Norm attaining compact operators

11th ILJU School of Mathematics: *Banach Spaces and related topics*

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Contents

★ Notation

X, Y real or complex Banach spaces

- $\mathbb{K}$ base field $\mathbb{R}$ or $\mathbb{C}$,
- $B_X = \{x \in X : \|x\| \leqslant 1\}$ closed unit ball of X,
- $S_X = \{x \in X : \|x\| = 1\}$ unit sphere of X,
- $\mathcal{L}(X, Y)$ bounded linear operators from X to Y,
  - $\|T\| = \sup\{\|T(x)\| : x \in S_X\}$ for $T \in \mathcal{L}(X, Y)$,
- $\mathcal{W}(X, Y)$ weakly compact linear operators from X to Y,
- $\mathcal{K}(X, Y)$ compact linear operators from X to Y,
- $\mathcal{F}(X, Y)$ bounded linear operators from X to Y with finite rank,
- if $Y = \mathbb{K}$, $X^* = \mathcal{L}(X, Y)$ topological dual of X,
- if $X = Y$, we just write $\mathcal{L}(X)$, $\mathcal{W}(X)$, $\mathcal{K}(X)$, $\mathcal{F}(X)$.

Observe that
$$\mathcal{F}(X, Y) \subset \mathcal{K}(X, Y) \subset \mathcal{W}(X, Y) \subset \mathcal{L}(X, Y).$$

References

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1 An overview on norm attaining operators

1.1 Introducing the topic

★ Norm attaining functionals and operators

Norm attaining functionals
$x^* \in X^*$ attains its norm when

$$\exists\, x \in S_X \ :\ |x^*(x)| \ =\ \|x^*\|$$

★ $\mathrm{NA}(X,\mathbb{K}) = \{x^* \in X^* \,:\, x^* \text{ attains its norm}\}$

Examples

- $\dim(X) < \infty \implies \mathrm{NA}(X,\mathbb{K}) = \mathcal{L}(X,\mathbb{K})$ (Heine-Borel).
- X reflexive $\implies \mathrm{NA}(X,\mathbb{K}) = \mathcal{L}(X,\mathbb{K})$ (Hahn-Banach).
- X non-reflexive $\implies \mathrm{NA}(X,\mathbb{K}) \neq \mathcal{L}(X,\mathbb{K})$ (James),
- but $\mathrm{NA}(X,\mathbb{K})$ separates the points of X (Hahn-Banach).

Norm attaining operators
$T \in \mathcal{L}(X,Y)$ attains its norm when

$$\exists\, x \in S_X \ :\ \|T(x)\| \ =\ \|T\|$$

★ $\mathrm{NA}(X,Y) = \{T \in \mathcal{L}(X,Y) \,:\, T \text{ attains its norm}\}$

Examples

- $\dim(X) < \infty \implies \mathrm{NA}(X,Y) = \mathcal{L}(X,Y)$ for every Y (Heine-Borel).
- $\mathrm{NA}(X,Y) \neq \emptyset$ (Hahn-Banach).
- X reflexive $\implies \mathcal{K}(X,Y) \subseteq \mathrm{NA}(X,Y)$ for every Y.
- X non-reflexive $\implies \mathrm{NA}(X,Y) \cap \mathcal{K}(X,Y) \neq \mathcal{K}(X,Y)$ for every Y.
- $\dim(X) = \infty \implies \mathrm{NA}(X,c_0) \neq \mathcal{L}(X,c_0)$ (see M.-Merí-Payá, 2006).

★ The problem of density of norm attaining functionals

Problem
Is $\mathrm{NA}(X,\mathbb{K})$ always dense in X^***?**

Theorem (E. Bishop & R. Phelps, 1961)
The set of norm attaining functionals is dense in X^* (for the norm topology).

Problem
Is $\mathrm{NA}(X,Y)$ always dense in $\mathcal{L}(X,Y)$**?**

The answer is **No** (as we will see in a minute).

Modified problem
When is $\mathrm{NA}(X,Y)$ dense in $\mathcal{L}(X,Y)$**?**

The study of this problem was initiated by J. Lindenstrauss in 1963, who provided the first negative and positive examples.

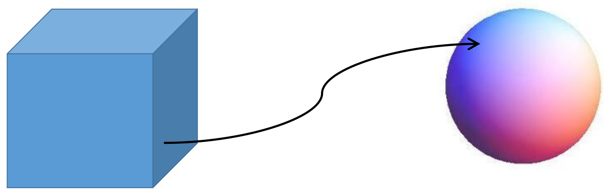
1.2 First results

★ An easy negative example

Example (Lindenstrauss, 1963)
Y strictly convex such that there is a non-compact operator from c_0 into Y.

$$\text{Then, } \mathrm{NA}(c_0, Y) \text{ is not dense in } \mathcal{L}(c_0, Y).$$

Lemma
If Y is strictly convex, then $\mathrm{NA}(c_0, Y) \subseteq \mathcal{F}(c_0, Y)$.



