ON THE AXIOMATIC DEFINITION OF REAL JB*-TRIPLES

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Abstract. In the last twenty years, a theory of real Jordan triples has been developed. In 1994 T. Dang and B. Russo introduced the concept of J*B-triple. These J*B-triples include real C*-algebras and complex JB*-triples. However, concerning J*B-triples, an important problem was left open. Indeed, the question was whether the complexification of a J*B-triple is a complex JB*-triple in some norm extending the original norm. T. Dang and B. Russo solved this problem for commutative J*B-triples.

In this paper we characterize those J*B-triples with a unitary element whose complexifications are complex JB*-triples in some norm extending the original one. We actually find a necessary and sufficient new axiom to characterize those J*B-triples with a unitary element which are J*B-algebras in the sense of [1] or real JB*-triples in the sense of [4].

1. Introduction

We recall that a real (respectively, complex) *Banach Jordan triple* is a real (respectively, complex) Banach space U with a continuous trilinear (respectively, bilinear in the outer variables and conjugate linear in the middle one) product

$$U \times U \times U \to U$$

$$(xyz) \mapsto \{x, y, z\}$$

satisfying

(1) $\{x, y, z\} = \{z, y, x\}$;

(2) Jordan Identity:

$$L(a, b)\{x, y, z\} - \{L(a, b)x, y, z\} =$$

$$-\{x, L(b, a)y, z\} + \{x, y, L(a, b)z\}$$

for all a, b, x, y, z in U, where $L(x, y)z := \{x, y, z\}$.

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A Banach Jordan triple U is *commutative* or *abelian* if

$$\{\{x,y,z\},u,v\} = \{x,y,\{z,u,v\}\} = \{x,\{y,z,u\},v\}$$

for all $x,y,z,u,v \in U$. An element $u \in U$ is said to be unitary if $L(u,u)$ coincides with the identity map on U.

A complex Jordan Banach triple $\mathcal{E}$ is said to be a *(complex) JB*-triple* if

(a) The map $L(a,a)$ from $\mathcal{E}$ to $\mathcal{E}$ is an hermitian operator with non negative spectrum for all a in $\mathcal{E}$;
(b) $\|\{a,a,a\}\| = \|a\|^3$ for all a in $\mathcal{E}$.

We recall that a bounded linear operator T on a complex Banach space is said to be *hermitian* if $\|\exp(i\alpha T)\| = 1$ for all real α.

Complex JB*-triples were introduced by W. Kaup in the study of Bounded Symmetric Domains in complex Banach spaces (see [5], [6]).

During the 80's and early 90's three notions of JB*-triples over the real field were introduced by H. Upmeier [9], T. Dang and B. Russo [3] and J. M. Isidro, W. Kaup and A. Rodríguez [4], respectively.

In 1994 T. Dang and B. Russo [3, Definition 1.3] gave the following axiomatic definition of a JB*-triple over the real field. A *J*B-triple* is a real Banach space E equipped with a structure of a real Banach Jordan triple which satisfies

(J*B1): $\|\{x,x,x\}\| = \|x\|^3$ for all x in E;
(J*B2): $\|\{x,y,z\}\| \leq \|x\|\,\|y\|\,\|z\|$ for all x,y,z in E;
(J*B3): $\sigma^{\mathbb{C}}_{L(E)}(L(x,x)) \subset [0,+\infty)$ for all $x \in E$;
(J*B4): $\sigma^{\mathbb{C}}_{L(E)}(L(x,y) - L(y,x)) \subset i\mathbb{R}$ for all $x,y \in E$.

Every closed subtriple of a J*B-triple is a J*B-triple (c.f. [3, Remark 1.5]). The class of J*B-triples includes real C*-algebras and complex JB*-triples. Moreover, in [3, Proposition 1.4] it is shown that complex JB*-triples are precisely those complex Banach Jordan triples whose underlying real Banach space is a J*B-triple. In [3] the following important problems concerning J*B-triples were left open

($P1$) Is the complexification of a J*B-triple a complex JB*-triple in some norm extending the original norm?
($P2$) Is the bidual of a J*B-triple a J*B-triple with a separately weak*-continuous triple product?

