Norm attaining compact operators

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Banach Spaces and related topics

Roadmap of the course

Notation

X, Y real or complex Banach spaces

- $\mathbb{K}$ base field $\mathbb{R}$ or $\mathbb{C}$,
- $B_X = \{x \in X \,:\, \|x\| \leqslant 1\}$ closed unit ball of X,
- $S_X = \{x \in X \,:\, \|x\| = 1\}$ unit sphere of X,
- $\mathcal{L}(X,Y)$ bounded linear operators from X to Y,
  - $\|T\| = \sup\{\|T(x)\| \,:\, x \in S_X\}$ for $T \in \mathcal{L}(X,Y)$,
- $\mathcal{W}(X,Y)$ weakly compact linear operators from X to Y,
- $\mathcal{K}(X,Y)$ compact linear operators from X to Y,
- $\mathcal{F}(X,Y)$ bounded linear operators from X to Y with finite rank,
- if $Y = \mathbb{K}$, $X^* = \mathcal{L}(X,Y)$ topological dual of X,
- if $X = Y$, we just write $\mathcal{L}(X)$, $\mathcal{W}(X)$, $\mathcal{K}(X)$, $\mathcal{F}(X)$.

Observe that

$$\mathcal{F}(X,Y) \subset \mathcal{K}(X,Y) \subset \mathcal{W}(X,Y) \subset \mathcal{L}(X,Y).$$

An overview on norm attaining operators

Section 1

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An overview on norm attaining operators

Section 1

Norm attaining functionals and operators

Norm attaining functionals

$x^* \in X^*$ attains its norm when

$$\exists\, x \in S_X \ : \ |x^*(x)| = \|x^*\|$$

★ $\mathrm{NA}(X,\mathbb{K}) = \{x^* \in X^* \ : \ x^* \text{ attains its norm}\}$

Examples

- $\dim(X) < \infty \implies \mathrm{NA}(X,\mathbb{K}) = \mathcal{L}(X,\mathbb{K})$ (Heine-Borel).
- X reflexive $\implies \mathrm{NA}(X,\mathbb{K}) = \mathcal{L}(X,\mathbb{K})$ (Hahn-Banach).
- X non-reflexive $\implies \mathrm{NA}(X,\mathbb{K}) \neq \mathcal{L}(X,\mathbb{K})$ (James),
- but $\mathrm{NA}(X,\mathbb{K})$ separates the points of X (Hahn-Banach).

Norm attaining functionals and operators

Norm attaining operators

$T \in \mathcal{L}(X,Y)$ attains its norm when

$$\exists\, x \in S_X \ :\ \|T(x)\| = \|T\|$$

★ $\mathrm{NA}(X,Y) = \{T \in \mathcal{L}(X,Y) \ :\ T$ attains its norm$\}$

Examples

- $\dim(X) < \infty \implies \mathrm{NA}(X,Y) = \mathcal{L}(X,Y)$ for every Y (Heine-Borel).
- $\mathrm{NA}(X,Y) \neq \emptyset$ (Hahn-Banach).
- X reflexive $\implies \mathcal{K}(X,Y) \subseteq \mathrm{NA}(X,Y)$ for every Y.
- X non-reflexive $\implies \mathrm{NA}(X,Y) \cap \mathcal{K}(X,Y) \neq \mathcal{K}(X,Y)$ for every Y.
- $\dim(X) = \infty \implies \mathrm{NA}(X,c_0) \neq \mathcal{L}(X,c_0)$ (see M.-Merí-Payá, 2006).

The problem of density of norm attaining functionals

Problem

Is $\mathrm{NA}(X, \mathbb{K})$ always dense in X^***?**

Theorem (E. Bishop & R. Phelps, 1961)

The set of norm attaining functionals is dense in X^* (for the norm topology).

Problem

Is $\mathrm{NA}(X, Y)$ always dense in $\mathcal{L}(X, Y)$**?**

The answer is **No** (as we will see in a minute).

Modified problem

When is $\mathrm{NA}(X, Y)$ dense in $\mathcal{L}(X, Y)$**?**

The study of this problem was initiated by J. Lindenstrauss in 1963, who provided the first negative and positive examples.

An overview on norm attaining operators

Section 1

An easy negative example

Example (Lindenstrauss, 1963)

Y strictly convex such that there is a non-compact operator from c_0 into Y.

$$\text{Then, } \mathrm{NA}(c_0, Y) \text{ is not dense in } \mathcal{L}(c_0, Y).$$

Lemma

If Y is strictly convex, then $\mathrm{NA}(c_0, Y) \subseteq \mathcal{F}(c_0, Y)$.

Example (Lindenstrauss, 1963)

There exists Z such that $\mathrm{NA}(Z, Z)$ is not dense in $\mathcal{L}(Z)$. Actually, $Z = c_0 \oplus_\infty Y$.

Lindenstrauss properties A and B

Observation

- The question now is for which X and Y the density holds.
- As this problem is too general, Lindenstrauss introduced two properties.

Definition

X, Y Banach spaces,

- X has (Lindenstrauss) property A when $\overline{\mathrm{NA}(X,Z)} = \mathcal{L}(X,Z) \quad \forall\, Z$
- Y has (Lindenstrauss) property B when $\overline{\mathrm{NA}(Z,Y)} = \mathcal{L}(Z,Y) \quad \forall\, Z$

First examples

- If X is finite-dimensional, then X has property A,
- $\mathbb{K}$ has property B (Bishop-Phelps theorem),
- c_0 fails property A,
- if Y is strictly convex and there is a non-compact operator from c_0 to Y, then Y fails property B.

Positive results I

Theorem (Lindenstrauss, 1963)

X, Y Banach spaces. Then

$$\{T \in \mathcal{L}(X,Y) \,:\, T^{**} : X^{**} \longrightarrow Y^{**} \text{ attains its norm}\}$$

is dense in $\mathcal{L}(X,Y)$.

Consequence

If X is reflexive, then X has property A.

An improvement (Zizler, 1973)

X, Y Banach spaces. Then

$$\{T \in \mathcal{L}(X,Y) \,:\, T^{*} : Y^{*} \longrightarrow X^{*} \text{ attains its norm}\}$$

is dense in $\mathcal{L}(X,Y)$.

Positive results II

Definitions (Lindenstrauss, Schachermayer)

Let Z be a Banach space. Consider for two sets $\{z_i \,:\, i \in I\} \subset S_Z$, $\{z_i^* \,:\, i \in I\} \subset S_{X^*}$ and a constant $0 \leqslant \rho < 1$, the following four conditions:

1. $z_i^*(z_i) = 1$, $\forall i \in I$;
2. $|z_i^*(z_j)| \leqslant \rho < 1$ if $i, j \in I, i \neq j$;
3. B_Z is the absolutely closed convex hull of $\{z_i \,:\, i \in I\}$ (i.e. $\|z^*\| = \sup\{|z^*(z_i)| \,:\, i \in I\}$);
4. B_{Z^*} is the absolutely weakly*-closed convex hull of $\{z_i^* \,:\, i \in I\}$ (i.e. $\|z\| = \sup\{|z_i^*(z)| \,:\, i \in I\}$).

