Maximal surfaces and harmonic diffeomorphisms

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Abstract

We survey some recent results on the structure of the space of maximal graphs with isolated singularities in the Lorentzian product $\mathbb{M} \times \mathbb{R}_1$, where $\mathbb{M}$ is an arbitrary $\mathfrak{n}$-dimensional compact Riemannian manifold, $\mathfrak{n} \geq 2$. As an application, when $\mathbb{M}$ has dimension 2, one has some consequences on the existence of harmonic diffeomorphisms between certain open Riemann surfaces. We pay a special attention to the case of domains in the Euclidean 2-sphere. Some open questions are proposed.

In this short article we survey some recent results we obtained in [1]. In Section 1, we present our results on the space of maximal hypersurfaces with a finite number of singularities in the Lorentzian manifold $\mathbb{M} \times \mathbb{R}_1$ where $\mathbb{M}$ is a compact Riemannian manifold and give an idea on the methods we use. In Section 2, we address the existence problem of harmonic diffeomorphisms between certain open Riemannian surfaces. Some existence results follow from the results of Section 1. Finally in Section 3, we comment on a related work of ours [2] and propose some open questions.

1 Maximal graphs with singularities

Let $\mathbb{M}$ be a compact Riemannian manifold without boundary of dimension $\mathfrak{n} \geq 2$. We denote by $\mathbb{M} \times \mathbb{R}_1$ the product space $\mathbb{M} \times \mathbb{R}$ endowed with the Lorentzian metric

$$\langle \cdot, \cdot \rangle = \pi_{\mathbb{M}}^*(\langle \cdot, \cdot \rangle_{\mathbb{M}}) - \pi_{\mathbb{R}}^*(dt^2),$$

where $\pi_{\mathbb{M}}$ and $\pi_{\mathbb{R}}$ denote the projections from $\mathbb{M} \times \mathbb{R}$ onto each factor. For simplicity, we write

$$\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{\mathbb{M}} - dt^2.$$

A smooth immersion $X : \Sigma \to \mathbb{M} \times \mathbb{R}_1$ of a connected $\mathfrak{n}$-dimensional manifold Σ is said to be *spacelike* if X induces a Riemannian metric $X^*(\langle \cdot, \cdot \rangle)$ on Σ.

*A. Alarcón is supported by Vicerrectorado de Política Científica e Investigación de la Universidad de Granada, and is partially supported by MCYT-FEDER grants MTM2007-61775 and MTM2011-22547, Junta de Andalucía Grant P09-FQM-5088, and the grant PYR-2012-3 CEI BioTIC GENIL (CEB09-0010) of the MICINN CEI Program.

Let $\Omega \subset \mathbb{M}$ be a connected domain and let $u : \Omega \to \mathbb{R}$ be a smooth function. Then the map

$$X^u : \Omega \to \mathbb{M} \times \mathbb{R}_1, \quad X^u(p) = (p, u(p)) \quad \forall p \in \Omega,$$

determines a smooth graph over Ω in $\mathbb{M} \times \mathbb{R}_1$. The metric induced on Ω by $\langle \cdot, \cdot \rangle$ via X^u is given by

$$\langle \cdot, \cdot \rangle_u := (X^u)^*(\langle \cdot, \cdot \rangle) = \langle \cdot, \cdot \rangle_{\mathbb{M}} - du^2,$$

hence X^u is spacelike if and only if $|\nabla u| < 1$ on Ω, where ∇u denotes the gradient of u in Ω and $|\nabla u|$ denotes its norm, both with respect to the metric $\langle \cdot, \cdot \rangle_{\mathbb{M}}$ in Ω. In this case the function u is said to be spacelike as well. If u is spacelike, then the mean curvature $H : \Omega \to \mathbb{R}$ of X^u is given by the equation

$$H = \frac{1}{\mathfrak{n}} \operatorname{Div} \left(\frac{\nabla u}{\sqrt{1 - |\nabla u|^2}} \right),$$