Problem ($P1$) was affirmatively solved by T. Dang and B. Russo in the particular case of a commutative J*B-triple [3, Theorem 3.11]. They proved that the complexification of every commutative J*B-triple E is a complex JB*-triple in some norm extending the norm on E.

In this paper we study both problems in the general case of J*B-triples with a unitary element. We characterize those J*B-triples having a unitary element whose complexifications are complex JB*-triples in some norm extending the original norm. As a consequence we solve $(P2)$ for the class of J*B-triples with a unitary element where problem $(P1)$ has an affirmative solution.

In 1997 real JB*-triples were introduced by J. M. Isidro, W. Kaup, and A. Rodríguez [4], as norm-closed real subtriples of complex JB*-triples. Isidro, Kaup, and Rodríguez [4, Proposition 2.2] proved that given a real JB*-triple E, then there exists a unique complex JB*-triple structure on its complexification with a norm extending the norm on E.

Let X be a complex Banach space. A *conjugation* on X is a conjugate-linear isometry on X of period 2. If τ is a conjugation on X, then X^τ will stand for the real Banach space of all τ-fixed elements of X. Real normed spaces which can be written as X^τ, for some conjugation τ on X, are called *real forms* of X.

If $\mathcal{E}$ is a complex JB*-triple, then conjugations on $\mathcal{E}$ preserve the triple product of E (c.f. [6, Proposition 5.5]), and hence the real forms of $\mathcal{E}$ are real JB*-triples. In [4] it is shown that actually every real JB*-triple can be regarded as a real form of a suitable complex JB*-triple.

The class of real JB*-triples includes all JB-algebras, all real C*-algebras, all J*B-algebras (compare [1]), and obviously all complex JB*-triples. In [4, Proposition 2.5] it is shown that every real JB*-triple is a J*B-triple but the converse is an open problem. Problem:

$(P1')$ Is every J*B-triple a real JB*-triple?

Problem $(P1')$ has been also stated in [4, page 318]. Actually problems $(P1)$ and $(P1')$ are equivalent.

In this paper (see Theorem 2.6) we also find a necessary and sufficient fifth axiom for a J*B-triple E with a unitary element to be a real JB*-triple in the sense of [4]. This new axiom assures that the operator $L(x,x)$ has non negative numerical range (see definition below). Actually, this new axiom characterizes those J*B-triples with a unitary element which are J*B-algebras in the sense of [1].

2. The results

Let X be a Banach space. Through the paper we denote by B_X, S_X, and X^* the closed unit ball, the unit sphere, and the dual space, respectively, of X. I_X will denote the identity operator on X, and if Y is another Banach space, then $L(X,Y)$ stands for the Banach space of all bounded linear operators from X to Y. We usually write $L(X)$ instead of $L(X, X)$.

Let E be a J*B-triple with a unitary element u. Then E is a unital real Jordan Banach algebra with product $x \circ y := \{x, u, y\}$ and involution $x^* := \{u, x, u\}$ (compare [9, Proposition 19.13]). By (J*B 2) and (J*B 1) it follows that $\|x^*\| = \|x\|$ and $\|x\|^3 = \|U_x(x^*)\|$ for all $x \in E$, where $U_x(y) := 2x \circ (x \circ y) - x^2 \circ y$. Throughout the paper we will denote by E_{sym} the set of all $*$-symmetric elements in E.

Let X be a Banach space, and u a norm-one element in X. The set of states of X relative to u, $D(X, u)$, is defined as the non empty, convex, and weak*-compact subset of X^* given by

$$D(X, u) := \{\Phi \in B_{X^*} : \Phi(u) = 1\}.$$

For $x \in X$, the *numerical range* of x relative to u, $V(X, u, x)$, is given by $V(X, u, x) := \{\Phi(x) : \Phi \in D(X, u)\}$. The *numerical radius* of x relative to u, $v(X, u, x)$, is given by

$$v(X, u, x) := \max\{|\lambda| : \lambda \in V(X, u, x)\}.$$

It is well known that a bounded linear operator T on a complex Banach space X is hermitian if and only if $V(L(X), I_X, T) \subseteq \mathbb{R}$ (compare [2, Corollary 10.13]).