- Z has property α if 1, 2, and 3 are satisfied (e.g. ℓ_1).
- Z has property β if 1, 2, and 4 are satisfied (e.g. c_0, ℓ_∞).

Theorem (Lindenstrauss, 1963; Schachermayer, 1983)

- Property α implies property A.
- Property β implies property B.

Positive results III

Examples

- The following spaces have property α:
  - ℓ_1,
  - finite-dimensional spaces whose unit ball has finitely many extreme points (up to rotation).
- The following spaces have property β:
  - every Y such that $c_0 \subset Y \subset \ell_\infty$,
  - finite-dimensional spaces such that the dual unit ball has finitely many extreme points (up to rotation).
- For finite-dimensional **real** spaces, property α and property β are equivalent.

Examples

- The following spaces have property A: ℓ_1 and **all** finite-dimensional spaces.
- The following spaces have property B: every Y such that $c_0 \subset Y \subset \ell_\infty$, finite-dimensional spaces such that the dual unit ball has finitely many extreme points (up to rotation).
- Every finite-dimensional space has property A, but the only known (in the 1960's) finite-dimensional real spaces with property B were the polyhedral ones. Only a little bit more is known nowadays…

Positive results IV

Theorem (Partington, 1982; Schachermayer, 1983; Godun-Troyanski, 1993)

- Every Banach space can be renormed with property β.
- Every Banach space admitting a long biorthogonal system (in particular, X separable) can be renormed with property α.

Consequence

- Every Banach space can be renormed with property B.
- Every Banach space admitting a long biorthogonal system (in particular, X separable) can be renormed with property A.

Remark (Shelah, 1984; Kunen, 1981)

Not every Banach space can be renormed with property α.
Indeed, there is K such that $C(K)$ cannot be renormed with property α.

Question

Can every Banach space be renormed with property A**?**

More negative results

Theorem (Lindenstrauss, 1963)

Let X be a Banach space with property A.

- If X admits a strictly convex equivalent norm, then B_X is the closed convex hull of its exposed points.
- If X admits an equivalent LUR norm, then B_X is the closed convex hull of its strongly exposed points.

Remark

In both cases, the author constructed isomorphisms which cannot be approximated by norm attaining operators.

Consequences

- The space $L_1(\mu)$ has property A if and only if μ is purely atomic.
- The space $C(K)$ with K compact metric has property A if and only if K is finite.

An overview on norm attaining operators

Section 1

The Radon-Nikodým property

Definitions

X Banach space.

- X has the Radon-Nikodým property (RNP) if the Radon-Nikodým theorem is valid for X-valued vector measures (with respect to every finite positive measure).
- $C \subset X$ is dentable if for every $\varepsilon > 0$ there is $x \in C$ which does not belong to the closed convex hull of $C \setminus (x + \varepsilon B_X)$.
- $C \subset X$ is subset-dentable if every subset of C is dentable.

Theorem (Rieffel, Maynard, Huff, David, Phelps, 1970's)

X RNP $\iff$ every bounded $C \subset X$ is dentable $\iff$ B_X subset-dentable.

Remark

In the book

J. Diestel and J. J. Uhl
Vector Measures
Math. Surveys **15**, AMS, Providence 1977.

there are more than 30 different reformulations of the RNP.

The RNP and property A: positive results

Theorem (Bourgain, 1977)

X Banach space, $C \subset X$ absolutely convex closed bounded subset-dentable, Y Banach space. Then

$$\{T \in \mathcal{L}(X,Y) \ : \ \text{the norm of } T \text{ attains its supremum on } C\}$$

is dense in $\mathcal{L}(X,Y)$.

★ In particular, RNP $\Longrightarrow$ property A.

Remark

It is actually shown that for every bounded linear operator there are arbitrary closed **compact** perturbations of it attaining the norm.

Non-linear Bourgain-Stegall variational principle (Stegall, 1978)

X, Y Banach spaces, $C \subset X$ bounded subset-dentable, $\varphi : C \longrightarrow Y$ uniformly bounded such that $x \longmapsto \|\varphi(x)\|$ is upper semicontinuous.
Then for every $\delta > 0$, there exists $x_0^* \in X^*$ with $\|x_0^*\| < \delta$ and $y_0 \in S_Y$ such that the function $x \longmapsto \|\varphi(x) + x^*(x)y_0\|$ attains its supremum on C.

The RNP and property A: negative results

Theorem (Bourgain, 1977)

$C \subset X$ separable, bounded, closed and convex,
$\{T \in \mathcal{L}(X,Y)\ :\ \text{the norm of } T \text{ attains its supremum on } C\}$ dense in $\mathcal{L}(X,Y)$.
$\Longrightarrow\ C$ is dentable.

★ In particular, if X is separable and has property A $\Longrightarrow$ B_X is dentable.

Remark

- Reformulation: if B_X is separable and not dentable $\Longrightarrow$ X fails property A.
- Actually, the operator found that cannot be approximated by norm attaining operators is an **isomorphism**.

A refinement (Huff, 1980)

X Banach space failing the RNP.

Then there exist X_1 and X_2 equivalent renorming of X such that

$$\mathrm{NA}(X_1, X_2) \text{ is NOT dense in } \mathcal{L}(X,Y).$$

The RNP and property A: characterization

Main consequence

Every renorming of X has property A $\iff$ X has the RNP.

Example

ℓ_1 has property A in every equivalent norm.

Another consequence

Every renorming of X has property B $\implies$ X has the RNP.

Example

Every Banach space containing c_0 can be renormed to fail property B.

Problem (solved in 1990's)

Does the RNP imply property B**?**

We will see in the next section that the answer is NO.

An overview on norm attaining operators

Section 1

The relation with the RNP I

Remark

- As we have shown, if Y has property B in every equivalent norm, then Y has the RNP.
- What about the converse?
- Even more, does there exists a reflexive space without property B?
- The known counterexamples of the 1960's and 1970's do no work for this question:

Example 1

Bourgain-Huff's counterexamples use spaces without the RNP as range.

Example 2 (Uhl, 1976)

- If Y has the RNP, then $\mathrm{NA}(L_1[0,1], Y)$ is dense in $\mathcal{L}(L_1[0,1], Y)$.
- If Y is strictly convex and $\mathrm{NA}(L_1[0,1], Y)$ is dense in $\mathcal{L}(L_1[0,1], Y)$, then Y has the RNP.

The relation with the RNP II

Remark

Lindenstrauss' counterexamples either use range spaces without the RNP or the domain space is c_0 and there is a non-compact operator from c_0 to the range space.

Operators from c_0

If $Y \not\supseteq c_0$, then $\mathcal{L}(c_0, Y) = \mathcal{K}(c_0, Y)$.

Remark (Johnson-Wolfe, 1979)

As we will see, $\mathrm{NA}(c_0, Y) \cap \mathcal{K}(c_0, Y)$ is dense in $\mathcal{K}(c_0, Y)$ for every Y.