where Div denotes the divergence operator on Ω with respect to $\langle \cdot, \cdot \rangle_{\mathbb{M}}$.
A smooth function $u : \Omega \to \mathbb{R}$ and its graph $X^u : \Omega \to \mathbb{M} \times \mathbb{R}_1$ are said to be *maximal* if u is spacelike and H vanishes identically on Ω.
Observe that the maximal graph equation is an elliptic one and, since $\mathbb{M}$ is compact, the maximum principle shows that an entire maximal graph in $\mathbb{M} \times \mathbb{R}_1$ is necessarily a slice $\mathbb{M} \times \{t\}$. Again the maximum principle shows there are no entire maximal graphs with only one singularity. Indeed if an entire graph has only one singularity then either a point where the graph function reaches its maximum or a point where it reaches its minimum is a regular one and then the maximum principle shows the function has to be constant, a contradiction. Therefore the interesting class of graphs to consider are those having at least 2 singularities. We are interested here in those having a finite number of singularities. Consider thus $\mathfrak{m} \in \mathbb{N}$, $\mathfrak{m} \geq 2$, and $p_1, \ldots, p_{\mathfrak{m}}$, $\mathfrak{m}$ distinct points in $\mathbb{M}$. Let also $t_1, \ldots, t_{\mathfrak{m}}$ be $\mathfrak{m}$ real numbers. We will first solve the following problem:
Find a function $u \in \mathcal{C}^0(\mathbb{M}) \cap \mathcal{C}^2(\mathbb{M} - \{p_1, \ldots, p_{\mathfrak{m}}\})$ such that

$$\begin{cases} |\nabla u| < 1 \qquad \text{in} \quad \mathbb{M} - \{p_1, \ldots, p_{\mathfrak{m}}\}, \\ \operatorname{Div} \left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}} \right) = 0 \qquad \text{in} \quad \mathbb{M} - \{p_1, \ldots, p_{\mathfrak{m}}\}, \\ u(p_i) = t_i, \qquad i = 1, \ldots, \mathfrak{m}. \end{cases} \tag{1}$$

This is a generalized Dirichlet problem for the maximal graph equation. As $|\nabla u| < 1$, it follows easily that the data $\{(p_i, t_i)\}_{i=1,\ldots,\mathfrak{m}} \subset \mathbb{M} \times \mathbb{R}$ has to satisfy the *spacelike condition*

$$|t_i - t_j| < \operatorname{dist}_{\mathbb{M}}(p_i, p_j), \quad \forall \{i, j\} \subset \{1, \ldots, \mathfrak{m}\} \text{ with } i \neq j.$$

It turns out the spacelike condition is also sufficient for the existence of a solution. The idea to prove this is as follows. One solves first the Dirichlet problem for the maximal graph equation on a sequence of regular domains exhausting $\mathbb{M} - \{p_1, \ldots, p_{\mathfrak{m}}\}$ in $\mathbb{M} \times \mathbb{R}_1$ with boundary values, for each $i = 1 \ldots, \mathfrak{m}$, the real number t_i on the boundary component corresponding to p_i. By [10, Theorem 5.1], one can solve the latter problem. The necessary condition for the existence of the solutions is guaranteed by the spacelike condition. By Arzela-Ascoli's theorem one has a subsequence converging to a weakly spacelike maximal function u verifying the *boundary condition* $u(p_i) = t_i$, $i = 1, \ldots, \mathfrak{m}$. By the results in [3, §6], the limit graph is a maximal graph except for a set of points which is the union of lightlike geodesics which extend up to the boundary. However the spacelike condition prevents the existence of such geodesics. The solution is therefore a spacelike maximal graph over $\mathbb{M} - \{p_i\}_{i=1,\ldots,\mathfrak{m}}$. Once the problem (1) is solved, the points $p_1, \ldots, p_{\mathfrak{m}}$ might be a priori regular. However, if we take $t_1 = \ldots = t_{\mathfrak{m}-1} \neq t_{\mathfrak{m}}$

and close enough to verify the spacelike condition, then the maximum principle for the maximal graph equation forces all these points to be singular, showing that the space of maximal graphs with singularities above the points $p_1, \ldots, p_{\mathfrak{m}}$ is not empty. More precisely, in [1], we proved the following

