FIRST DIRICHLET EIGENVALUE AND EXIT TIME MOMENTS: A SURVEY

VICENT GIMENO* AND ANA HURTADO**

ABSTRACT. In this paper we summarize recent results concerning the connection between the L^1-moment spectrum associated to a domain D in a Riemannian manifold and its Dirichlet spectrum. In particular, we will expose how to obtain estimations of the first Dirichlet eigenvalue of D in a Riemannian manifold with controlled geometry (or in a submanifold of it) via the study of the so-called moment spectrum.

1. INTRODUCTION

In Riemannian Geometry, the study of partial differential equations and differential operators on Riemannian manifolds plays a central role. One of the most celebrated differential operator, used to built up partial differential equations, is the Laplacian. Given a complete Riemannian manifold (M, g), the Laplacian of a C^2-function $f : M \to \mathbb{R}$ is given in local coordinates $(x^1, \cdots, x^n)$ as

$$\Delta f = \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} \left(\sqrt{\det g}\, g^{ij} \frac{\partial f}{\partial x^j} \right).$$

With the Laplacian, the classical heat equation, for instance, can be formulated in a complete Riemannian manifold. In particular can be studied the problem of finding a heat diffusion function $v : [0, T) \times M \to \mathbb{R}$ solution to the following problem

$$\left\{ \begin{array}{lcl} \frac{\partial v}{\partial t} & = & \Delta v, \\ v(0, x) & = & v_0(x), \end{array} \right. \tag{1.1}$$

where v_0 is a given real function defined on M. It is well-known, see [24], that when v_0 is a bounded positive continuous function equation (1.1) has a solution given by

$$v(x, t) = \int_M p_t(x, y) v_0(y) dV(y),$$

where $p_t(x, y)$ is the heat kernel of M. Similarly, the heat problem with Dirichlet boundary condition in a precompact domain $D \subset M$ with smooth boundary has solution

$$v(x, t) = \int_D p_t^D(x, y) v_0(y) dV(y),$$

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where $p_t^D(x,y)$ is the Dirichlet heat kernel of D and is given by (see [12] for instance)

$$p_t^D(x,y) = \sum_{i=1}^{\infty} e^{-i\lambda_i(D)t}\phi_i(x)\phi_i(y).$$

Here $\{\phi_i\}$ is a complete orthonormal basis of $L^2(D)$ made up of eigenfunctions for $-\Delta$ with Dirichlet boundary condition on D and $\{\lambda_i(D)\}$ are the corresponding eigenvalues.

Recall that given an open precompact domain D with smooth boundary ∂D, if there exists a C^∞ function f on D, not identically zero, solution of the Dirichlet boundary value problem

$$\begin{aligned} \Delta f + \lambda f = 0 \ \text{ on } \ D \\ f|_{\partial D} = 0, \end{aligned} \tag{1.2}$$

for a real number λ, then λ is called an eigenvalue of $-\Delta$ with respect to the Dirichlet boundary condition and f an eigenfunction. In this problem, the eigenvalues are positive with finite multiplicity and we obtain a discrete increasing sequence of eigenvalues $\{\lambda_i(D)\}$ satisfying

$$0 < \lambda_1(D) < \lambda_2(D) \leq \cdots \quad \text{with} \quad \lambda_i \to \infty \quad \text{when} \quad i \to \infty,$$

with each distinct eigenvalue repeated according with its multiplicity. Then we can construct a complete orthonormal basis of $L^2(D)$ consisting on eigenfunctions $\{\phi_i\}$ with corresponding eigenvalues $\{\lambda_i(D)\}$.

The lowest eigenvalue $\lambda_1(D)$ is called, unsurprisingly, the *first eigenvalue* of D, and it has a positive associated eigenfunction $\phi_1 : D \to \mathbb{R}$. The influence of geometric properties, such as curvature bounds and isoperimetric inequalities, on the first eigenvalue, the spectrum of the Laplacian, or the heat kernel, has been widely studied along the last century. See [12, 23, 24] for a detailed discussion.

On the other hand, with the heat kernel a (sub)Markov process X_t on M, called *Brownnian motion*, can be constructed with the transition density p_t. The influence of geometric properties on the behaviour of the Brownian motion including recurrence, transience, and stochastic completeness has also been largely studied. We refer to [23] for a comprehensive survey on the topic. Furthermore, let $\mathbb{P}^x$, $x \in M$, be the associated family of probability measures weighting those Brownian paths beginning at x. The first exit time τ_D of a bounded domain $D \subset M$ for X_t is

$$\tau_D = \inf\{t \geq 0 \,:\, X_t \notin D\}.$$

Hence, by using the expectation operator $\mathbb{E}^x$ with respect to $\mathbb{P}^x$, the *moment spectrum of* D can be defined as the L^1-norm of the k-th moment of τ_D, see [33]:

$$\operatorname{mspec}(D) = \{\mathcal{A}_k(D)\}_{k\in\mathbb{N}} = \{\|\mathbb{E}^x[\tau_D^k]\|_{L^1}\}_{k\in\mathbb{N}},$$

with

$$\|\mathbb{E}^x[\tau_D^k]\|_{L^1} = \int_D \mathbb{E}^x[\tau_D^k]dV.$$

In this survey we discuss the relation between the moment spectrum of D and the first eigenvalue $\lambda_1(D)$ of $-\Delta$ with Dirichlet boundary.