Example 3

$$\text{If } Y \text{ has RNP, then } \mathrm{NA}(c_0, Y) \text{ is dense in } \mathcal{L}(c_0, Y).$$

Negative results: Gowers' counterexample

Theorem (Gowers, 1990)

ℓ_p does not have property B for any $1 < p < \infty$.

The construction

Let X be the space of sequences (a_i) such that

$$\lim_{N\to\infty} \left(\sum_{i=1}^{N} a_i^* \Big/ \sum_{i=1}^{N} \frac{1}{i} \right) = 0$$

(where (a_i^*) is the decreasing rearrangement of $(|a_i|)$), endowed with the norm

$$\|(a_i)\| = \max_{N\in\mathbb{N}} \left(\sum_{i=1}^{N} a_i^* \Big/ \sum_{i=1}^{N} \frac{1}{i} \right).$$

- X is a Banach space,
- the formal inclusion $T : X \longrightarrow \ell_p$ is bounded,
- for $x_0 \in S_X$ there is $n \in \mathbb{N}$ and $\delta > 0$ such that $\|x_0 \pm \delta e_n\| \leqslant 1$,
- so, if $S \in \mathrm{NA}(X, \ell_p)$, then there is $n \in \mathbb{N}$ such that $S(e_n) = 0$.
- Therefore, $\mathrm{dist}(T, \mathrm{NA}(X, \ell_p)) \geqslant 1$.

Negative results: strictly convex spaces

Theorem (Acosta, 1999)

Every infinite-dimensional strictly convex space fails property B.

The domain space

Fix $w = (w_n) \in \ell_2 \setminus \ell_1$ decreasing, positive, with $w_1 < 1$, and let $Z(w)$ be the Banach space of sequences z of scalars with norm

$$\|z\| := \|(1-w)z\|_\infty + \|wz\|_1 < \infty.$$

Let $X(w) = \overline{\operatorname{lin}}\{e_n \,:\, n \in \mathbb{N}\} \subset Z(w)^*$.

- (e_n) is a one-unconditional normalized basis of $X(w)$, $X(w)^* \equiv Z(w)$,
- $B_{X(w)} = \left\{u \in X(w) \,:\, \left\|\dfrac{u}{1-w}\right\|_1 \leqslant 1\right\} + \left\{v \in X(w) \,:\, \left\|\dfrac{v}{w}\right\|_\infty \leqslant 1\right\}$,
- $B_{X(w)} = \overline{\operatorname{co}}\left\{\theta_m(1-w_m)e_m + \displaystyle\sum_{i=1}^{n} \theta_i w_i e_i \,:\, m, n \in \mathbb{N},\ |\theta_i| = 1\ \forall i\right\}$,
- If $x_0 \in S_{X(w)}$ and $N \in \mathbb{N}$, there is $n \geqslant N$ and $\delta > 0$ such that $\|x_0 \pm \delta e_n\| \leqslant 1$.

Negative results: strictly convex spaces II

The domain space (recalling)

Fix $w = (w_n) \in \ell_2 \setminus \ell_1$ decreasing, positive, with $w_1 < 1$, consider $X(w)$:

- $B_{X(w)} = \overline{\mathrm{co}}\left\{\theta_m(1-w_m)e_m + \sum_{i=1}^n \theta_i w_i e_i \ : \ m,n \in \mathbb{N},\ |\theta_i| = 1\ \forall i\right\}$,
- If $x_0 \in S_{X(w)}$ and $N \in \mathbb{N}$, there is $n \geqslant N$ and $\delta > 0$ such that $\|x_0 \pm \delta e_n\| \leqslant 1$.

The argument

Y infinite-dimensional strictly convex.

- By Dvoretzky-Rogers theorem, there is $(y_n) \subset S_Y$ such that $\sum_{n\geqslant 1} w_n y_n$ converges unconditionally, so $\left\{\sum_{n=1}^\infty \theta_n w_n y_n \ : \ |\theta_n| \leqslant 1\ \forall n\right\}$ is bounded,
- hence $T(e_n) = y_n$ defines a bounded linear operator on $X(w)$.
- If $S \in \mathrm{NA}(X(w), Y)$, then there exists $n \in \mathbb{N}$ such that $S(e_n) = 0$,
- so $\|T - S\| \geqslant \|T(e_n) - S(e_n)\| = \|y_n\| = 1$. Therefore, Y fails property B.

Consequence

Y separable having property B in every equivalent norm $\implies$ Y is finite-dimensional.

★ What's about the converse?

Negative results: $L_1(\mu)$ spaces

Theorem (Acosta, 1999)

Every infinite-dimensional $L_1(\mu)$ space fails property B.

The domain space

Fix $w = (w_n) \in \ell_2 \setminus \ell_1$ decreasing, positive, with $w_1 < 1$, consider $X(w)$:

- $B_{X(w)} = \overline{\text{co}}\left\{\theta_m(1-w_m)e_m + \sum_{i=1}^n \theta_i w_i e_i \ : \ m,n \in \mathbb{N},\ |\theta_i| = 1\ \forall i\right\}$,
- For $x^* \in \mathrm{NA}(X(w),\mathbb{K})$, $w\chi_{\text{supp}(x^*)} \in \ell_1$.

The argument

- By Dvoretzky-Rogers theorem, there is $(f_n) \subset S_{L_1(\mu)}$ such that $\sum_{n\geqslant 1} w_n f_n$ converges unconditionally, so $\left\{\sum_{n=1}^\infty \theta_n w_n f_n \ : \ |\theta_n| \leqslant 1\ \forall n\right\}$ is bounded;
- so $T(e_n) = f_n$ defines a bounded linear operator on $X(w)$.
- If $S \in \mathrm{NA}(X(w), L_1(\mu))$, then there exists $I \subset \mathbb{N}$ with $w\chi_I \notin \ell_1$ such that

$$\sum_{n\in I} w_n \|Se_n\| \leqslant \|S\|.$$

- As $\|Te_n\| = 1\ \forall n$, we have $\|T-S\| \geqslant 1$. Therefore, $L_1(\mu)$ fails property B.

An overview on norm attaining operators

Section 1

Some classical spaces: positive results

Example (Johnson-Wolfe, 1979)

In the real case, $\mathrm{NA}(C(K_1), C(K_2))$ is dense in $\mathcal{L}(C(K_1), C(K_2))$.

Example (Iwanik, 1979)

$\mathrm{NA}(L_1(\mu), L_1(\nu))$ is dense in $\mathcal{L}(L_1(\mu), L_1(\nu))$.

Theorem (Schachermayer, 1983)

Every weakly compact operator from $C(K)$ can be approximated by (weakly compact) norm attaining operators.

Consequence (Schachermayer, 1983)

$\mathrm{NA}(C(K), L_p(\mu))$ is dense in $\mathcal{L}(C(K), L_p(\mu))$ for $1 \leqslant p < \infty$.

Example (Finet-Payá, 1998)

$\mathrm{NA}(L_1[0,1], L_\infty[0,1])$ is dense in $\mathcal{L}(L_1[0,1], L_\infty[0,1])$.

Some classical spaces: negative results

Example (Schachermayer, 1983)

$\mathrm{NA}(L_1[0,1], C[0,1])$ is NOT dense in $\mathcal{L}(L_1[0,1], C[0,1])$.