It is well known that for an element a of a unital Banach algebra A with unit u the following assertions hold

(*a*) $V(A, u, a) = V(L(A), I_A, L_a) = V(L(A), I_A, R_a)$, where L_a (respectively, R_a) is the map given by $x \mapsto ax$ (respectively, $x \mapsto xa$);

(*b*) If B is a subalgebra of A with $\{a, u\} \subset B$, then $V(B, u, a) = V(A, u, a)$.

Remark 2.1. Let X be a complex Banach space and τ a conjugation on X. We define a conjugation $\widetilde{\tau}$ on $L(X)$ by $\widetilde{\tau}(T) := \tau T \tau$. If T is a $\widetilde{\tau}$-invariant element of $L(X)$, then we have $T(X^\tau) \subseteq X^\tau$, and hence we can consider $\Lambda(T) := T|_{X^\tau}$ as a bounded linear operator on the real Banach space X^τ. Since the mapping $\Lambda : L(X)^{\widetilde{\tau}} \to L(X^\tau)$ is a linear contraction sending I_X to I_{X^τ}, we get

$$V(L(X^\tau), I_{X^\tau}, \Lambda(T)) \subseteq V(L(X)^{\widetilde{\tau}}, I_X, T)$$

for all $T \in L(X)^{\widetilde{\tau}}$. On the other hand, by the Hahn-Banach Theorem, we have

$$V(L(X)^{\widetilde{\tau}}, I_X, T) = V(L(X)_{\mathbb{R}}, I_X, T)$$

for every $T \in L(X)^{\widetilde{\tau}}$. It follows

$$V(L(X^\tau), I_{X^\tau}, \Lambda(T)) \subseteq \Re e\ V(L(X), I_X, T)$$

for all $T \in L(X)^{\tilde{\tau}}$.

By [8, Lemma 1.6] we know that if we denote by $\mathcal{H}$ the real Banach space of all hermitian operators on X which lie in $L(X)^{\tilde{\tau}}$, then for every $\Phi \in D(L(X), I_X)$, there exists $\Psi \in D(L(X^\tau, I_{X^\tau})$ such that $\Phi(T) = \Psi(\Lambda(T))$ for every T in $\mathcal{H}$. As a consequence

$$V(L(X), I_X, T) = V(L(X^\tau), I_{X^\tau}, \Lambda(T))$$

for every $T \in \mathcal{H}$.

Let E be a J*B-triple with a unitary element. The next proposition allows us to calculate the norm of a symmetric element $x \in E$ from the numerical radius of the map $L(x, x)$.

Proposition 2.2. *Let E be a J*B-triple with a unitary element u. Then for every $x \in E_{sym}$ the following assertions hold*

$$V(L(E), I_E, L(x, x)) \subset [0, +\infty),$$

and

$$\|x\|^2 = \|x^2\| = v(E, u, x^2) = v(L(E), I_E, L(x, x)).$$

Moreover

$$\|2x \circ x^*\| = v(E, u, L_{2x\circ x^*}) = v(L(E), I_E, L(x, x) + L(x^*, x^*))$$

for all $x \in E$.