In section 3 it is shown that the moment spectrum determines the first eigenvalue of the Laplacian, in the sense that the knowledge of $\operatorname{mspec}(D)$ implies the knowledge of $\lambda_1(D)$. More precisely, in theorem 3.1, theorem 3.6 and equation (3.8) explicit formulae are provided for the expression of $\lambda_1(D)$ in terms of $\operatorname{mspec}(D)$.

We must remark here that in some cases is harder to solve the eigenvalue problem and determine the first eigenvalue than to obtain the whole moment spectrum. For instance there is no analytic formula for the first eigenvalue of a geodesic ball even in the Hyperbolic space of constant sectional curvature of dimension 2.

The problem of finding the moment spectrum can be drastically simplified in the presence of a large group of isometries. In section 2.3 it is shown how to compute the moment spectrum of a geodesic ball in a rotationally symmetric manifold. Hence, the first eigenvalue of a rotationally symmetric geodesic ball can be computed (see theorem 3.6 and equation (3.8) again). This computation is used to obtain in section 3 theorem 3.9, where several upper bounds for the first eigenvalue of a geodesic ball are obtained in terms only of the area functions of the geodesic spheres in the line of [21]. Finally, in section 4 we explain how to use this bridge between moment spectrum and first eigenvalue to obtain upper and lower bounds for the first eigenvalue from geometric comparisons. These comparisons can be either intrinsic or extrinsic. In the intrinsic case, the first eigenvalue of a domain in a manifold with bounded sectional (or Ricci) curvatures is estimated by the first eigenvalue of a geodesic ball in a suitable model space. In the extrinsic one, we consider domains in a submanifold with controlled mean curvature immersed in an ambient manifold with bounded curvatures.

2. Preliminaries: Poisson hierarchy, Green operator, Moment spectrum and Model Spaces

2.1. **Poisson Hierarchy.** Classically, from mathematical physics, given a precompact domain $D \subset \mathbb{R}^2$ with smooth boundary, the *torsional rigidity* $\mathcal{A}_1(D)$ of D is the torque required per unit angle of twist and per unit length when twisting an elastic beam of uniform cross section D, see [1, 41]. The torsional rigidity of D can be calculated as the L_1-norm of the expectation of the first exit time function

$$\mathcal{A}_1(D) = \|\mathbb{E}^x[\tau_D]\|_1 = \int_D \mathbb{E}^x[\tau_D]dV.$$

Given a precompact domain $D \subset M$ with smooth boundary, the mean exit time function

$$E : D \to \mathbb{R}, \quad x \mapsto \mathbb{E}^x[\tau_D],$$

is characterized (see [20]), as the solution of the following second order PDE, with Dirichlet boundary data

$$\begin{aligned} \Delta E + 1 &= 0, \text{ in } D, \\ E|_{\partial D} &= 0. \end{aligned} \tag{2.1}$$

This characterization of the first moment of the exit time can be extended to any k-th moment. There exits a sequence $\{u_k\}$ of smooth functions such that

$$\mathcal{A}_k(D) = \|\mathbb{E}^x[\tau_D^k]\|_1 = \int_D u_k(x)dV(x).$$

This sequence $\{u_k\}$ is the so-called, see [19], *Poisson hierarchy for D*. The functions $\{u_k\}$ of the Poisson hierarchy can be obtained inductively as the solution of the following boundary value problems on D: First we let,

$$u_0(x) = 1, \quad \text{for} \quad \text{all} \quad x \in D,$$

and then for $k \geq 1$,

$$\begin{aligned} \Delta u_k + k u_{k-1} &= 0, \text{ on } D, \\ u_k|_{\partial D} &= 0. \end{aligned} \tag{2.2}$$

We are going to focus our study in $\{\mathcal{A}_k(D)\}_{k=1}^{\infty}$ which is the L^1-moment spectrum of D. But in some parts of this survey we will also consider the L^p-moment spectrum of D, which can be defined as the following sequence of integrals:

$$\mathcal{A}_{p,k}(D) := \left(\int_D (u_k(x))^p dV\right)^{\frac{1}{p}},\ k = 1, 2, ..., \infty.$$

2.2. **Green Operator.** Let D be a bounded open subset with smooth boundary $\partial D \neq \emptyset$ of a Riemannian manifold (M, g). The Green operator $G^D : L_2(D) \to L_2(D)$ is given by

$$G^D(f)(x) = \int_0^\infty \int_D p_t^D(x, y)\, f(y) dV\, dt = \int_D g^D(x, y)\, f(y) dV,$$

where $p_t^D(x, y)$ is the heat kernel of the operator $-\Delta$ and

$$g^D(x, y) = \int_0^\infty p_t^D(x, y) dt$$

is the Green function of D. The Green operator is a bounded self-adjoint operator in $L_2(D)$ and it is the inverse of $-\Delta$. Thus for any $f \in L_2(D)$ there is a unique solution $u = G^D(f)$ to the equation $-\Delta u = f$. Applying G^D to the equation

$$\Delta u + \lambda_i(D) u = 0,$$

the eigenvalue $\lambda_i(D)$ of D is given by $\lambda_i(D) = u/G^D(u)$. This kind of quotients for a suitable function u can be used to obtain estimations of the eigenvalue $\lambda_1(D)$ as we will see in Section 3.