Consequence

$C[0,1]$ does not have property B and it was the first "classical" example.

Example (Aron-Choi-Kim-Lee-M., 2015; M., 2014)

$$\left.\begin{array}{c} Z = C[0,1] \oplus_1 L_1[0,1] \\ \text{or} \\ Z = C[0,1] \oplus_\infty L_1[0,1] \end{array}\right\} \quad \Longrightarrow \quad \mathrm{NA}(Z,Z) \text{ not dense in } \mathcal{L}(Z).$$

An overview on norm attaining operators

Section 1

Main open problems

The main open problem

★ Do finite-dimensional spaces have Lindenstrauss property B**?**

(Stunning) open problem

Do finite-dimensional Hilbert spaces have Lindenstrauss property B**?**

Open problem

Characterize the topological compact spaces K such that $C(K)$ has property B.

Open problem

X Banach space without the RNP, does there exists a renorming of X such that $\mathrm{NA}(X,X)$ is not dense in $\mathcal{L}(X,X)$**?**

Remark

If $X \simeq Z \oplus Z$, then the above question has a positive answer (use Bourgain-Huff).

Norm attaining compact operators

Section 2

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Norm attaining compact operators

Section 2

Posing the problem for compact operators

Question

Can every compact operator be approximated by norm-attaining operators?

Observations

- In all the negative examples of the previous section, the authors constructed NON COMPACT operators which cannot be approximated by norm attaining operators.
- Actually, the idea of the proofs is to use that the operator which is not going to be approximated is not compact or, even, it is an isomorphism.
- In most examples, it was even known that compact operators attaining the norm are dense.

Where was it explicitly possed?

- Diestel-Uhl, *Rocky Mount. J. Math.*, 1976.
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- Johnson-Wolfe, *Studia Math.*, 1979.
- Acosta, *RACSAM* (survey), 2006.

More observations on compact operators

Question

Can every compact operator be approximated by norm-attaining operators?

Observations

- If X is reflexive, then ALL compact operators from X into Y are norm attaining. (Indeed, compact operators carry weak convergent sequences to norm convergent sequences.)
- It is known from the 1970's that whenever $X = C_0(L)$ or $X = L_1(\mu)$ (and Y arbitrary) or $Y = L_1(\mu)$ or $Y^* \equiv L_1(\mu)$ (and X arbitrary), $\Longrightarrow \ \mathrm{NA}(X,Y) \cap \mathcal{K}(X,Y)$ is dense in $\mathcal{K}(X,Y)$.
- On the other hand, for a non reflexive space X and an arbitrary Y, we do not know whether there is any norm attaining operator from X to Y with rank greater than one.
- Actually, we do not know whether there exists a Banach space X such that $\mathrm{NA}(X,\ell_2)$ is contained in the set of rank-one operators.

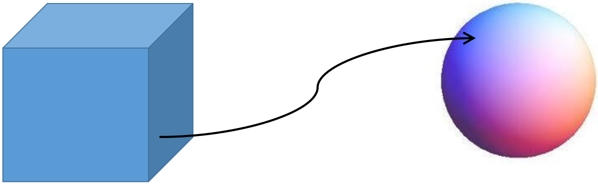
Norm attaining compact operators

Section 2

Extending a result by Lindenstrauss

X, Y Banach spaces, $T \in \mathcal{L}(X,Y)$ and $x_0 \in S_X$ with $\|T\| = \|Tx_0\| = 1$.

- If x_0 is not extreme point of B_X, there is $z \in X$ such that $\|x_0 \pm z\| \leqslant 1$, so $\|Tx_0 \pm Tz\| \leqslant 1$.
- If Tx_0 is an extreme point of B_Y, then $Tz = 0$.



Extending a result by Lindenstrauss

X, Y Banach spaces, $T \in \mathcal{L}(X,Y)$ and $x_0 \in S_X$ with $\|T\| = \|Tx_0\| = 1$.

- If x_0 is not extreme point of B_X, there is $z \in X$ such that $\|x_0 \pm z\| \leqslant 1$, so $\|Tx_0 \pm Tz\| \leqslant 1$.
- If Tx_0 is an extreme point of B_Y, then $Tz = 0$.

Geometrical lemma, Lindenstrauss

X, Y Banach spaces. Suppose that

- for every $x_0 \in S_X$, $\operatorname{lin}\{z \in X \,:\, \|x_0 \pm z\| \leqslant 1\}$ has finite codimension,
- Y is strictly convex.

Then, $\mathrm{NA}(X,Y) \subseteq \mathcal{F}(X,Y)$.

First consequence (recalling, Lindenstrauss, 1963)

- $\mathrm{NA}(c_0,Y) \subseteq \mathcal{F}(c_0,Y)$ if Y is strictly convex.
- Therefore, c_0 fails property A.

Extending a result by Lindenstrauss (II)

Proposition (extension of Lindenstrauss result)

$X \leqslant c_0$. For every $x_0 \in S_X$, $\operatorname{lin}\{z \in X \,:\, \|x_0 \pm z\| \leqslant 1\}$ has finite codimension.

Proof.

- as $x_0 \in c_0$, there exists m such that $|x_0(n)| < 1/2$ for every $n \geqslant m$;
- let $Z = \big\{z \in X \,:\, x_0(i) = 0 \text{ for } 1 \leqslant i \leqslant m\big\}$ (finite codimension in X);
- for $z \in Z$ with $\|z\| \leqslant 1/2$, one has $\|x_0 \pm z\| \leqslant 1$.

Main consequence

$X \leqslant c_0$, Y strictly convex. Then $\mathrm{NA}(X,Y) \subseteq \mathcal{F}(X,Y)$.

Question

What's next? How to use this result?

Grothendieck's approximation property

Definition (Grothendieck, 1950's)

Z has the approximation property (AP) if for every $K \subset Z$ compact and every $\varepsilon > 0$, there exists $F \in \mathcal{F}(Z)$ such that $\|Fz - z\| < \varepsilon$ for all $z \in K$.

Basic results

X, Y Banach spaces.

- (Grothendieck) Y has AP $\iff$ $\overline{\mathcal{F}(Z,Y)} = \mathcal{K}(Z,Y)$ for all Z.
- (Grothendieck) X^* has AP $\iff$ $\overline{\mathcal{F}(X,Z)} = \mathcal{K}(X,Z)$ for all Z.
- (Grothendieck) X^* AP $\implies$ X AP.
- (Enflo, 1973) There exists $X \leqslant c_0$ without AP.
- (Davie, 1973) There exists $X \leqslant \ell_p$ without AP for $1 \leqslant p < 2$.
- (Szankowski, 1976) There exists $X \leqslant \ell_p$ without AP for $2 < p < \infty$.

The first example

Theorem

There exists a **compact** operator which cannot be approximated by norm attaining operators.

