iii) and iv) are equivalent. By using [7] a C^*-algebra has the DP iff for any weakly null sequence (x_n) in the space, it holds that $(x_n^* x_n)$ is also weak null. Therefore, if A has the DP, for any weak null sequence $(T_n) \subset A$, it holds that $T_n^* T_n$ converges weakly to zero, an so, for any $h \in H$, it holds $(T_n^* T_n h|h) \to 0$, that is, $\|T_n h\| \to 0$.

On the other hand, if every weakly null sequence $(T_n) \subset A$ converges to zero in the strong operator topology, then it implies that $(\|T_n^* T_n h\| \to 0$ for any $h \in H$, and so $(T_n^* T_n h_1 | \, h_2) \to 0$ for every $h_1, h_2 \in H$. Since $A \subset K(H)$, this implies that $T_n^* T_n$ converges to zero weakly and by the result of [7], A has the DP.

In order to prove that i) and ii) are equivalent, a similar argument can be used, but replacing the result by Chu-Iochum by [12, Theorem 3.1]. □

Remark 2.12. Theorem 2.9 can be also used to prove the equivalence between i) and ii), since the second condition just says that the pointwise evaluation at elements of H are DP1 operators on the algebra and in this case $A = \{T^t : T \in A\}$.

There are examples of subalgebras of $K(H)$ having the DP1 that are not DP. For instance, let us consider the subspace A generated by the subset

$$\{e_n^* \otimes e_1 : n \in \mathbb{N}\},$$

where $\{e_n\}$ is an orthonormal system of an infinite dimensional Hilbert space H and we denote by

$$(e_n^* \otimes e_1)(x) = (x|e_n)e_1.$$

It is immediate that A is an subalgebra of $K(H)$ isometric to ℓ_2, hence it has the Kadec-Klee property, and so the DP1. Of course, ℓ_2 does not have the DP.

For some special subalgebras of $K(H)$, we will also obtain a characterization of DP1. To begin with, for a reflexive Banach space X, the following result provides necessary conditions in a subalgebra of $K(X)$ to have the DP1.

Proposition 2.13. *Let X be a reflexive Banach space and $A \subset K(X)$ a closed subalgebra of $K(X)$ having the DP1. Then the operators R_S, L_S are DP1 operators for any $S \in A$, where $R_S, L_S : A \longrightarrow A$ are given by*

$$R_S(T) = TS, \qquad L_S(T) = ST, \qquad \forall T \in A.$$

Proof. See Proposition 2.2 and Remark 2.3. □

The following result is a partial converse for a class of subalgebras of $K(H)$ satisfying a very mild condition (compare this result to [15, Theorem 7]).

Proposition 2.14. *Let X be a reflexive Banach space with Schauder basis and $A \subset K(X)$ be a closed subalgebra of $K(X)$ satisfying the $\mathcal{P}$-property. Suppose that the linear subspaces generated by*

$$\{Tx : x \in X, T \in A\}$$

and

$$\{T^*x^* : x^* \in X^*, T \in A\}$$

are dense in X and X^*, respectively and assume that the operators $L_T, R_T : A \longrightarrow A$ are DP1. Then A has the DP1.

Proof. By Corollary 2.8 we have to check that the evaluation operators at elements $x \in X$ and $x^* \in X^*$ are DP1 when they are restricted to A.

Hence, let us fix $x \in X$ and a sequence $(T_n) \subset A$ such that (T_n) is weakly null and for some $S \in S_A$ it holds that $1 = \|S + T_n\|$ for every n. We have to show that $\{T_n x : n \in \mathbb{N}\}$ is relatively compact in X.