In the sequel, we will omit the superscript D in the notation of the Green operator G^D by simplicity in the notation.

2.3. **Radial Green Operator and Poisson Hierarchy for Geodesic Balls of Model spaces.** The problem of finding $\{u_k\}$ and $\{\mathcal{A}_{p,k}(D)\}$ for a given smoothly bounded precompact domain $D \subset M$ is, in general, hard to solve. This problem can be simplified when certain symmetries are assumed. In particular, the problem can be completely and explicitly solved when the metric tensor of M is rotationally symmetric around a point $p \in M$. We will say that the metric tensor of M is *rotationally symmetric* around p if the metric spheres S_R of radius R centered at p have maximal dimension of the isometry group, and hence, in the case of being simply connected are isometric to some $n-1$ dimensional sphere of constant sectional curvature. Rotationally symmetric metrics can be constructed by using warping products as follows:

Definition 2.1 (See [22, 23, 39])**.** *A $w-$model M_w^m is a product manifold*

$$[0, \Lambda) \times \mathbb{S}_1^{m-1}/\sim, \quad (t_1, \theta_1) \sim (t_2, \theta_2) \quad \text{if} \quad (t_1, \theta_1) = (t_2, \theta_2) \quad \text{or} \quad t_1 = t_2 = 0,$$

endowed with the warping product metric

$$dr \otimes dr + w^2(r) d\theta \otimes d\theta,$$

where $d\theta \otimes d\theta$ is the standard metric of constant sectional curvature 1 *in the sphere $\mathbb{S}_1^{m-1}$, and w is the warping function $w : [0, \Lambda) \to \mathbb{R}_+ \cup \{0\}$ with $w(0) = 0$, $w'(0) = 1$, $w^{(k)}(0) = 0$ for all even derivation orders k, and $w(r) > 0$ for all $r > 0$. The point $o_w = \pi^{-1}(0)$, where π denotes the projection onto $[0, \Lambda)$, is called the* center point *of the model space. The value Λ is called the* radius of the model space.

The conditions on w and its derivatives at 0 in the above definition are required to ensure a smooth metric tensor on o_w. Observe that the metric of a model space is rotationally symmetric around the center point o_w and moreover, if the model space has radius $\Lambda = \infty$, then the center point o_w is a pole of M_w^m.

Remark 2.2. *The simply connected space forms $\mathbb{K}^m(b)$ of constant curvature b can be constructed as $w-$models with warping functions*

$$w(r) = w_b(r) = \begin{cases} \frac{1}{\sqrt{b}}\sin(\sqrt{b}\, r) & \text{if } b > 0 \\ \quad r & \text{if } b = 0 \\ \frac{1}{\sqrt{-b}}\sinh(\sqrt{-b}\, r) & \text{if } b < 0, \end{cases} \tag{2.3}$$

and radius $\Lambda = \infty$ for $b \leq 0$ and $\Lambda = \pi/\sqrt{b}$ for $b \geq 0$. Note that for $b > 0$ the warping function w_b induces a smooth metric tensor on

$$\left[0,\, \pi/\sqrt{b}\right] \times \mathbb{S}_1^{m-1}/\sim,$$

where $(t_1, \theta_1) \sim (t_2, \theta_2)$ if $(t_1, \theta_1) = (t_2, \theta_2)$, or $t_1 = t_2 = 0$, or $t_1 = t_2 = \pi/\sqrt{b}$. For $b \leq 0$ any center point is a pole.

A complete description of these model spaces can be found in [22, 23, 31, 32, 37]. The sectional curvatures in the radial directions from the center point are determined by the radial function $-\frac{w''}{w}(r(p))$ for any $p \in M_w^m$. Moreover, the mean curvature of the distance sphere of radius R from the center point is a radial function given by

$$\eta_w(R) = \frac{w'(R)}{w(R)} = \ln'(w(R)). \tag{2.4}$$

The solution to the boundary value problem (2.2) defined on the geodesic R-ball B_R^w of radius R centered at $o_w \in M_w^m$ is computed in [27] and the functions u_k are given by

$$u_k(r) = k \int_r^R \frac{\int_0^t w^{m-1}(s)\, u_{k-1}(s)\, ds}{w^{m-1}(t)}\, dt. \tag{2.5}$$

Moreover, by applying the Divergence theorem

$$\mathcal{A}_k(B_R^w) = -\frac{1}{k+1} u'_{k+1}(R) \operatorname{Vol}(S_R^w), \tag{2.6}$$

where S_R^w is the geodesic R-sphere in M_w^m. Hence, the Poisson hierarchy $\{u_k\}$ and the moment spectrum $\{\mathcal{A}_k(B_R^w)\}$ can be always explicitly computed and it depends only on the warping function w. Observe moreover that the Poisson hierarchy $\{u_k\}$ consists of radial functions for geodesic balls of a model space. This is in fact a particular case of a boarder phenomena: given an isometry $\phi : M \to M$ such that for a some precompact domain $D \subset M$ with smooth boundary

$$\phi(D) = D, \quad \phi(\partial D) = \partial D,$$

any function $u_k : D \to \mathbb{R}$ satisfying (2.2) must be invariant under the action of ϕ, *i.e.*,

$$u_k = u_k \circ \phi. \tag{2.7}$$

The above equality can be deduced because since ϕ is an isometry, $\Delta(u_k \circ \phi) = (\Delta u_k) \circ \phi$ and therefore $u_k \circ \phi$ satisfies (2.2) as well. Finally the uniqueness of the solution of (2.2) implies equality (2.7).

