THE ALTERNATIVE DUNFORD-PETTIS PROPERTY IN C*-ALGEBRAS AND VON NEUMANN PREDUALS

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ABSTRACT. A Banach space X is said to have the alternative Dunford-Pettis property if, whenever a sequence $x_n \to x$ weakly in X with $\|x_n\| \to \|x\|$, we have $\rho_n(x_n) \to 0$ for each weakly null sequence (ρ_n) in X^*. We show that a C*-algebra has the alternative Dunford-Pettis property if and only if every one of its irreducible representations is finite dimensional so that, for C*-algebras, the alternative and the usual Dunford-Pettis properties coincide as was conjectured by Freedman. We further show that the predual of a von Neumann algebra has the alternative Dunford-Pettis property if and only if the von Neumann algebra is of type I.

Amongst several characterisations (see [6]) a convenient formulation of the Dunford-Pettis property (DP) in Banach spaces is that a Banach space E is said to have the DP if and only if whenever (x_n) and (ρ_n) are weakly null sequences in E and E^*, respectively, then $\rho_n(x_n) \to 0$. The reader is referred to [6] for a comprehensive survey of the Dunford-Pettis property.

All $C(\Omega)$-spaces, alias commutative C*-algebras, have the DP [8]. Subsequent investigations of general C*-algebras in [3, 4] and [11] culminate in the proof [4] that a C*-algebra has the DP if and only if it has no infinite dimensional irreducible representations. By [3] and [11] together with the observation [2], for the predual of a von Neumann algebra M to have the DP it is both necessary and sufficient that M be type I finite.

In an interesting development [7] introduced and studied the alternative Dunford-Pettis property, known in abbreviation as the DP1, referred to above. We recall that a Banach space E is said to have the DP1 if whenever a sequence $x_n \to x$ weakly in E with $\|x_n\| = \|x\| = 1$, for all $n \in \mathbb{N}$, and (ρ_n) is a weakly null sequence in E^*, then $\rho_n(x_n) \to 0$.

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Several equivalent formulations of the DP1 are given in [7, 1.4] analogous to those of the DP given in [6, pages 17-18]. We remark that the DP1 is inherited by 1-complemented subspaces (cf Remark 3).

By confining the DP condition to the unit sphere of norm one elements, the DP1 allows greater freedom. In contrast to the DP, the DP1 is not isomorphism invariant [7, 1.6]. That the DP implies the DP1 is clear. The space of trace class operators on an infinite dimensional Hilbert space provides a counter-example to the opposite implication, [7, 2.4]. Nevertheless, by [7, 3.5], the DP and the DP1 are equivalent for von Neumann algebras. The suggestion in [7] that the predual of every von Neumann algebra has the DP1 was negated recently with the discovery [12] that this fails in the case of type II von Neumann algebras.

The purpose of the present note is to show that the DP and the DP1 are equivalent for all C*-algebras and that a von Neumann algebra is of type I if and only if its predual has the DP1.

We use standard C*-algebra notation and we refer the reader to [13] and [15] for any undefined terms. If x is a positive element in a C*-algebra, A, the norm closure of xAx is denoted by $A(x)$. Thus, $A(x)^{**} = r(x)A^{**}r(x)$, where $r(x)$ is the range projection of x in A^{**}.

Lemma 1. *Let A be a C*-algebra having the DP1. Let x be a non-zero positive element of A and let $e = 1 - r(x)$. Then $A \cap eA^{**}e$ has the DP.*

Proof. We may suppose that $\|x\| = 1$. Let I denote $A \cap eA^{**}e$. Let (x_n) and (ρ_n) be weakly null sequences in I and I^*, respectively. We shall show that $\rho_n(x_n) \to 0$. We may suppose that $\|x_n\| \leq 1$, for all n.