Theorem I. *Let $\mathbb{M} = (\mathbb{M}, \langle \cdot, \cdot \rangle_{\mathbb{M}})$ be a compact Riemannian manifold without boundary of dimension $\mathfrak{n} \in \mathbb{N}$, $\mathfrak{n} \geq 2$, and denote by $\mathbb{M} \times \mathbb{R}_1$ the product manifold $\mathbb{M} \times \mathbb{R}$ endowed with the Lorentzian metric $\langle \cdot, \cdot \rangle_{\mathbb{M}} - dt^2$. Let $\mathfrak{m} \in \mathbb{N}$, $\mathfrak{m} \geq 2$, and let $\mathfrak{A} = \{(p_i, t_i)\}_{i=1,\ldots,\mathfrak{m}}$ be a subset of $\mathbb{M} \times \mathbb{R}_1$ such that*

- *$p_i \neq p_j$ and*
- *$|t_i - t_j| < \mathrm{dist}_{\mathbb{M}}(p_i, p_j)$, $\forall \{i, j\} \subset \{1, \ldots, \mathfrak{m}\}$ with $i \neq j$.*

Then there exists exactly one entire graph $\Sigma(\mathfrak{A})$ over $\mathbb{M}$ in $\mathbb{M} \times \mathbb{R}_1$ such that

- *$\mathfrak{A} \subset \Sigma(\mathfrak{A})$ and*
- *$\Sigma(\mathfrak{A}) - \mathfrak{A}$ is a spacelike maximal graph over $\mathbb{M} - \{p_i\}_{i=1,\ldots,\mathfrak{m}}$.*

Moreover the space $\mathfrak{G}_{\mathfrak{m}}$ of entire maximal graphs over $\mathbb{M}$ in $\mathbb{M} \times \mathbb{R}_1$ with precisely $\mathfrak{m}$ singularities, endowed with the topology of uniform convergence, is non-empty, and there exists a $m!$-sheeted covering, $\overline{\mathfrak{G}}_{\mathfrak{m}} \to \mathfrak{G}_{\mathfrak{m}}$, where $\overline{\mathfrak{G}}_{\mathfrak{m}}$ is an open subset of $(\mathbb{M} \times \mathbb{R})^{\mathfrak{m}}$.

The case of entire maximal graphs with a finite number of singularities in the 3-dimensional Lorentz-Minkowski space $\mathbb{L}^3$ was previously treated in [8]. The study in [8] uses different tools and exploits the Weierstrass representation for maximal surfaces which is not available in the general case.

2 Harmonic diffeomorphisms between domains of the Euclidean 2-sphere

Let $\mathcal{R}$ be a Riemann surface and let $N = (N, h)$ be a smooth Riemannian manifold. Given a smooth map $f : \mathcal{R} \to N$, a conformal Riemannian metric g on $\mathcal{R}$, and a domain $\Omega \subset \mathcal{R}$ with piecewise C^1 boundary $\partial\Omega$, the quantity