In a model space M^m_w, the Green operator for radial functions on B^w_R is given by

$$G(u)(r) = \int_r^R \frac{\int_0^t w^{m-1}(s)\, u(s) ds}{w^{m-1}(t)}\, dt, \tag{2.8}$$

and the moment functions u_k can be written recursively via the Green operator as $u_k(r) = kG(u_{k-1})(r)$. Indeed, in [10] it is proved that for any bounded open subset D with smooth boundary $\partial D \neq \emptyset$ in a Riemannian manifold, the sequence of the Poisson hierarchy for D can be written as

$$u_k = k! G^k(1),$$

where G is the Green operator on D and $G^k = G \circ \cdots \circ G$ k-times.

In this context the Green operator has been used to obtain bounds of $\lambda_1(B^w_R)$ as can be seen in [6, 7, 10, 28, 42].

3. First Eigenvalue and Moment Spectrum

In this section we collect some results that show how to obtain estimations of the first Dirichlet eigenvalue of a domain D in terms of its moment spectrum. The relationship between the moment spectrum and the Dirichlet spectrum is given by the equality (see [35] and see [10] for the weighted version):

$$\mathcal{A}_k(D) = k! \sum_{i=1}^{\infty} \frac{a_i^2}{\lambda_i^k}, \tag{3.1}$$

where $a_i = \int_D \phi_i \, dV$ and $\{\phi_i\}$ is a complete orthonormal basis in $L^2(D)$ of eigenfunctions of $-\Delta$ with associated eigenvalues $\{\lambda_i\}$. In fact, using this formula, in [35] it is proved that

Theorem 3.1. *The first Dirichlet eigenvalue $\lambda_1(D)$ of any smoothly bounded precompact domain $D \subseteq M$ can be directly extracted from the corresponding exit time moment spectrum $\{A_k(D)\}_{k=k_0}^{\infty}$ as follows:*

$$\lambda_1(D) = \sup \left\{ \eta \geq 0 \, : \, \lim_{k\to\infty} \sup \left(\frac{\eta}{2}\right)^n \frac{\mathcal{A}_k(D)}{\Gamma(n+1)} < \infty \right\}.$$

In [28], Markvorsen, Palmer, and the second author study the problem of determining the first eigenvalue of a geodesic ball B^w_R in a model space M^m_w with pole o_w. For the radial functions $u_k(r)$ defined on $D = B^w_R$ and given by (2.5), they proved that, since $u_k(r) = kG(u_{k-1}(r))$ for the Green operator G in (2.8), the functions $\frac{k\, u_{k-1}}{u_k}(r)$ are increasing. Then, applying Barta's inequality:

$$\frac{k\, u_{k-1}}{u_k}(o_w) \leq \lambda_1(B^w_R) \leq \frac{k\, \mathcal{A}_{k-1}(B^w_R)}{\mathcal{A}_k(B^w_R)}. \tag{3.2}$$

For k even the upper bound was generalized in [19] for a bounded open domain D with smooth boundary in a complete Riemannian manifold. When $k = 1$, the above inequalities reads

$$\frac{1}{\int_0^R q_w(t)\, dt} \leq \lambda_1(B^w_R) \leq \frac{\mathrm{vol}(B^w_R)}{\mathcal{A}_1(B^w_R)}, \tag{3.3}$$

where $q_w(t)$ is the isoperimetric quotient defined by

$$q_w(t) := \frac{\int_0^t w^{m-1}(s)\, ds}{w^{m-1}(t)}. \tag{3.4}$$

The lower bound was obtained in [11] for geodesic balls in the n-dimensional sphere $\mathbb{S}^n(1)$, and later it was generalized for an arbitrary M_w^m in [7]. Since, in this case,

$$\int_0^R q_w(t)\,dt = \max_{x\in B_R^w} u_1(x),$$

this bound is also related with the following result of Del Grosso-Marchetti proved in [25], see also [12]:

Proposition 3.2. *Let $D\subset M$ be a smoothly bounded precompact domain. Then*

$$\lambda_1(D) \geq \frac{1}{\max\limits_{x\in D} u_1(x)}.$$

By using the specific expression of the Green function, it is proved in [10] that in the particular case of geodesic balls in rotationally symmetric manifolds can be stated the following

Theorem 3.3. *Let B_R^w be the geodesic ball of radius R in M_w^m. Then,*

$$\max_{x\in B_R^w} u_1(x) = \sum_{i=1}^{\infty} \frac{1}{\lambda_i^{\mathrm{rad}}(B_R^w)}$$

where $\left\{\lambda_i^{\mathrm{rad}}(B_R^w)\right\}_{i=1}^{\infty}$ is the set of eigenvalues associated to radial eigenfunctions.