Proof. Let $x \in E_{sym}$. We denote by

$$C(x) := \overline{Span}^{\|\cdot\|}\{x^n : n = 0, 1, 2, \ldots\}$$

the closed Jordan subalgebra of E generated by x and u. Since $x^* = x$ we conclude that $C(x)$ is a commutative closed subtriple of E. By [3, Remark 1.5], it follows that $C(x)$ is a commutative J*B-triple with a unitary element. Now Theorem 3.11 in [3] assures us that there exists a commutative complex JB*-triple $B(x)$ and a conjugation τ on $B(x)$ such that $C(x) = B(x)^\tau$. Since $B(x)$ is a commutative complex JB*-triple with a unitary element u, we can conclude that $B(x)$ is a commutative C*-algebra with product $z \circ y = \{z, u, y\}$ and involution $z^* = \{u, z, u\}$. Moreover, the conjugation τ on $B(x)$ satisfies $\tau(z^*) = (\tau(z))^*$ for all $z \in B(x)$. By Remark 2.1, the identity $L(x, x) = L_{x^2}$, and since $C(x)$ is a subalgebra of E containing $\{x, u\}$, it follows that

$$V(L(B(x)), I_{B(x)}, L(x, x)) = V(L(C(x)), I_{C(x)}, L(x, x))$$

$$= V(L(C(x)), I_{C(x)}, L_{x^2}) = V(C(x), u, x^2) = V(E, u, x^2)$$

$$= V(L(E), I_E, L(x, x)). \tag{1}$$

Since $B(x)$ is a complex JB*-triple, and hence $L(x,x)$ is hermitian and has non-negative spectrum, it follows from (1) and [2, Lemma 38.3] that

$$V(L(E), I_E, L(x,x)) = V(L(B(x)), I_{B(x)}, L(x,x)) \subset [0, +\infty).$$

Since $B(x)$ is a commutative C*-algebra and $x \in B(x)$ is an hermitian element ($x^* = \{u, x, u\} = x$), Sinclair's theorem (c.f. [2, Theorem 10.17]) and equality (1) assure that

$$\|L(x,x)\| = \|x\|^2 = \|x^2\| = v(B(x), u, x^2)$$
$$= v(L(B(x)), I_{B(x)}, L_{x^2}) = v(L(B(x)), I_{B(x)}, L(x,x))$$
$$= v(L(E), I_E, L(x,x)) = v(E, u, x^2).$$

Finally, let $x \in E$. From the equality $L(x,x) + L(x^*, x^*) = L_{2x \circ x^*}$ and following the arguments given above, we conclude that

$$\|2x \circ x^*\| = v(B(x \circ x^*), u, 2x \circ x^*) = v(L(B(x \circ x^*)), I_{B(x \circ x^*)}, L_{2x \circ x^*})$$
$$= v(L(C(x \circ x^*)), I_{C(x \circ x^*)}, L_{2x \circ x^*}) = v(C(x \circ x^*)), u, 2x \circ x^*)$$
$$= v(E, u, 2x \circ x^*) = v(L(E), I_E, L(x,x) + L(x^*, x^*)).$$

□

Remark 2.3. Let E be a J*B-triple, $x \in E$ and let $C(x)$ the closed real subtriple of E generated by x. Then $C(x)$ is a commutative real J*B-triple (compare [3, Remark 1.5]). Therefore, by [3, Theorem 3.11], there is a commutative (complex) JB*-triple $\widehat{C}(x)$ and a conjugation τ on $\widehat{C}(x)$ such that $C(x) = \widehat{C}(x)^\tau$. Now, since $B(x)$ is a complex JB*-triple, and hence $L(x,x)$ is hermitian and has non-negative spectrum, it follows from Remark 2.1 that

$$V(L(C(x)), I_{C(x)}, L(x,x)) = V(L(B(x)), I_{B(x)}, L(x,x)) \subset [0, +\infty).$$

Since the mapping $T \mapsto T|_{C(x)}$ is a linear contraction from $L(E)$ to $L(C(x))$ sending I_E to $I_{C(x)}$ we have

$$v(L(C(x)), I_{C(x)}, L(x,x)) \leq v(L(E), I_E, L(x,x)).$$

Finally, by Sinclair's theorem we have

$$\|L(x,x)\| = v(L(B(x)), I_{B(x)}, L(x,x)) = v(L(C(x)), I_{C(x)}, L(x,x))$$
$$\leq v(L(E), I_E, L(x,x)) \leq \|L(x,x)\| = \|x\|^2.$$

The following corollary shows that the symmetric part of a J*B-triple with a unitary element is a JB-algebra.