Proof:

- consider $X \leqslant c_0$ without AP (Enflo);
- X^* does not has AP
  $\Longrightarrow$ there exists Y and $T \in \mathcal{K}(X,Y)$ such that $T \notin \overline{\mathcal{F}(X,Y)}$;
- we may suppose $Y = \overline{T(X)}$, which is separable;
- so Y admits an equivalent strictly convex renorming (Klee);
- we apply the extension of Lindenstrauss result: $\mathrm{NA}(X,Y) \subseteq \mathcal{F}(X,Y)$;
- therefore, $T \notin \overline{\mathrm{NA}(X,Y)}$.

Two useful definitions

Definitions

X and Y Banach spaces.

- X has property AK when $\overline{\mathrm{NA}(X,Z)\cap\mathcal{K}(X,Z)} = \mathcal{K}(X,Z) \quad \forall\, Z$;
- Y has property BK when $\overline{\mathrm{NA}(Z,Y)\cap\mathcal{K}(Z,Y)} = \mathcal{K}(Z,Y) \quad \forall\, Z$.

Some basic results

- Finite-dimensional spaces have property AK;
- $Y = \mathbb{K}$ has property BK;
- Real finite-dimensional polyhedral spaces have property BK.

Our negative example (recalling)

There exists $X \leqslant c_0$ failing AK and there exits Y failing BK.

Norm attaining compact operators

Section 2

More examples: Domain space

Proposition (what we have proved so far...)

$X \leqslant c_0$ such that X^* fails AP $\implies$ X does not have AK.

Example by Johnson-Schechtman, 2001

Exists X subspace of c_0 **with Schauder basis** such that X^* fails the AP.

Corolary

There exists a Banach space X with Schauder basis failing property AK.

More examples: Range space

Strictly convex spaces

Y strictly convex without AP $\implies$ Y fails BK.

Lemma (Grothendieck)

Y has AP iff $\mathcal{F}(X,Y)$ is dense in $\mathcal{K}(X,Y)$ for every $X \leqslant c_0$.

Subspaces of $L_1(\mu)$

$Y \leqslant L_1(\mu)$ (complex case) without AP $\implies$ Y fails BK.

Observation (Globevnik, 1975)

Complex $L_1(\mu)$ spaces are complex strictly convex:

$$f, g \in L_1(\mu),\ \|f\| = 1 \text{ and } \|f + \theta g\| \leqslant 1\, \forall \theta \in B_{\mathbb{C}} \implies g = 0.$$

More examples: Domain=Range

Theorem

There exists a Banach space Z and a compact operator from Z to Z which cannot be approximated by norm attaining operators.

Proposition

X and Y Banach spaces, $Z = X \oplus_1 Y$ or $Z = X \oplus_\infty Y$.
$\mathrm{NA}(Z,Z) \cap \mathcal{K}(Z)$ dense in $\mathcal{K}(Z)$ $\Longrightarrow$ $\mathrm{NA}(X,Y) \cap \mathcal{K}(X,Y)$ dense in $\mathcal{K}(X,Y)$.

Proof. Fix $T_0 \in K(X,Y)$ with $\|T_0\| = 1$ and $0 < \varepsilon < 1/2$.

- Define $S_0 \in K(Z,Z)$ by $S_0(x,y) = (0, T_0(x))$ for every $(x,y) \in X \oplus_\infty Y$, $\|S_0\| = 1$,
- there exists $S \in NA(Z,Z)$ such that $\|S_0 - S\| < \varepsilon$, take $(x_0, y_0) \in S_X \times B_Y$ such that $\|S(x_0,y_0)\| = \|S\|$.
- $\|P_X S\| = \|P_X S - P_X S_0\| \leqslant \|S - S_0\| < \varepsilon$, so $\|P_Y S(x_0,y_0)\| = \|P_Y S\| = \|S\|$.
- Take $x_0^* \in S_{X^*}$ such that $x_0^*(x_0) = 1$ and define the operator $T \in \mathcal{K}(X,Y)$ by
$$T(x) = P_Y S\big(x, x_0^*(x) y_0\big) \qquad \big(x \in X\big).$$
- $\|T\| \leqslant \|P_Y S\|$ and $\|T(x_0)\| = \|P_Y S(x_0,y_0)\| = \|P_Y S\|$, so $T \in NA(X,Y)$.
- $\|T_0 - T\| \leqslant \|P_Y S_0 - P_Y S\| \leqslant \|S_0 - S\| < \varepsilon$.

Norm attaining compact operators

Section 2

Property AK

Definition (recalling)

X Banach space. X has property AK when $\overline{\mathrm{NA}(X,Z)\cap\mathcal{K}(X,Z)}=\mathcal{K}(X,Z) \quad \forall\, Z$.

First positive examples

- (Lindenstrauss-Schachermayer) Property α implies property AK;
- (Godun-Troyanski) so every separable Banach space can be renormed to have property AK;
- (Bourgain) RNP implies property AK (in every equivalent norm);
- Property AK is stable by ℓ_1-sums.

Negative examples

Every subspace of c_0 whose dual fails AP;

Question

Are there more positive examples**?**

Leading open problem

Problem

$$X^* \text{ AP} \implies X \text{ AK}\textbf{?}$$

Observation

Known positive results on property AK are partial answers to the above question, as strong forms of the AP for the dual are involved.

Old known examples

- (Diestel-Uhl, 1976) $L_1(\mu)$ has AK;
- (Johnson-Wolfe, 1979) $C_0(L)$ has AK.

Our next aim is to prove these results and some more.

An interesting new example

$$\text{If } X^* \text{ has AP and } X \text{ has property A} \implies X \text{ has property AK.}$$

Positive results on property AK

Problem

$$X^* \text{ AP} \implies X \text{ AK}\textbf{?}$$

Partial answer:

(Johnson-Wolfe) With a strong approximation property of the dual...

Suppose there exists a net of contractive projections $(P_\alpha)_\alpha$ in X with finite rank such that $\lim_\alpha P_\alpha^* = \mathrm{Id}_{X^*}$ in SOT. Then, X has AK.

Proof. Fix $T \in \mathcal{K}(X,Y)$.

- $TP_\alpha(B_X) = T(B_{P_\alpha(X)})$ (we need $P_\alpha^2 = P_\alpha$ and $\|P_\alpha\| = 1$).
- Then, TP_α attains the norm.
- As T^* is compact, $P_\alpha^* T^* \longrightarrow T^*$ in norm, so $TP_\alpha \longrightarrow T$ in norm.

Positive results on property AK

Problem

$$X^* \text{ AP} \implies X \text{ AK}\textbf{?}$$

Partial answer:

(Johnson-Wolfe) With a strong approximation property of the dual...

Suppose there exists a net of contractive projections $(P_\alpha)_\alpha$ in X with finite rank such that $\lim_\alpha P_\alpha^* = \mathrm{Id}_{X^*}$ in SOT. Then, X has AK.

Consequences

- (Diestel-Uhl) $L_1(\mu)$ has AK.
- (Johnson-Wolfe) $C_0(L)$ has AK.
- X with monotone and shrinking basis $\implies$ X has AK.
- X with monotone unconditional basis, $X \not\supseteq \ell_1 \implies X$ has AK.
- $X^* \equiv \ell_1 \implies X$ has AK (using a result by Gasparis).
- $X \leqslant c_0$ with monotone basis $\implies$ X has AK (using a result by Godefroy–Saphar).