The upper bound in (3.2) gives a relation between the first Dirichlet eigenvalue and the torsional rigidity of a geodesic ball in a rotationally symmetric space. For domains in Euclidean space this is the classical inequality of Pólya's (see [41], see [3, 4] for further developments) and it was generalized in [19] for a general bounded open domain D in a complete Riemannian manifold. Indeed, they prove that:

Theorem 3.4 ([19])**.** *Let $D\subseteq M$ be a bounded open domain with smooth boundary in a complete Riemannian manifold M. Then for every k,*

$$\lambda_1(D) \leq \frac{(k!)^2}{(2k-1)!}\frac{\mathcal{A}_{2k-1}(D)}{\mathcal{A}_k^2(D)}\,\mathrm{Vol}(D).$$

Relating the quotient before with the variance of the random variable τ_D^k given by the k-power of the first exit time from D, they also obtain that

Corollary 3.5 ([19])**.** *Let D as before. For $k\in\mathbb{N}$, let $\mathrm{Var}_k(D)$ be the L^1-norm of the variance of τ_D^k:*

$$\mathrm{Var}_k(D) = \int_D (u_{2k}-u_k)\,dV. \tag{3.5}$$

Then,

$$\lambda_1(D) \leq \frac{(2k)!-(k!)^2}{(2k-1)!}\frac{\mathcal{A}_{2k-1}(D)}{\mathrm{Var}_k(D)}.$$

It is also established in [19] a lower bound for the first Dirichlet eigenvalue using the quantity a_1^2 appearing in (3.1) and the L^1-moment spectrum of a bounded open domain D with smooth boundary. Indeed, for all $k\leq 1$,

$$\left(\frac{k!a_1^2}{\mathcal{A}_k(D)}\right)^{1/k} \leq \lambda_1(D). \tag{3.6}$$

In the previous results, the value of the first eigenvalue of a domain is bounded from above or from below by geometric quantities involving the isoperimetric quotient, the torsional rigidity or the L^1-moment spectrum of the domain D. But the exact value of $\lambda_1(D)$ can be obtained as a limit of some of these estimations. Indeed, the bounds in (3.2) improves as k increases and we can obtain better and better estimates of $\lambda_1(B_R^w)$ since the functions u_k can be explicitly computed in this case (see for instance [7, 10, 11, 28, 42] for approximations in this line). Moreover, the exact value of $\lambda_1(B_R^w)$ is obtained in the limit:

Theorem 3.6 ([28])**.** *Let B_R^w be the geodesic ball of radius R and center the pole o_w in M_w^m. Then,*

$$\lambda_1(B_R^w) = \lim_{k\to\infty} \frac{k\, u_{k-1}(0)}{u_k(0)} = \lim_{k\to\infty} \frac{k\, \mathcal{A}_{k-1}(B_R^w)}{\mathcal{A}_k(B_R^w)},$$

where u_k are the functions defined by (2.5) *and $\{\mathcal{A}_k(B_R^w)\}$ is the L^1-moment spectrum of B_R^w. Moreover, the radial C^2-function $g_\infty(r) := \lim_{k\to\infty} \frac{u_k(r)}{u_k(0)}$ is an eigenfunction of the first eigenvalue.*

This result has been generalized in [10] for a bounded open domain D with smooth boundary in a (weighted) Riemannian manifold using the relation (3.1), and it is also generalized in [18]. By using the expression of the Green operator $G : L^2(D) \to L^2(D)$ in an orthonormal basis in $L_2(D)$ of eigenfunctions of $-\Delta$, and studying the convergence of the sequence $k \mapsto \frac{\|G^k(f)\|_{L^2}}{\|G^{k+1}(f)\|_{L^2}}$, in [10] Bessa, Jorge, and the first author proved the following relation between the Green operator and the spectrum of the Laplacian.

Theorem 3.7. *Let D be a bounded open subset with smooth boundary $\partial D \neq \emptyset$ in a Riemannian manifold (M^m, g). Let G be the Green operator. Then, for any positive $f \in L^2(D)$,*

$$\lambda_1(D) = \lim_{k\to\infty} \frac{\|G^k(f)\|_{L^2}}{\|G^{k+1}(f)\|_{L^2}}. \tag{3.7}$$

Indeed, the previous theorem is stated in [10] in the boarder setting of weighted manifolds. Observe, moreover, that using $f = 1$ in the above theorem, we can state that L^2-norm of the Poisson hierarchy is related with the first eigenvalue. In fact,

$$\begin{aligned}\lambda_1(D) &= \lim_{k\to\infty} (k+1) \frac{\|u_k\|_{L^2}}{\|u_{k+1}\|_{L^2}} = \lim_{k\to\infty} (k+1) \frac{\mathcal{A}_{2,k}(D)}{\mathcal{A}_{2,k-1}(D)} \\ &= \lim_{k\to\infty} \frac{k\mathcal{A}_{2,k-1}(D)}{\mathcal{A}_{2,k}(D)}.\end{aligned} \tag{3.8}$$

More recently Sarrion-Pedralva and the first author used in [21] the above techniques to obtain an upper bound for the first eigenvalue of a geodesic ball B_R centered at a point p of a Riemmanian m-dimensional manifold. The main idea is to built up a model space M_w^m with geodesic spheres of the same volume than the geodesic spheres included in B_R. Then using the first eigenfunction of B_R^w in this model space and the Rayleigh quotient they show that