Corollary 2.4. *Let E be a J*B-triple with a unitary element u. Then E_{sym} is a JB-algebra.*

Proof. By Proposition 2.2 the equality

$$\|x\|^2 = v(L(E), I_E, L(x,x)) = v(E, u, x^2) \tag{2}$$

holds for all $x \in E_{sym}$.

Now let $x, y \in E_{sym}$ and $\phi \in S_{E^*}$ with $\phi(u) = 1$. By Proposition 2.2, we also have $V(E, u, z^2) = V(L(E), I_E, L(z,z) = L_{z^2}) \subset [0, +\infty)$, for all $z \in E_{sym}$. In particular $\phi(z^2) \geq 0$ for all $z \in E_{sym}$. Thus, it follows that

$$\|x^2 + y^2\| \geq \phi(x^2 + y^2) \geq \phi(x^2).$$

Finally, taking supreme on $\{\phi \in S_{E^*} : \phi(u) = 1\}$, we conclude by (2) that

$$\|x^2 + y^2\| \geq \|x^2\|,$$

which shows that E_{sym} is a JB-algebra. □

Remark 2.5. Let E be a J*B-triple. Suppose that the complexification of E admits a complex JB*-triple structure with a norm extending the original norm of E. Then E is clearly a real JB*-triple. Let us denote by $\widehat{E} := E \oplus iE$ the complexification of E, by τ the canonical conjugation on $\widehat{E}$ satisfying $E = \widehat{E}^\tau$ and let $x \in E$. Since $\widehat{E}$ is a complex JB*-triple, it follows that $L(x,x)$ is an hermitian operator with non-negative spectrum, and hence, by [2, Lemma 38.3],

$$V(L(\widehat{E}), I_{\widehat{E}}, L(x,x)) \subset [0, +\infty).$$

By Remark 2.1 we also have

$$V(L(E), I_E, L(x,x)) = V(L(\widehat{E}), I_{\widehat{E}}, L(x,x)) \subset [0, +\infty).$$

Thus it seems natural to add the axiom

$$V(L(E), I_E, L(x,x)) \subset [0, +\infty) \text{ for all } x \in E \tag{3}$$

to the structure of the J*B-triple E, in order to show that E is also a real JB*-triple.

We can now state our main result which shows that a J*B-triple E with a unitary element is a real JB*-triple if and only if E satisfies axiom (3). Previously, we recall (c.f. [1]) that a J*B-algebra is a real Jordan algebra A with unit and an involution * equipped with a complete algebra norm such that $\|U_x(x^*)\| = \|x\|^3$ and $\|x^*x\| \leq \|x^*x + y^*y\|$ for all $x, y \in A$.

Theorem 2.6. *Let E be a J*B-triple with a unitary element u. The following assertions are equivalent*

(a) $V(L(E), I_E, L(x,x)) \subset [0, +\infty)$ *for all* $x \in E$;

(b) *E is a J*B-algebra with product* $x \circ y := \{x, u, y\}$ *and involution* $x^* := \{u, x, u\}$;

(c) *E is a real JB*-triple;*
(d) *The complexification of E is a complex JB*-triple in some norm extending the original norm on E.*

Proof. $(a) \Rightarrow (b)$ As we have seen before, E is a real Jordan Banach algebra with product $x \circ y := \{x, u, y\}$ and involution $x^* := \{u, x, u\}$. We have also seen that $\|U_x(x^*)\| = \|x\|^3$, for all $x \in E$. Hence only the inequality

$$\|x^* \circ x\| \leq \|x^* \circ x + y^* \circ y\|$$

has to be shown for all $x, y \in E$.

By Proposition 2.2, it follows that

$$\|2z \circ z^*\| = v(E, u, 2z \circ z^*) = v(L(E), I_E, L(z, z) + L(z^*, z^*)) \tag{4}$$

for all $z \in E$.