Norm attaining compact operators

Section 2

Property BK

Definition (recalling)

Y Banach space. Y has property BK when $\overline{\mathrm{NA}(Z,Y)\cap\mathcal{K}(Z,Y)} = \mathcal{K}(Z,Y) \quad \forall\, Z$.

First positive examples

- (Lindenstrauss) Property β implies property BK;
- (Partington) so every Banach space can be renormed to have property BK.
- (Cascales-Guirao-Kadets) $A(\mathbb{D})$ has BK (actually, every uniform algebra).
- Property BK is stable by c_0- and ℓ_∞-sums.

Negative examples

- Every strictly convex space without AP;
- every subspace of the complex $L_1(\mu)$ spaces without AP.

Question

Are there more positive examples**?**

Positive results on property BK I

Main open question

$$\text{AP} \implies \text{BK?}$$

A partial answer (Johnson-Wolfe)

- If Y is polyhedral (real) and has AP $\implies$ Y has BK.
- X (complex) space with AP such that the norm of every finite-dimensional subspace can be calculated as the maximum of a finite set of functionals $\implies$ Y has BK.

Example (Johnson-Wolfe)

$$Y \leqslant c_0 \text{ (real or complex) with AP} \implies Y \text{ has BK.}$$

A somehow reciprocal to the problem...

Y separable with BK for every equivalent norm $\implies$ Y has AP.

Positive results on property BK II

Main open question

$$\text{AP} \implies \text{BK}\textbf{?}$$

Another partial answer (Johnson-Wolfe)

Y Banach space. Suppose there exists a uniformly bounded net of projections $(Q_\alpha)_\alpha$ in Y such that $\lim_\alpha Q_\alpha = \mathrm{Id}_Y$ in SOT and $Q_\alpha(Y)$ has property BK.
Then, Y has property BK.

Proof. X Banach space, $T \in \mathcal{K}(X,Y)$.

- $Q_\alpha T$ converges in norm to T (by compactness of T),
- $Q_\alpha T$ arrives to $Q_\alpha(X)$, which has property BK,
- so each $Q_\alpha T$ can be approximated by norm-attaining compact operators.

Positive results on property BK II

Main open question

$$\text{AP} \implies \text{BK}\textbf{?}$$

Another partial answer (Johnson-Wolfe)

Y Banach space. Suppose there exists a uniformly bounded net of projections $(Q_\alpha)_\alpha$ in Y such that $\lim_\alpha Q_\alpha = \mathrm{Id}_Y$ in SOT and $Q_\alpha(Y)$ has property BK.
Then, Y has property BK.

Examples (Johnson-Wolfe)

- Y predual of $L_1(\mu)$ (real or complex) $\implies$ Y has BK;
- in particular, real or complex $C_0(L)$ spaces have property BK;
- real $L_1(\mu)$ spaces have property BK.

Norm attaining compact operators

Section 2

Some open problems

Main open problem

★ Can every finite-rank operator be approximated by norm-attaining operators **?**

Open problem

X Banach space, does there exist a norm-attaining rank-two operator from X to a Hilbert space**?**

Another main open problem

★ X^* AP $\implies$ X AK**?**

Open problem

$X \leqslant c_0$ with the metric AP, does it have AK**?**

Open problem

X such that $X^* \equiv L_1(\mu)$, does X have AK**?**

Open problem

Y subspace of the real $L_1(\mu)$ without the AP, does Y fail property BK**?**

Numerical radius attaining operators

Section 3

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Numerical radius attaining operators

Section 3

Numerical range: Hilbert spaces

Hilbert space numerical range (Toeplitz, 1918)

- A $n \times n$ real or complex matrix

$$W(A) = \Big\{(Ax \mid x) \ : \ x \in \mathbb{K}^n, \ (x \mid x) = 1\Big\}.$$

- H real or complex Hilbert space, $T \in \mathcal{L}(H)$,

$$W(T) = \Big\{(Tx \mid x) \ : \ x \in H, \ \|x\| = 1\Big\}.$$

Some properties

H Hilbert space, $T \in \mathcal{L}(H)$:

- $W(T)$ is convex.
- In the complex case, $\overline{W(T)}$ contains the spectrum of T.
- If T is normal, then $\overline{W(T)} = \overline{\mathrm{co}}\,\mathrm{Sp}(T)$.

Numerical range: Banach spaces

Banach space numerical range (Bauer 1962; Lumer, 1961)

X Banach space, $T \in \mathcal{L}(X)$,

$$V(T) = \left\{ x^*(Tx) \,:\, x^* \in S_{X^*},\, x \in S_X,\, x^*(x) = 1 \right\}$$

Some properties

X Banach space, $T \in \mathcal{L}(X)$:

- $V(T)$ is connected (not necessarily convex).
- In the complex case, $\overline{V(T)}$ contains the spectrum of T.
- In fact,

$$\overline{\text{co}}\,\text{Sp}(T) = \bigcap \overline{\text{co}}\,V(T),$$

  the intersection taken over all numerical ranges $V(T)$ corresponding to equivalent norms on X.

Some motivations for the numerical range

For Hilbert spaces

- It is a comfortable way to study the spectrum.
- It is useful to work with some concept like hermitian operator, skew-hermitian operator, dissipative operator. . .
- It is useful to estimate spectral radii of small perturbations of matrices.

For Banach spaces

- It allows to carry to the general case the concepts of hermitian operator, skew-hermitian operator, dissipative operators. . .
- It gives a description of the Lie algebra corresponding to the Lie group of all onto isometries on the space.
- It gives an easy and quantitative proof of the fact that Id is an strongly extreme point of $B_{\mathcal{L}(X)}$ (MLUR point).

Numerical radius

Numerical radius

X Banach space, $T \in \mathcal{L}(X)$. The numerical radius of T is

$$v(T) = \sup\left\{|x^*(Tx)| \ : \ x^* \in S_{X^*},\ x \in S_X,\ x^*(x) = 1\right\}.$$

★ Notation: $\Pi(X) = \left\{(x, x^*) \ : \ x^* \in S_{X^*},\, x \in S_X,\, x^*(x) = 1\right\}$.
With this notation, $v(T) = \sup\left\{|x^*(Tx)| \ : \ (x, x^*) \in \Pi(X)\right\}$.

Remark

The numerical radius is a continuous seminorm in $\mathcal{L}(X)$. Actually, $v(\cdot) \leqslant \|\cdot\|$.

Numerical radius attaining operators

$T \in \mathcal{L}(X)$ attains its numerical radius when

$$\exists\ (x, x^*) \in \Pi(X) \ : \ |x^*T(x)| = v(T)$$

★ $\mathrm{NRA}(X) = \{T \in \mathcal{L}(X) \ : \ T \text{ attains its numerical radius}\}$

Numerical radius attaining operators

Section 3

Numerical radius attaining operators: first results

Numerical radius attaining operators

X Banach space, $T \in \mathcal{L}(X)$ attains its numerical radius when

$$\exists\ (x, x^*) \in \Pi(X)\ :\ |x^*T(x)| = \sup\Big\{|y^*(Ty)|\ :\ (y, y^*) \in \Pi(X)\Big\}.$$

Some examples

- If $\dim(X) < \infty$, then $\mathrm{NRA}(X) = \mathcal{L}(X)$ ($\Pi(X)$ is compact).
- Even in $X = \ell_2$ there are (diagonal) operators which do not attain their numerical radius.
- Suppose $v(T) = \|T\|$:
  - $T \in \mathrm{NRA}(X) \implies T \in \mathrm{NA}(X, X)$,
  - $T \in \mathrm{NA}(X, X) \not\Longrightarrow T \in \mathrm{NRA}(X, X)$.