Via [13, 3.21.2], there is a projection f (possibly zero) in A^{**} such that

$$I^{**} = fA^{**}f.$$

For each n, let $\widetilde{\rho}_n$ in A^* be defined by

$$\widetilde{\rho}_n(a) := \rho_n(faf).$$

Then $(\widetilde{\rho}_n)$ is a weakly null sequence in A^*. Further, $x_n + x \to x$ weakly in A. We have $ex = xe = 0$ and, for each n, $x_n = ex_ne$, giving

$$x_n^* x = x_n x^* = 0.$$

Therefore

$$\|x_n + x\|^2 = \|x_n^* x_n + x^* x\| = 1.$$

Since A has the DP1, we have $\widetilde{\rho}_n(x_n + x) \to 0$, and hence, $\rho_n(x_n) = \widetilde{\rho}_n(x_n) \to 0$, as required. □

Theorem 2. *The following are equivalent for a C^*-algebra A.*

(a) *A has the DP1;*
(b) *A has the DP;*
(c) *All irreducible representations of A are finite dimensional.*

Proof. The equivalence of (b) and (c) was proved in [4] and we already know that (b) implies (a). It remains to show (a) implies (c).

Suppose that A satisfies (a) and that $\pi : A \to B(H)$ is an irreducible representation. We shall show that H is finite dimensional. The result will then follow from known results mentioned previously. We may suppose H has dimension greater than one, else there is nothing to prove. In which case, by functional calculus, $\pi(A)$ contains non-zero positive elements u and v such that $uv = 0$. It follows from [1, 2.3 and 2.4] that we may further choose positive elements x and y in A such that $\pi(x) = u$, $\pi(y) = v$ and $xy = 0$.

By [13, 4.1.5], π restricts to an irreducible representation of $A(y)$ and we note that $A(y)$ is contained in $A \cap eA^{**}e$, where $e = 1 - r(x)$. Hence, by Lemma 1, together with the equivalence of (b) and (c) we see that $v = \pi(y)$ has finite rank, as has u, by symmetry. In particular, by [13, 6.1.4], $\pi(A)$ contains the space of all compact operators on H. So the above argument implies that the elements in every pair of non zero positive orthogonal compact operators on H are of finite rank. Therefore, H is finite dimensional. □

We now turn to von Neumann algebras.

Remark 3. *Let M be a von Neumann algebra. If $P : M \to M$ is a normal contractive projection onto a weak*-closed subalgebra, N, and $P_* : M_* \to M_*$ is the induced predual contractive projection, then N_* is isometric to $P_*(M_*)$ via $\rho \mapsto \rho \circ P$. Thus, if M_* has the DP1 then so does N_*. In particular, for any projection e of M, $(eMe)_*$ has the DP1 if M_* has the DP1.*

In the proof of the next proposition $F(H)$ denotes the space of all finite rank operators on a Hilbert space, H, and $K(H)$ denotes the space of all compact operators.

Proposition 4. *Let M be a type I von Neumann algebra. Then M_* has the DP1.*

Proof. By homogeneous decomposition [15, 5.1.27] together with [7, 1.9] we may suppose that M coincides with the von Neumann tensor product

$$N \overline{\otimes} B(H),$$

where N is an abelian von Neumann algebra and H is a complex Hilbert space. Since the C*-algebra tensor product, A, of N and $K(H)$ is weak*-dense in M, we have M is *-isomorphic to $A^{**}z$ for some central projection z in A^{**}. Thus M_* is isometric to $(A^{**}z)_*$ which is complemented in A^*. Hence, if A^* has the DP1 then so does M_*. We shall show that A^* has the DP1.

To this end, let (ρ_n) be a sequence of states in A^* converging weakly to a state ρ in A^*, and let (x_n) be a weakly null sequence in A^{**} (that is, in the $\sigma(A^{**}, A^{***})$-topology). By [7, 2.1.(b)], it is enough to show that $\rho_n(x_n) \to 0$. We may suppose that $\|x_n\| \leq 1$ for all n.