By assumption, there are elements $y_1, \ldots, y_n$ in X and $S_1, \ldots, S_n$ in A satisfying

$$\left\|\sum_{i=1}^{m} S_i y_i - x\right\| < \varepsilon.$$

Hence

$$\left\|\sum_{i=1}^{m} T_n S_i y_i - T_n x\right\| < \varepsilon \|T_n\|, \forall n \in \mathbb{N}.$$

Since $\{T_n S_i : n \in \mathbb{N}\}$ is relatively compact, because R_{S_i} is a DP1 operator, also the subset $\{T_n S_i y_i : n \in \mathbb{N}\}$ is relatively compact and this property is preserved by finite sums, therefore by using that

$$\{T_n x : n \in \mathbb{N}\} \subset \left\{\sum_{i=1}^{m} T_n S_i y_i : n \in \mathbb{N}\right\} + \varepsilon \ \sup\{\|T_n\|\} B_X,$$

then $\{T_n x : x \in X\}$ is relatively compact. By a similar argument, if we use the denseness of the subspace generated by $\{T^* x^* : x^* \in X^*, T \in A\}$ it follows that $\{T_n^* x^* : n \in \mathbb{N}\}$ is relatively compact for any $x^* \in X^*$. By Corollary 2.8, the subalgebra A has the DP1. □

Corollary 2.15. *Let X be a reflexive Banach space with Schauder basis and $A \subset K(X)$ a closed subalgebra of $K(X)$ in the hypothesis of Proposition 2.14. Then A has the DP1 if, and only if, the operators $L_T, R_T : A \longrightarrow A$ are DP1.*

When in the proof of Proposition 2.14 we replace Corollary 2.8 by Theorem 2.9, we obtain the following result.

Proposition 2.16. *Let $A \subset K(H)$ be a closed subalgebra of $K(H)$ satisfying that the linear subspaces generated by*

$$\{Th : h \in H, T \in A\}$$

and

$$\{T^*h : h \in H, T \in A\}$$

are dense in H. Assume that the operators $L_T, R_T : A \longrightarrow A$ are DP1 for any T in A. Then A has the DP1.

Corollary 2.17. *Let $A \subset K(H)$ be a closed subalgebra of $K(H)$ satisfying the hypothesis of Proposition 2.16. Then A has the DP1 if, and only if, the operators $L_T, R_T : A \longrightarrow A$ are DP1.*

Remark 2.18. We recall that a Banach space Y is said to have the Schur property if every weakly compact subset of Y is norm compact. It is known that the dual Y^* of a Banach space Y has the Schur property if and only if Y has the DP and does not contain an isomorphic copy of ℓ^1 (see for instance [10, Theorem 3]). Since $K(H)$ does not contain an isomorphic copy of ℓ^1 ([11]), it follows that a closed subspace M of $K(H)$ satisfies the DP if, and only if, M^* has the Schur property if, and only if, for every $h \in H$ the evaluation operators given by

$$\begin{array}{cc} M \longrightarrow H & M \longrightarrow H \\ S \mapsto Sh & S \mapsto S^*h \end{array}$$

are completely continuous (see [15, Corollary 4] and also [3]).

Since the DP1 (respectively, the KKP) can be obtained by confining the DP (respectively, the Schur) condition to the unit sphere, we might expect a relation between DP1 in a closed subspace M of $K(H)$ and the KKP in M^*. However, the KKP in M^* does not imply the DP1 in M. Indeed, let $\{e_n\}$ be an orthonormal sequence in a Hilbert space H. The closed subspace M of $K(H)$ generated by $\{e_1^* \otimes e_1, e_n^* \otimes e_2 : n \geq 2\}$ is isometric to $\mathbb{R} \oplus_\infty \ell_2$ and $e_n^* \otimes e_m(h) := (h|e_n)e_m$. Therefore M does not satisfy the $DP1$ property (see [12, Example 1.6]), however M^* satisfies the KKP since it is isometric to $\mathbb{R} \oplus^{\ell_1} \ell_2$ (see [12, Theorem 1.9]). Another example is $M = K(H)$. The dual of $K(H)$, the Banach space of all trace class operators on H, satisfies the KKP ([9]) and $K(H)$ does not satisfy the DP1 whenever H is infinite dimensional.

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