Example (Lindenstrauss, 1963)
There exists Z such that $\mathrm{NA}(Z, Z)$ is not dense in $\mathcal{L}(Z)$. Actually, $Z = c_0 \oplus_\infty Y$.

★ Lindenstrauss properties A and B

Observation

- The question now is for which X and Y the density holds.
- As this problem is too general, Lindenstrauss introduced two properties.

Definition
X, Y Banach spaces,

- X has (Lindenstrauss) property A when $\overline{\mathrm{NA}(X, Z)} = \mathcal{L}(X, Z) \quad \forall\, Z$
- Y has (Lindenstrauss) property B when $\overline{\mathrm{NA}(Z, Y)} = \mathcal{L}(Z, Y) \quad \forall\, Z$

First examples

- If X is finite-dimensional, then X has property A,
- $\mathbb{K}$ has property B (Bishop-Phelps theorem),
- c_0 fails property A,
- if Y is strictly convex and there is a non-compact operator from c_0 to Y, then Y fails property B.

★ Positive results I

Theorem (Lindenstrauss, 1963)
X, Y Banach spaces. Then

$$\{T \in \mathcal{L}(X,Y) \,:\, T^{**} : X^{**} \longrightarrow Y^{**} \text{ attains its norm}\}$$

is dense in $\mathcal{L}(X,Y)$.

Observation
Given $T \in \mathcal{L}(X,Y)$, there is $S \in \mathcal{K}(X,Y)$ such that $[T+S]^{**} \in \mathrm{NA}(X^{**}, Y^{**})$.

Consequence
If X is reflexive, then X has property A.

An improvement (Zizler, 1973)
X, Y Banach spaces. Then

$$\{T \in \mathcal{L}(X,Y) \,:\, T^{*} : Y^{*} \longrightarrow X^{*} \text{ attains its norm}\}$$

is dense in $\mathcal{L}(X,Y)$.

★ Positive results II

Definitions (Lindenstrauss, Schachermayer)
Let Z be a Banach space. Consider for two sets $\{z_i \,:\, i \in I\} \subset S_Z$, $\{z_i^* \,:\, i \in I\} \subset S_{X^*}$ and a constant $0 \leqslant \rho < 1$, the following four conditions:

1. $z_i^*(z_i) = 1$, $\forall i \in I$;
2. $|z_i^*(z_j)| \leqslant \rho < 1$ if $i, j \in I, i \neq j$;
3. B_Z is the absolutely closed convex hull of $\{z_i \,:\, i \in I\}$
(i.e. $\|z^*\| = \sup\{|z^*(z_i)| \,:\, i \in I\}$);
4. B_{Z^*} is the absolutely weakly*-closed convex hull of $\{z_i^* \,:\, i \in I\}$
(i.e. $\|z\| = \sup\{|z_i^*(z)| \,:\, i \in I\}$).

- Z has property α if 1, 2, and 3 are satisfied (e.g. ℓ_1).
- Z has property β if 1, 2, and 4 are satisfied (e.g. c_0, ℓ_∞).

Theorem (Lindenstrauss, 1963; Schachermayer, 1983)

- Property α implies property A.
- Property β implies property B.

★ Positive results III

Examples

- The following spaces have property α:
  - ℓ_1,
  - finite-dimensional spaces whose unit ball has finitely many extreme points (up to rotation).
- The following spaces have property β:
  - every Y such that $c_0 \subset Y \subset \ell_\infty$,
  - finite-dimensional spaces such that the dual unit ball has finitely many extreme points (up to rotation).
- For finite-dimensional **real** spaces, property α and property β are equivalent.

Examples

- The following spaces have property A: ℓ_1 and **all** finite-dimensional spaces.
- The following spaces have property B: every Y such that $c_0 \subset Y \subset \ell_\infty$, finite-dimensional spaces such that the dual unit ball has finitely many extreme points (up to rotation).
- Every finite-dimensional space has property A, but the only known (in the 1960's) finite-dimensional real spaces with property B were the polyhedral ones. Only a little bit more is known nowadays...

★ Positive results IV

Theorem (Partington, 1982; Schachermayer, 1983; Godun-Troyanski, 1993)

- Every Banach space can be renormed with property β.
- Every Banach space admitting a long biorthogonal system (in particular, X separable) can be renormed with property α.

Consequence

- Every Banach space can be renormed with property B.
- Every Banach space admitting a long biorthogonal system (in particular, X separable) can be renormed with property A.

Remark (Shelah, 1984; Kunen, 1981)

Not every Banach space can be renormed with property α. Indeed, there is K such that $C(K)$ cannot be renormed with property α.

Question

Can every Banach space be renormed with property A**?**

★ More negative results

Theorem (Lindenstrauss, 1963)
Let X be a Banach space with property A.

- If X admits a strictly convex equivalent norm, then B_X is the closed convex hull of its exposed points.
- If X admits an equivalent LUR norm, then B_X is the closed convex hull of its strongly exposed points.