Let $0 < \varepsilon < 1$. We may choose a in A such that $0 \leq a \leq 1$ and $\rho(a) > 1-\varepsilon$. We may further choose x in the algebraic tensor product, $N \otimes F(H)$, such that $x \geq 0$ and $\|a - x\| < \varepsilon$. Since x must lie in $N \otimes eB(H)e$ for some finite rank projection e, we see that $A(x)^{**}$ is a subalgebra of a type I_n algebra, for some $n < \infty$, so that $A(x)^*$ has the DP. Hence, since (xx_nx) is weakly null in $A(x)^{**}$, we have $\rho_n(xx_nx) \to 0$. Thus, for all n large enough,

$$\begin{aligned}|\rho_n(ax_na)| &\leq |\rho_n(xx_nx)| + |\rho_n(ax_n(a-x))| + |\rho_n((a-x)x_nx)| \\ &< 2\varepsilon + \varepsilon(1+\varepsilon) < 4\varepsilon.\end{aligned}$$

Further, since $\rho_n(a) \to \rho(a)$ and $\rho(a) > 1 - \varepsilon$, we get $\rho_n((1-a)^2) \leq \rho_n(1-a) < \varepsilon$, for all large n. By the Cauchy-Schwarz inequality we deduce that, for all n large enough,

$$|\rho_n(ax_n(1-a))|^2 \leq \rho_n(ax_nx_n^*a)\rho_n((1-a)^2) < \varepsilon$$

and similarly that

$$|\rho_n((1-a)x_n)|^2 < \varepsilon,$$

which, in turn, gives

$$|\rho_n(x_n)| \leq |\rho_n(ax_na)| + |\rho_n((1-a)x_n)| + |\rho_n(ax_n(1-a))| < 4\varepsilon + 2\varepsilon^{\frac{1}{2}}.$$

Therefore, $\rho_n(x_n) \to 0$, as required. □

The following was proved in [12]. A version is included here for completeness and for the convenience of the reader.

Proposition 5. [12, Theorem 3] *Let M be a type II von Neumann algebra. Then M_* does not have the DP1.*

Proof. Passing to a non-zero finite projection we may suppose M is type II_1. Let τ be a normal trace on M and let (s_n) be an infinite sequence of anti-commuting symmetries in M. The real Banach space, H, generated by the (s_n) is isometric to a Hilbert space containing (s_n) as an orthonormal system and we have $\tau(s_n) = 0$ for all n [10, §6]. Let

e_n denote the projection $\frac{1}{2}(1+s_n)$ and let τ_n denote the normal state defined by

$$\tau_n(x) := 2\tau(e_n x) \ (= 2\tau(e_n x e_n)).$$

Since (s_n) is an orthonormal system in H, so that (s_n) is weakly null in H and hence in M, it follows that $\tau_n \to \tau$ weakly in M_*. But $\tau_n(s_n) = 1$, for all n. Thus M_* does not have the DP1. □

In order to show that only preduals of type I von Neumann algebras can have the DP1 it remains to deal with type III von Neumann algebras. For this we shall need a remarkable theorem [9, 11.1] of Haagerup and Størmer. The required argument follows the pattern of the proof of [5, Corollary 9] of Connes and Størmer in their affirmative settling of Dell'Antonio's conjecture. Nevertheless, we feel obliged to provide details.

Theorem 6. *Let M be a von Neumann algebra. Then M_* has the DP1 if and only if M is of type I.*

Proof. By [15, 5.1.19], Proposition 4, and Proposition 5 we may assume M is of type III. We may further suppose M is σ-finite. By [9, 11.1], there is a faithful normal state ρ of M such that its centralizer, M_ρ, is type II_1. By [16, Lemma 1.5.8] and [17, page 309], there is a normal contractive projection from M onto M_ρ. Since, by Proposition 5, $(M_\rho)_*$ does not have DP1 neither does M_*. □

Recalling that a C*-algebra A is said to be *postliminal* if $\pi(A)$ contains the space of all compact operators on H for every irreducible representation, $\pi : A \to B(H)$, and that A is postliminal if and only if A^{**} is type I, we have the following automatic corollary.

Corollary 7. *Let A be a C*-algebra. Then A^* has the DP1 if and only if A is postliminal.*

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