$$E_\Omega(f) = \frac{1}{2} \int_\Omega |df|^2 dV_g \tag{2}$$

is called the *energy* of f over Ω. Here dV_g denotes the volume element of $(\mathcal{R}, g)$, and $|\cdot|$ the norm on (N, h). The energy integral does not depend on the choice of the conformal metric g. A smooth map $f : \mathcal{R} \to N$ is said to be *harmonic* if it is a critical point of the energy functional, that is, if for any relatively compact domain $\Omega \subset \mathcal{R}$ and any smooth variation $F : \mathcal{R} \times (-\epsilon, \epsilon) \to N$ of f supported in Ω (i.e., F is a smooth map, $f_0 = f$, and $f_t|_{\mathcal{R}-\Omega} = f|_{\mathcal{R}-\Omega}$ $\forall t \in (-\epsilon, \epsilon)$, where $f_t := F(\cdot, t) : \mathcal{R} \to N$ and $\epsilon > 0$), the first variation $\frac{d}{dt} E_\Omega(f_t)|_{t=0}$ is zero.
As the energy integral (2) depends only on the conformal structure of $\mathcal{R}$, the harmonicity of a map from a Riemann surface to a Riemannian manifold is a well defined notion. On the other hand, the harmonicity of a map is not preserved under conformal changes of the metric of the target manifold. One may consult the surveys [6, 7, 12] for more information on harmonic maps.
In 1952, Heinz [11] proved there is no harmonic diffeomorphism from the unit complex disk $\mathbb{D}$ onto the complex plane $\mathbb{C}$, with the euclidean metric. Later, Schoen and Yau [15] asked whether Riemannian surfaces which are related by a harmonic diffeomorphism are quasiconformally related, and proposed to investigate whether there is a harmonic diffeomorphism from $\mathbb{C}$ onto the hyperbolic plane $\mathbb{H}^2$. Markovic [13] answered the first question in the negative, by showing

an example consisting of a pair of Riemann surfaces of infinite topological type. He also gave conditions under which the question has a positive answer in the case of surfaces of finite topology. Finally, Collin and Rosenberg [5] gave a harmonic diffeomorphism from $\mathbb{C}$ onto $\mathbb{H}^2$, disproving the conjecture by Schoen and Yau [15]. To do that, they constructed an entire minimal graph Σ over $\mathbb{H}^2$ in the Riemannian product $\mathbb{H}^2 \times \mathbb{R}$, with the conformal type of $\mathbb{C}$. Then the vertical projection $\Sigma \to \mathbb{H}^2$ is a surjective harmonic diffeomorphism.
Motivated by the problem of Schoen and Yau and the above results, we addressed similar questions using maximal surfaces.
Suppose $\mathbb{M}$ is a compact Riemannian surface and let $\mathfrak{A} = \{(p_i, t_i)\}_{i=1,\dots,\mathfrak{m}}$ be a subset of $\mathbb{M} \times \mathbb{R}_1$ as in Theorem I. Let u be the graph function defining the maximal graph $\Sigma(\mathfrak{A})$, given by Theorem I, with singularities at each of the points (p_i, t_i), $i = 1, \dots, \mathfrak{m}$.
Set $\Omega = \mathbb{M} - \{p_1, \dots, p_{\mathfrak{m}}\}$. With the notation of Section 1, since $u : \Omega \to \mathbb{R}$ is maximal then $X^u : (\Omega, \langle\cdot,\cdot\rangle_u) \to (\mathbb{M} \times \mathbb{R}_1, \langle\cdot,\cdot\rangle)$ is a harmonic map. In particular

$$\mathrm{Id} : (\Omega, \langle\cdot,\cdot\rangle_u) \to (\Omega, \langle\cdot,\cdot\rangle_{\mathbb{M}})$$