Main problem here

When is $\mathrm{NRA}(X)$ dense in $\mathcal{L}(X)$**?**

The study of this problem was initiated in the PhD dissertation of B. Sims of 1972, where some positive results were given.

Some positive results

Proposition (Berg-Sims, 1984)

X uniformly convex $\implies \mathrm{NRA}(X)$ dense in $\mathcal{L}(X)$.

Proposition (Acosta-Payá, 1989)

For every Banach space X, $\{T \in \mathcal{L}(X) : T^{**} \in \mathrm{NRA}(X^{**})\}$ is dense.

Theorem (Acosta-Payá, 1993)

If X has the RNP, then $\mathrm{NRA}(X)$ is dense in $\mathcal{L}(X)$.

Examples (Cardasi, 1985)

$C(K)$ and $L_1(\mu)$ (real case) satisfy the density of numerical radius attaining operators.

Proposition (Acosta, 1991 & 1993)

Property α and property β (real case) implies the density of numerical radius attaining operators.

- Consequence: every real space can be renormed to get the density of numerical radius attaining operators.

Some negative results

Example (Payá, 1992)

There is a Banach space Z for which $\mathrm{NRA}(Z)$ is not dense in $\mathcal{L}(Z)$.

- $Z = c_0 \oplus_\infty Y$, where Y is a concrete strictly convex renorming of c_0.

Example (Acosta-Aguirre-Payá, 1992)

For $Z = G \oplus_\infty \ell_2$ (G from Gowers' counterexample), $\mathrm{NRA}(Z)$ is not dense in $\mathcal{L}(Z)$.

Example (Kim-Lee-M., 2016?)

For $Z = c_0 \oplus_1 Y$ ($Y \simeq c_0$ strictly convex), $\mathrm{NRA}(Z)$ is not dense in $\mathcal{L}(Z)$.

- $\mathrm{NRA}(c_0 \oplus_1 Y)$ dense in $\mathcal{L}(c_0 \oplus_1 Y) \implies \mathrm{NA}(c_0, Y)$ dense in $\mathcal{L}(c_0, Y)$.

Example (Capel-M.-Merí, preprint)

For $Z = L_1[0,1] \oplus_1 C[0,1]$ and $Z = L_1[0,1] \oplus_\infty C[0,1]$, $\overline{\mathrm{NRA}(Z)} \neq \mathcal{L}(Z)$.

- $v(T) = \|T\|$ for every $T \in \mathcal{L}(Z)$, and $\mathrm{NA}(Z, Z)$ is not dense in $\mathcal{L}(Z)$.

None of these examples produce a **compact** operator outside $\overline{\mathrm{NRA}(Z)}$.

Numerical radius attaining operators

Section 3

The counterexample

Example

Given $1 < p < 2$, there are a subspace X of c_0 and a quotient Y of ℓ_p such that $\mathcal{K}(X \oplus_\infty Y)$ is not contained in the closure of $\mathrm{NRA}(X \oplus_\infty Y)$.

The proof needs five steps:

- use that the norm of Y^* is smooth enough (lemma 1);
- use that X is strongly flat (lemma 2);
- calculate numerical radius of operators on ℓ_∞-sums (lemma 3);
- glue these three results and use numerical radius attaining operators (proposition ★)
- use the AP and finish the proof (proof of the example).

Step 1: using the smoothness of Y^*

Smoothness and duality mapping

Let Z be a Banach space.

- The norm of Z is smooth if it is Gâteaux differentiable at every $z \in Z \setminus \{0\}$.
- The normalized duality mapping $J_Z : Z \longrightarrow 2^{Z^*}$ of Z is given by

$$J(z) = \{z^* \in Z^* \ : \ z^*(z) = \|z^*\|^2 = \|z\|^2\} \qquad (z \in Z).$$

- If the norm of Z is smooth, J is single-valued and the map $\widetilde{J}_Z : Z \setminus \{0\} \longrightarrow S_{Z^*}$ given by

$$\widetilde{J}_Z(z) = J\left(\frac{z}{\|z\|}\right) = \frac{J(z)}{\|J(z)\|} \qquad (z \in Z \setminus \{0\})$$

  is well defined.
- $\widetilde{J}_Z(z)$ can be alternatively defined as the unique $z^* \in S_{Z^*}$ such that $z^*(z) = \|z\|$.
- If the norm of Z is C^2-smooth, then $\widetilde{J}_Z$ is Fréchet differentiable on $Z \setminus \{0\}$.

Step 1: using the smoothness of Y^* II

Smoothness and pre-duality mapping

Let Y be a reflexive Banach space whose dual norm is C^2-smooth.
Then $\widetilde{J}_{Y^*} : Y^* \setminus \{0\} \longrightarrow S_Y$ is Fréchet differentiable.

- $\widetilde{J}_{Y^*}(y^*)$ is the unique $y \in S_Y$ such that $y^*(y) = \|y^*\|$.

Lemma 1

Y (reflexive) space such that the norm of Y^* is C^2-smooth on $Y^* \setminus \{0\}$, X Banach space. Suppose that $A \in \mathcal{L}(Y)$, $B \in \mathcal{L}(X, Y)$, and $(y_0, y_0^*) \in \Pi(Y)$ satisfy that

$$|y^*(Ay)| + \|B^*y^*\| \leqslant |y_0^*(Ay_0)| + \|B^*y_0^*\|$$

for all $(y, y^*) \in \Pi(Y)$. Then,

$$\lim_{t\to 0} \frac{\|B^*y_0^* + tB^*h^*\| + \|B^*y_0^* - tB^*h^*\| - 2\|B^*y_0^*\|}{t} = 0$$

for every $h^* \in S_{Y^*}$.

Step 2: using that X is strongly flat

Strongly flat

X Banach space, $x_0 \in S_X$.

- $\mathrm{Flat}(x_0) = \{x \in X \,:\, \|x_0 \pm x\| \leqslant 1\}$;
- X is strongly flat if $\operatorname{codim}\left(\overline{\operatorname{lin}}\,\mathrm{Flat}(x_0)\right) < \infty$.

Lemma 2

X strongly flat Banach space, Y Banach space.
Suppose that for $B \in \mathcal{L}(X,Y)$ there is $y_0^* \in S_{Y^*}$ such that

$$\lim_{t\to 0^+} \frac{\|B^*y_0^* + tB^*h^*\| + \|B^*y_0^* - tB^*h^*\| - 2\|B^*y_0^*\|}{t} \leqslant 0$$

for every $h^* \in S_{Y^*}$ and that $B^*y_0^*$ attains its norm on X.
Then, B has finite-rank.