Theorem 3.8. [21]*Let (M, g) be a Riemannian manifold, let $p \in M$ with injectivity radius* $\mathrm{inj}(p)$*, let B_R be the geodesic ball of radius R centered at p, and let $\mathrm{A}(t) :=$* $\mathrm{vol}(\mathrm{S}_t)$ *be the area function of the geodesic spheres centered at p. If $R < \mathrm{inj}(p)$, then the model space M_w^m with warping function*

$$w(t) = \left(\frac{\mathrm{vol}(\mathrm{S}_t)}{\mathrm{vol}(\mathbb{S}_1^{m-1})}\right)^{\frac{1}{m-1}} \tag{3.9}$$

has smooth metric tensor in B_R^w and

$$\lambda_1(B_R) \leq \lambda_1(B_R^w). \tag{3.10}$$

Furthermore, equality (3.10) *is attained if and only if, for any* $t \in (0, R)$*, the mean curvature pointed inward* $\mathrm{H}_{\mathrm{S}_t}$ *of the geodesic sphere* S_t *of radius* t *centered at* p *is a radial function. Namely, equality* (3.10) *is attained, if and only if, there exists a smooth function* $h(t)$ *such that*

$$\mathrm{H}_{\mathrm{S}_t} = h(t) \quad \text{for any} \quad 0 < t < R.$$

Since the warping function given in (3.9) of the model space in the above theorem depends only on the area function of the geodesic spheres, we can rewrite the expression of the radial Green operator in (2.8), and the Poisson hierarchy of B_R^w in (2.5) in terms of this area function as follows:

$$G(u)(r) := \int_r^R \frac{\int_0^t A(s)u(s)ds}{A(t)}dt, \quad u_k(r) = k\int_r^R \frac{\int_0^t A(s)\,u_{k-1}(s)\,ds}{A(t)}\,dt. \tag{3.11}$$

This allows us to provide the following theorem which makes use of almost every result listed in in this survey for the comparison of the first eigenvalue and the Poisson hierarchy for geodesic balls in rotationally symmetric model spaces.

Theorem 3.9. *Let* (M, g) *be a Riemannian manifold, let* $p \in M$ *with injectivity radius* $\mathrm{inj}(p)$*, and let* B_R *be the geodesic ball of radius* R *centered at* p*. Let us denote by* $\{\mathcal{A}_{1,k}\}_{k=0}^\infty$*,* $\{\mathcal{A}_{2,k}\}_{k=0}^\infty$ *and* $\{\mathcal{V}_k\}_{k=0}^\infty$ *the following sequences constructed recursively from the area function,*

$$\begin{aligned}
\mathcal{A}_{1,k} &:= \textstyle\int_0^R u_k(t)A(t)dt,\\
\mathcal{A}_{2,k} &:= \left(\textstyle\int_0^R u_k^2(t)A(t)dt\right)^{1/2},\\
\mathcal{V}_k &:= \textstyle\int_0^R \left(u_{2k}(t) - u_k(t)\right)A(t)dt,
\end{aligned}$$

with

$$u_k(r) = k\int_r^R \frac{\int_0^t A(s)\,u_{k-1}(s)\,ds}{A(t)}\,dt, \quad u_0 = 1.$$

If $R < \mathrm{inj}(p)$*, then, for any* $k \geq 1$*,*

$$\lambda_1(\mathrm{B}_R) \leq \tfrac{k\mathcal{A}_{1,k-1}}{\mathcal{A}_{1,k}}, \tag{3.12}$$

$$\lambda_1(\mathrm{B}_R) \leq \tfrac{(k!)^2}{(2k-1)!}\tfrac{\mathcal{A}_{1,2k-1}}{\mathcal{A}_{1,k}^2}\,\mathcal{A}_{1,0}, \tag{3.13}$$

$$\lambda_1(\mathrm{B}_R) \leq \tfrac{(2k)!-(k!)^2}{(2k-1)!}\tfrac{\mathcal{A}_{1,2k-1}}{\mathcal{V}_k}. \tag{3.14}$$

Moreover,

$$\begin{aligned}
\lambda_1(\mathrm{B}_R(p)) &\leq \lim_{k\to\infty}\frac{k\mathcal{A}_{2,k-1}}{\mathcal{A}_{2,k}},\\
\lambda_1(\mathrm{B}_R(p)) &\leq \lim_{k\to\infty}\frac{k\mathcal{A}_{1,k-1}}{\mathcal{A}_{1,k}},
\end{aligned} \tag{3.15}$$

where equality (3.15) *is attained in if and only if, for any* $t \in (0, R)$*, the mean curvature pointed inward* $\mathrm{H}_{\mathrm{S}_t}$ *of the geodesic sphere* S_t *of radius* t *centered at* p *is a radial function.*

Remark 3.10. *Unless in this survey we are focus our attention in the first Dirichlet eigenvalue, estimates of the following eigenvalues in terms of the moment spectrum can also be obtained, see* [10, 18, 19].

4. Comparison Results for the First Eigenvalue

An important problem in Riemannian geometry is to find good upper and lower estimates of the first Dirichlet eigenvalue of a domain in a Riemannian manifold M. A way to obtain such estimations, in contrast with the techniques used in the above section, is to establish comparison results. If we assume that M has controlled geometry, in the sense that some of its curvatures (or another interesting geometric quantity) are bounded from above or from below by the curvatures of a model space, one can obtain inequalities for the Laplacian of distinguished functions that provide estimations not only for the first Dirichlet eigenvalue, but also for other geometric objects which are related with the Laplacian operator. These estimations are usually given in terms of the corresponding quantity for a geodesic ball in the model space that we are using to establish the comparison. Classical results in this direction are Cheng's comparison theorems for the first eigenvalue of a geodesic ball B_R in a manifold with sectional curvatures (respectively, Ricci curvatures) bounded from above (respectively, from below) by a given constant [13, 14]. In this case, $\lambda_1(B_R)$ is bounded by the first eigenvalue of a geodesic ball in a simply connected real space form of constant sectional curvature.