is a harmonic diffeomorphism, and

$$u : (\Omega, \langle\cdot,\cdot\rangle_u) \to \mathbb{R}$$

is a harmonic function.
It remains to understand the conformal structure of $(\Omega, \langle\cdot,\cdot\rangle_u)$. Let $p \in \pi_{\mathbb{M}}(\mathfrak{A})$ and let A be an annular end of $(\Omega, \langle\cdot,\cdot\rangle_u)$ corresponding to p. Then A is conformally equivalent to an annulus $A(r, 1) := \{z \in \mathbb{C} \mid r < |z| \leq 1\}$ for some $0 \leq r < 1$. Identify $A \equiv A(r, 1)$ and notice that u extends continuously to $S(r) = \{z \in \mathbb{C} \mid |z| = r\}$ with $u|_{S(r)} = u(p)$. By [4], $X^u(A)$ is tangent to either the upper or the lower light cone at $X^u(p)$ in $\mathbb{M} \times \mathbb{R}_1$. In particular p is either a strict local minimum or a strict local maximum of u. Then, up to a shrinking of A, we can assume that $u|_{S(1)}$ is constant, where $S(1) = \{z \in \mathbb{C} \mid |z| = 1\}$. Since $u|_A$ is harmonic, bounded, and non-constant, then $r > 0$ and so A has *hyperbolic* conformal type. This proves that

Claim 2.1. $(\Omega, \langle\cdot,\cdot\rangle_u)$ *is conformally an open Riemann surface with the same genus as* $\mathbb{M}$ *and* $\mathfrak{m}$ *hyperbolic ends.*

In particular, one has the following

Corollary 2.1. *Let* $\mathbb{M}$ *be a compact Riemannian surface, let* $\mathfrak{m} \geq 2$, *and let* $\{p_1, \dots, p_{\mathfrak{m}}\} \subset \mathbb{M}$. *Then there exist an open Riemann surface* $\mathcal{R}$ *and a harmonic diffeomorphism* $\phi : \mathcal{R} \to \mathbb{M} - \{p_1, \dots, p_{\mathfrak{m}}\}$ *such that every end of* $\mathcal{R}$ *is of hyperbolic type.*

Let $\mathbb{S}^2$ and $\overline{\mathbb{C}}$ denote the 2-dimensional Euclidean unit sphere and the Riemann sphere, respectively. A domain in $\overline{\mathbb{C}}$ is said to be a *circular domain* if every connected component of its boundary is a circle.
By Koebe's uniformization theorem, any finitely connected planar domain is conformally equivalent to a domain in $\overline{\mathbb{C}}$ whose frontier consists of points and circles. In this setting the corollary above gives the following existence result for harmonic diffeomorphism between hyperbolic and parabolic domains in $\mathbb{S}^2$.

Corollary 2.2. *Let* $\mathfrak{m} \in \mathbb{N}$, $\mathfrak{m} \geq 2$, *and let* $\{p_1, \dots, p_{\mathfrak{m}}\} \subset \mathbb{S}^2$.
Then there exist a circular domain U *in* $\overline{\mathbb{C}}$ *and a harmonic diffeomorphism* $U \to \mathbb{S}^2 - \{p_1, \dots, p_{\mathfrak{m}}\}$.

In the opposite direction, one would like to know whether there is a harmonic diffeomorphism from $\overline{\mathbb{C}} - \{z_1, \dots, z_{\mathfrak{m}}\}$ onto a circular domain in $\mathbb{S}^2$ endowed with the spherical metric. It turns out this does not exist and we actually have proved a slightly more general result. To state it, we need to recall some basic notions on Riemann surfaces.

Let us recall the following classification of Riemann surfaces. A compact Riemann surface (without boundary) is said to be elliptic. An open Riemann surface (i.e., non-compact and without boundary) is said to be *hyperbolic* if it carries non-constant negative subharmonic functions, and it is said to be *parabolic* otherwise. For instance, the spherical domain $\overline{\mathbb{C}} - \{z_1, \dots, z_{\mathfrak{m}}\}$ is parabolic.
Using Bochner type formulas due to Schoen and Yau [15], we have obtained in [1] the following

Proposition 2.1. *Let $\mathcal{R}$ be a parabolic open Riemann surface, let N be an oriented Riemannian surface, and let $\phi : \mathcal{R} \to N$ be a harmonic local diffeomorphism. Suppose either that N has Gaussian curvature $K_N > 0$ or that $K_N \geq 0$ and N has no flat open subset.*
Then ϕ is either holomorphic or antiholomorphic.

Summarizing, in the special case of the Euclidean 2-sphere, we have the following

Theorem II.