Let $\phi \in S_{E^*}$ with $\phi(u) = 1$. By hypothesis, $V(L(E), I_E, L(x, x))$ is contained in $[0, +\infty)$ for all $x \in E$, then we can conclude by the identity $L(x, x) + L(x^*, x^*) = L_{2x \circ x^*}$ that $\phi(x \circ x*) \geq 0$ for all $x \in E$. Therefore, given $x, y \in E$ we have

$$\|x \circ x^* + y \circ y^*\| \geq \phi(x \circ x^* + y \circ y^*) \geq \phi(x \circ x^*).$$

Finally, taking supreme on $\{\phi \in S_{E^*} : \phi(u) = 1\}$, we conclude by (4) that

$$\|x^* \circ x\| \leq \|x^* \circ x + y^* \circ y\|.$$

$(b) \Rightarrow (c)$ By [1, Theorem 4.4] it follows that the norm on E can be extended to the complexification $\widehat{E} = E \oplus iE$ of E such that $(\widehat{E}, \circ, ^*, \|.\|)$ is a JB*-algebra. Actually [1, Theorem 4.4] shows that E is a real form of its complexification. Since every JB*-algebra is a complex JB*-triple ([9, Proposition 20.35]), we conclude that E is a real JB*-triple.

The implications $(c) \Rightarrow (d)$ and $(d) \Rightarrow (a)$ follow from [4, Proposition 2.2] and Remark 2.5, respectively.

□

The next definition is motivated by the above theorem.

Definition 2.7. *A numerically positive real J*B-triple is a J*B-triple E satisfying the following fifth axiom*

$$V(L(E), I_E, L(x, x)) \subset [0, +\infty) \textit{ for all } x \in E \textit{ (J*B 5).}$$

Clearly every real JB*-triple is a numerically positive real J*B-triple and every numerically positive real J*B-triple is a J*B-triple. By Theorem 2.6 we know that if E is a Banach Jordan triple with a unitary element then problem $(P1)$ has an affirmative solution for E if and only if E is a real JB*-triple if and only if E is a numerically positive real J*B-triple.

It seems natural to ask if we have found an axiomatic definition of real JB*-triples, but for the moment, we do not know if every numerically positive real J*B-triple is a real JB*-triple.

Now we deal with problem ($P2$). By [4, Lemma 4.2 and Theorem 4.4] we know that the bidual of a real JB*-triple is a real JB*-triple with a separate weak*-continuous triple product. Thus the following corollary is derived from Theorem 2.6.

Corollary 2.8. *Let E be a J*B-triple with a unitary element. Suppose that $V(L(E), I_E, L(x,x)) \subset [0, +\infty)$ for all $x \in E$, i.e., E is a numerically positive real J*B-triple with a unitary element. Then the bidual of E is again a J*B-triple with a unitary element and separately weak*-continuous triple product.*

If E is a real JB*-triple which is also a Banach dual space, then E has a unique predual and the triple product of E is separately weak*-continuous (c.f. [7]). Now applying Theorem 2.6, we obtain the following corollary.

Corollary 2.9. *Let E be a numerically positive real J*B-triple with a unitary element. Suppose that E is a Banach dual space. Then E has a unique predual and the triple product of E is separately weak*-continuous.*

We leave open the following problems

($P3$) Is the complexification of a numerically positive real J*B-triple a complex JB*-triple in some norm extending the original norm? This problem is equivalent to ask whether a numerically positive real J*B-triple is a real JB*-triple.

($P4$) Is the bidual of a numerically positive real J*B-triple a numerically positive real J*B-triple with a separately weak*-continuous triple product?

($P5$) Let E be a numerically positive real J*B-triple which is also a Banach dual space. Does E have a unique predual? Is the triple product on E separately weak*-continuous?

We have shown that problems ($P3$), ($P4$), and ($P5$) have an affirmative solution whenever E has a unitary element.

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