There is a vast literature in this subject; in this section we do not aim to give an exhaustive list of comparison results for the first eigenvalue of the Laplacian, but just to focus our attention in those which are most related to the ones in the previous section.

Let us consider a complete Riemannian manifold (M^m, g) and fix $p \in M$. For any $x \in M - \{p\}$ the sectional curvature of the two-planes $\sigma_x \in T_xM$ that contain the tangent vector to a minimal geodesic from p to x is called the p-radial sectional curvature of M. In a model space M^m_w, the o_w-radial curvatures are determined by the radial function $-w''(r)/w(r)$. In [28] it is established a generalization of Cheng's eigenvalue comparison theorem for Riemannian manifolds with radial sectional curvatures bounded by the curvatures of M^m_w. If g_∞ is the eigenfunction of $\lambda_1(B^w_R)$ given in Theorem 3.6, then we can obtain a radial function on M by composing g_∞ with the distance function from the point p: $g(r) := g_\infty \circ r$. Comparing the Laplacian in M of the function g with the Laplacian in M^m_w of g_∞ it can be proved the following.

Theorem 4.1 ([28]). *Let B_R be a geodesic ball of a complete Riemannian manifold M^m with a pole p and suppose that the p-radial sectional curvatures of M^m are bounded from below (respectively from above) by the o_w-radial sectional curvatures of a w-model space M^m_w. Then*

$$\lambda_1(B_R) \leq (\geq) \lambda_1(B^w_R) = \lim_{k\to\infty} \frac{k\mathcal{A}_{k-1}(B^w_R)}{\mathcal{A}_k(B^w_R)}, \tag{4.1}$$

where B^w_R is the o_w-centered geodesic ball in M^m_w.

An alternative proof of this theorem can be done using the description of $\lambda_1(D)$ given by McDonald and Meyers in Theorem 3.1 and the isoperimetric type inequalities for the exit time moment spectrum established in [27] that asserts that under the hypothesis of theorem 4.1,

$$\frac{\mathcal{A}_k\left(\mathrm{B}_R\right)}{\mathrm{vol}\left(\mathrm{S}_R\right)} \geq (\leq) \frac{\mathcal{A}_k\left(\mathrm{B}^\omega_R\right)}{\mathrm{vol}\left(\mathrm{S}^\omega_R\right)}, \quad \text{for all } k \geq 0. \tag{4.2}$$

Using this second strategy, if the model manifold M^m_w is *strictly balanced* in the sense that $q_w(r)\eta_w(r) > 1/m$ for all radius r, then equality in (4.1) for some fixed radius R_0 implies that B_{R_0} and $B^w_{R_0}$ are isometric.

In a similar direction, it is shown in [34] that if a complete Riemannian manifold M^m satisfies the *moment comparison condition with constant curvature space form* $\mathbb{K}^m(b)$, $\mathcal{A}_k(D) \leq \mathcal{A}_k(B)$ for all smoothly bounded domain $D \subset M$ with compact closure and all $k \in \mathbb{N}$, where B is the geodesic ball in $\mathbb{K}^m(b)$ with the same volume of D, then

$$\lambda_1(B) \leq \lambda_1(D).$$

We point out that in [28] the comparison result in Theorem 4.1 is formulated in a more general context. The authors consider an unbounded complete and closed submanifold P^n with controlled mean curvature in a Riemannian manifold N^m with radial sectional curvatures bounded by a radial function. In this context, an estimation of the first Dirichlet eigenvalue of a extrinsic ball D_R, namely, the intersection of a geodesic ball of the ambient manifold with the submanifold P, is obtained in terms of the first eigenvalue of a geodesic ball in a suitable model space. However the construction of this model space of comparison in terms of the assumptions over P and M is very technical, so for clarity of exposition, we have only stated the corresponding intrinsic version. The details of the general statement can be seen in [28]. As a particular case of these more general statements for a Cartan-Hadamard manifold, applying the techniques in [15] (see also [8]), we have that

Theorem 4.2 ([28]). *Let M^m be a Cartan-Hadamard manifold, with sectional curvatures bounded from above by a constant $K_M \leq b \leq 0$. Let $p \in M$ be a pole in M and r the distance function from p. Let $P^n \subseteq M^m$ be a complete and non-compact properly immersed submanifold with*

$$-\langle \nabla r(x), H_P(x) \rangle \leq h(r(x)),$$

for all $x \in P^n$, where H_P is the mean curvature of P^n and $h(r)$ is a radial smooth function. Suppose that

$$(n-1) \cdot \sqrt{-b}\coth(R\sqrt{-b}) \geq n \cdot \sup_{r \in [0,R]} h(r), \tag{4.3}$$

where we read $\sqrt{-b}\coth(R\sqrt{-b})$ to be $1/R$ when $b = 0$. For any given extrinsic ball $D_R(p)$ in P^n we then have the following inequality:

$$\lambda_1(D_R) \geq \frac{1}{4}\left((n-1) \cdot \sqrt{-b}\coth(R\sqrt{-b}) \; - n \cdot \sup_{r \in [0,R]} h(r)\right)^2. \tag{4.4}$$