(i) *For any $\mathfrak{m} \in \mathbb{N}$, $\mathfrak{m} \geq 2$, and any subset $\{p_1, \dots, p_{\mathfrak{m}}\} \subset \mathbb{S}^2$ there exist a circular domain $U \subset \overline{\mathbb{C}}$ and a harmonic diffeomorphism $U \to \mathbb{S}^2 - \{p_1, \dots, p_{\mathfrak{m}}\}$.*

(ii) *There exists no harmonic diffeomorphism $\mathbb{D} \to \mathbb{S}^2 - \{p\}$.*

(iii) *For any $\mathfrak{m} \in \mathbb{N}$, any subset $\{z_1, \dots, z_{\mathfrak{m}}\} \subset \overline{\mathbb{C}}$, and any pairwise disjoint closed discs $D_1, \dots, D_{\mathfrak{m}}$ in $\mathbb{S}^2$ there exists no harmonic diffeomorphism $\overline{\mathbb{C}} - \{z_1, \dots, z_{\mathfrak{m}}\} \to \mathbb{S}^2 - \cup_{j=1}^{\mathfrak{m}} D_j$.*

Item (i) is Corollary 2.2 and (iii) is a particular case of Proposition 2.1. The proof of (ii) uses the fact that a harmonic diffeomorphism from a simply connected Riemann surface $\mathcal{S}$ into $\mathbb{S}^2$ can be realized as the Gauss map of a surface of positive constant Gaussian curvature in $\mathbb{R}^3$ such that the conformal structure induced by its second fundamental form coincides with the conformal structure of $\mathcal{S}$. On the other hand, we are able to show that if the Gauss map of a surface of positive constant curvature in $\mathbb{R}^3$ is a diffeomorphism onto $\mathbb{S}^2 - \{p\}$, then the conformal structure induced by the second fundamental form of the surface is that of $\mathbb{C}$.

3 Some comments and open questions

Given a surface in $\mathbb{R}^3$ with constant mean curvature, it is a known fact [14] that its Gauss map into $\mathbb{S}^2$ is harmonic. Also, for a surface of positive Gaussian curvature, its second fundamental form determines a Riemannian metric on it and hence a conformal structure, called the *extrinsic* conformal structure. It is also a known fact [9] that its Gauss map into $\mathbb{S}^2$ is harmonic for the extrinsic conformal structure. A natural question is therefore whether a harmonic diffeomorphism as those in Theorem II-(i) can be realized as the Gauss map of either a constant mean curvature or a constant Gaussian curvature surface in $\mathbb{R}^3$. We have answered these questions in our paper [2].
We end this paper with some questions related to the work described here.

(1) With the notation of Theorem I, determine the open subset $\overline{\mathfrak{G}}_{\mathfrak{m}} \subset (\mathbb{M} \times \mathbb{R})^{\mathfrak{m}}$. In other words, for which data $\mathfrak{A} = \{(p_i, t_i)\}_{i=1,\dots,\mathfrak{m}} \subset \mathbb{M} \times \mathbb{R}_1$ satisfying the spacelike condition, does the maximal graph $\Sigma(\mathfrak{A})$ given by Theorem I have exactly $\mathfrak{m}$ singularities ?

(2) Given a finite subset $\{p_1, \dots, p_{\mathfrak{m}}\} \subset \mathbb{S}^2, \mathfrak{m} \geq 2$, study the structure of the space of circular domains $\mathcal{U}$ in $\overline{\mathbb{C}}$ for which there is a harmonic diffeomorphism $\mathcal{U} \to \mathbb{S}^2 - \{p_1, \dots, p_{\mathfrak{m}}\}$.

(3) Let g be a Riemannian metric on $\mathbb{S}^2$ and $p \in \mathbb{S}^2$. Is it true that there is no harmonic diffeomorphism $\mathbb{D} \to (\mathbb{S}^2, g) - \{p\}$? This is true for the round metric (cf. (ii) of Theorem II).

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