Step 3: numerical radius and ℓ_∞-sums

Lemma 3 (Payá, 1992)

X, Y Banach spaces, $Z = X \oplus_\infty Y$ and P_X, P_Y natural projections.
For $T \in \mathcal{L}(Z)$, we have

1. $v(T) = \max\{v(P_X T), v(P_Y T)\}$;
2. $T \in \mathrm{NRA}(Z)$ and $v(P_Y T) > v(P_X T) \implies P_Y T \in \mathrm{NRA}(Z)$;
3. $v(P_Y T) = \sup\{|y^*(P_Y T(y + x))| \,:\, (y, y^*) \in \Pi(Y),\ x \in B_X\}$;
4. $P_Y T \in \mathrm{NRA}(Z) \iff$ the supremum above is attained.

Step 4: gluing the thee results and using $\mathrm{NRA}(Z)$

Proposition ★

Y such that the norm of Y^* is C^2-smooth on $Y^* \setminus \{0\}$, X strongly flat, $Z = X \oplus_\infty Y$.
For $A \in \mathcal{L}(Y)$ and $B \in \mathcal{L}(X,Y)$, define $T \in \mathcal{L}(Z)$ by

$$T(x+y) = A(y) + B(x) \qquad \big(x \in X,\ y \in Y\big).$$

If $T \in \mathrm{NRA}(Z)$, then B is of finite-rank.

Step 5: The AP and the proof of the example

Example

Given $1 < p < 2$, there are a subspace X of c_0 and a quotient Y of ℓ_p such that $\mathcal{K}(X \oplus_\infty Y)$ is not contained in the closure of $\mathrm{NRA}(X \oplus_\infty Y)$.

- Take Y quotient of ℓ_p without the AP;
- consider $X \leqslant c_0$ such that exists $S \in \mathcal{K}(X,Y) \setminus \overline{\mathcal{F}(X,Y)}$;
- define $T \in \mathcal{K}(Z)$ by $T(x+y) = Sx$;
- work with Proposition ★ to get that $T \notin \overline{\mathrm{NRA}(Z)}$.

Some results on the AP

- (Davie, 1973) There exists $Y \leqslant \ell_q$ without AP for $2 < q < \infty$.
- (Grothendieck) Y reflexive, Y^* AP $\iff$ Y AP.
- (Grothendieck) Y has AP $\iff$ $\overline{\mathcal{F}(X,Y)} = \mathcal{K}(X,Y)$ for every $X \leqslant c_0$.

Step 5: The AP and the proof of the example

Example

Given $1 < p < 2$, there are a subspace X of c_0 and a quotient Y of ℓ_p such that

$\mathcal{K}(X \oplus_\infty Y)$ is not contained in the closure of $\mathrm{NRA}(X \oplus_\infty Y)$.

- Take Y quotient of ℓ_p without the AP;
- consider $X \leqslant c_0$ such that exists $S \in \mathcal{K}(X,Y) \setminus \overline{\mathcal{F}(X,Y)}$;
- define $T \in \mathcal{K}(Z)$ by $T(x+y) = Sx$;
- work with Proposition ★ to get that $T \notin \overline{\mathrm{NRA}(Z)}$.

Proposition ★

Y such that the norm of Y^* is C^2-smooth on $Y^* \setminus \{0\}$, X strongly flat, $Z = X \oplus_\infty Y$. For $A \in \mathcal{L}(Y)$ and $B \in \mathcal{L}(X,Y)$, define $T \in \mathcal{L}(Z)$ by

$$T(x+y) = A(y) + B(x) \qquad \big(x \in X,\ y \in Y\big).$$

If $T \in \mathrm{NRA}(Z)$, then B is of finite-rank.

Numerical radius attaining operators

Section 3

Some positive results I

The positive results to get density of numerical radius attaining operators also works for compact operators:

Positive results

X Banach space satisfying one of the following conditions:

- X has RNP,
- X has property α,
- X is real and has property β.

Then $\mathrm{NRA}(X) \cap \mathcal{K}(X)$ is dense in $\mathcal{K}(X)$.

In all the proofs, every operator is perturbed by a compact operator to get a numerical radius attaining one.

Some positive results II: CL-spaces

Definition (Fullerton, 1961)

A Banach space X is a CL-space if B_X is the absolutely convex hull of every maximal convex subset of S_X.

Examples

Real or complex $C(K)$ spaces and real $L_1(\mu)$ spaces are CL-spaces.

Theorem (Acosta, 1990)

X CL-space. Then:

- For every $T \in \mathcal{L}(X)$, $v(T) = \|T\|$;
- $T \in \mathrm{NA}(X,X) \iff T \in \mathrm{NRA}(X)$.

Main consequence

$X = C(K)$ (real or complex) or $X = L_1(\mu)$ (real) $\implies \overline{\mathrm{NRA}(X) \cap \mathcal{K}(X)} = \mathcal{K}(X)$.

Some positive results II: CL-spaces

Definition (Fullerton, 1961)

A Banach space X is a CL-space if B_X is the absolutely convex hull of every maximal convex subset of S_X.

Examples

Real or complex $C(K)$ spaces and real $L_1(\mu)$ spaces are CL-spaces.

Theorem (Acosta, 1990)

X CL-space. Then:

- For every $T \in \mathcal{L}(X)$, $v(T) = \|T\|$;
- $T \in \mathrm{NA}(X, X) \iff T \in \mathrm{NRA}(X)$.

Another consequence

$X = C[0,1] \oplus_1 L_1[0,1]$ (real) or $X = C[0,1] \oplus_\infty L_1[0,1]$ (real)

$\Longrightarrow \mathrm{NRA}(X) \cap \mathcal{K}(X)$ dense in $\mathcal{K}(X)$.

★ Recall that $\mathrm{NRA}(X)$ is NOT dense in $\mathcal{L}(X)$.

Numerical radius attaining operators

Section 3

Open problems

Open problem

X Banach space without the RNP, does there exists a renorming of X such that $\mathrm{NRA}(X)$ is not dense in $\mathcal{L}(X)$?

Open problem

X Banach space without the RNP, does there exists a renorming of X such that $\mathrm{NRA}(X) \cap \mathcal{K}(X)$ is not dense in $\mathcal{K}(X)$?

Open problem

Do we have $\overline{\mathrm{NRA}(X) \cap \mathcal{K}(X)} = \mathcal{K}(X)$ for X such that $X^* \equiv L_1(\mu)$?

Open problem

Suppose that $v(T) = \|T\|$ for every $T \in \mathcal{L}(X)$ and $\mathrm{NA}(X, X)$ is dense in $\mathcal{L}(X)$. Does $\mathrm{NRA}(X)$ have to be dense in $\mathcal{L}(X)$?

Open problem

Suppose that $v(T) = \|T\|$ for every $T \in \mathcal{K}(X)$ and $\mathrm{NA}(X, X) \cap \mathcal{K}(X)$ is dense in $\mathcal{K}(X)$. Does $\mathrm{NRA}(X) \cap \mathcal{K}(X)$ have to be dense in $\mathcal{K}(X)$?