Notice that when P is a minimal submanifold we can take as bounding function $h(r(x)) = 0$. In the intrinsic setting, when $P^n = N^m$, $H_P = 0$, and $D_R = B_R$ the classical result of McKean [36] for the fundamental tone of a Cartan-Hadamard manifold is recovered. Moreover, Bessa and Montenegro observe in [8] an improvement of the bound (4.4) in the intrinsic setting as follows:

Theorem 4.3. *Under the intrinsic conditions with M^m having sectional curvatures bounded from above by $b \leq 0$:*

$$\lambda_1(B_R) \geq \frac{1}{4}\left(\max\left(\frac{m}{R}\;,\;(m-1) \cdot \sqrt{-b}\coth(R\sqrt{-b})\right)\right)^2. \tag{4.5}$$

Comparison results for the first eigenvalue of a domain can be also formulated assuming bounds for the mean curvature of the geodesic spheres of the manifold M instead of bounding the sectional or Ricci curvatures. Given a point $p \in M$, for $R < \mathrm{inj}(p)$, we denote by H_{S_R} the pointed inward mean curvatures of the R-geodesic spheres centered at p in M. Notice that in a model space M^m_w, $H_{S^w_R} = \eta_w(R) = w'(R)/w(R)$. With this weaker hypothesis, in [9] the first eigenvalue of a geodesic ball is compared with the first eigenvalue of a geodesic ball in a model space and recently, it is shown in [38] the following.

Theorem 4.4. *Let M^m be a complete manifold and let $p \in M$ such that* $\operatorname{inj}(p) \leq \operatorname{inj}(o_w)$ *for o_w the center point of a rotationally symmetric space M_w^m. Assume that for $R < \operatorname{inj}(p)$ the mean curvature of the geodesic spheres in M satisfies that*

$$H_{S_t} \geq (\leq) H_{S_t^w} = \frac{w'(t)}{w(t)}, \tag{4.6}$$

for all $0 < t \leq R$. Then,

$$\lambda_1(B_R) \geq (\leq) \lambda_1(B_R^w). \tag{4.7}$$

Equality in (4.7) *implies that*

$$H_{S_t} = H_{S_t^w}, \tag{4.8}$$

for all $0 < t \leq R$, and then we have the following equalities:

(1) $u_k = u_k^w \circ r_p$ *on B_R for all $k \geq 1$, where r_p is the distance function from the point p in M and $\{u_k\}$ and $\{u_k^w\}$ are the Poisson hierarchy for B_R and B_R^w, respectively. Hence, $u_k = u_k^w \circ r_p$ on B_t for all $k \geq 1$ and for all $0 < t \leq R$.*
(2) $\operatorname{vol}(B_t) = \operatorname{vol}(B_t^w)$ *and* $\operatorname{vol}(S_t) = \operatorname{vol}(S_t^w)$ *for all $0 < t \leq R$.*
(3) $\mathcal{A}_k(B_t) = \mathcal{A}_k(B_t^w)$*, for all $k \geq 1$ and for all $0 < t \leq R$.*

The proof follows the lines of the proof of Theorem 4.1, so the authors previously shown that inequality (4.2) for the exit time moment spectrum is still valid under this weaker hypothesis on the mean curvature of geodesic spheres. In this setting, the equality between the first eigenvalues of the geodesic spheres of the manifold M and the model space M_w^m gives the equality between the mean curvatures of the corresponding geodesic spheres instead of the isometry of the geodesic balls as happens in Theorem 4.1 where the radial sectional curvatures of M are bounded for the radial sectional curvatures of M_w^m. Observe that this is coherent since bounds on the sectional curvatures of the manifold implies bounds on the mean curvature of its geodesic spheres, but the reciprocal is not true. Indeed, in [9] the authors construct a family of complete smooth metrics on $\mathbb{R}^m$ non-isometric to the constant sectional curvatures b metrics of the simply connected space forms $\mathbb{K}^m(b)$ such that the geodesic balls B_R and $B_R^{w_b}$ have the same first eigenvalue and the geodesic spheres S_t and $S_t^{w_b}$ for $0 < t \leq R$ have the same mean curvatures (see also the examples in [10] and [38]).

Finally, as a consequence of the comparison results obtained in [38] and the study of the equality case in the comparison, it is shown that

Corollary 4.5. [38] *Under the assumptions of Theorem 4.4, the following equalities are equivalent:*

(1) $\lambda_1(B_R) = \lambda_1(B_R^w)$.
(2) $\mathcal{A}_k(B_R) = \mathcal{A}_k(B_R^w)$*, for all $k \geq 1$.*
(3) $u_k = u_k^w \circ r_p$ *on B_R for all $k \geq 1$, where r_p is the distance function from the point p in M and $\{u_k\}$ and $\{u_k^w\}$ are the Poisson hierarchy for B_R and B_R^w, respectively.*

Moreover, equality $H_{S_t} = H_{S_t^w}$ for all $0 < t \leq R$ implies any (and hence all) of the equalities (1)*,* (2)*, and* (3)*.*

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Departament de Matemàtiques- IMAC, Universitat Jaume I, Castelló, Spain.
Email address: `gimenov@uji.es`

Departamento de Geometría y Topología and Excellence Research Unit "Modeling Nature" (MNat), Universidad de Granada, E-18071, Spain.
Email address: `ahurtado@ugr.es`