Remark
In both cases, the author constructed isomorphisms which cannot be approximated by norm attaining operators.

Consequences

- The space $L_1(\mu)$ has property A if and only if μ is purely atomic.
- The space $C(K)$ with K compact metric has property A if and only if K is finite.

1.3 Property A

★ The Radon-Nikodým property

Definitions
X Banach space.

- X has the Radon-Nikodým property (RNP) if the Radon-Nikodým theorem is valid for X-valued vector measures (with respect to every finite positive measure).
- $C \subset X$ is dentable if for every $\varepsilon > 0$ there is $x \in C$ which does not belong to the closed convex hull of $C \setminus (x + \varepsilon B_X)$.
- $C \subset X$ is subset-dentable if every subset of C is dentable.

Theorem (Rieffel, Maynard, Huff, David, Phelps, 1970's)
$$X \text{ RNP} \iff \text{every bounded } C \subset X \text{ is dentable} \iff B_X \text{ subset-dentable.}$$

Remark
In the book
J. Diestel and J. J. Uhl, *Vector Measures*, Math. Surveys **15**, AMS, Providence, 1977.
there are more than 30 different reformulations of the RNP.

★ The RNP and property A: positive results

Theorem (Bourgain, 1977)
X Banach space, $C \subset X$ absolutely convex closed bounded subset-dentable, Y Banach space. Then
$$\{T \in \mathcal{L}(X,Y) \,:\, \text{the norm of } T \text{ attains its supremum on } C\}$$
is dense in $\mathcal{L}(X,Y)$.

★ In particular, RNP $\implies$ property A.

Remark
It is actually shown that for every bounded linear operator there are arbitrary closed **compact** perturbations of it attaining the norm.

Non-linear Bourgain-Stegall variational principle (Stegall, 1978)
X, Y Banach spaces, $C \subset X$ bounded subset-dentable, $\varphi : C \longrightarrow Y$ uniformly bounded such that $x \longmapsto \|\varphi(x)\|$ is upper semicontinuous. Then for every $\delta > 0$, there exists $x_0^* \in X^*$ with $\|x_0^*\| < \delta$ and $y_0 \in S_Y$ such that the function $x \longmapsto \|\varphi(x) + x^*(x)y_0\|$ attains its supremum on C.

★ The RNP and property A: negative results

Theorem (Bourgain, 1977)
$C \subset X$ separable, bounded, closed and convex,
$\{T \in \mathcal{L}(X,Y) \,:\, \text{the norm of } T \text{ attains its supremum on } C\}$ dense in $\mathcal{L}(X,Y)$.
$\Longrightarrow$ C is dentable.

★ In particular, if X is separable and has property A $\Longrightarrow$ B_X is dentable.

Remark

- Reformulation: if B_X is separable and not dentable $\Longrightarrow$ X fails property A.
- Actually, the operator found that cannot be approximated by norm attaining operators is an **isomorphism**.

A refinement (Huff, 1980)
X Banach space failing the RNP. Then there exist X_1 and X_2 equivalent renorming of X such that $\mathrm{NA}(X_1, X_2)$ is NOT dense in $\mathcal{L}(X_1, X_2)$.

★ The RNP and property A: characterization

Main consequence
Every renorming of X has property A $\Longleftrightarrow$ X has the RNP.

Example
ℓ_1 has property A in every equivalent norm.

Another consequence
Every renorming of X has property B $\Longrightarrow$ X has the RNP.

Example
Every Banach space containing c_0 can be renormed to fail property B.

Problem (solved in 1990's)
Does the RNP imply property B? We will see in the next section that the answer is NO.

1.4 Property B

★ The relation with the RNP I

Remark

- As we have shown, if Y has property B in every equivalent norm, then Y has the RNP.
- What about the converse?
- Even more, does there exists a reflexive space without property B?

- The known counterexamples of the 1960's and 1970's do no work for this question:

Example 1
Bourgain-Huff's counterexamples use spaces without the RNP as range.

Example 2 (Uhl, 1976)

- If Y has the RNP, then $\mathrm{NA}(L_1[0,1],Y)$ is dense in $\mathcal{L}(L_1[0,1],Y)$.
- If Y is strictly convex and $\mathrm{NA}(L_1[0,1],Y)$ is dense in $\mathcal{L}(L_1[0,1],Y)$, then Y has the RNP.

★ The relation with the RNP II

Remark
Lindenstrauss' counterexamples either use range spaces without the RNP or the domain space is c_0 and there is a non-compact operator from c_0 to the range space.

Operators from c_0
If $Y \not\supseteq c_0$, then $\mathcal{L}(c_0,Y)=\mathcal{K}(c_0,Y)$.

Remark (Johnson-Wolfe, 1979)
As we will see, $\mathrm{NA}(c_0,Y)\cap\mathcal{K}(c_0,Y)$ is dense in $\mathcal{K}(c_0,Y)$ for every Y.

Example 3
If Y has RNP, then $\mathrm{NA}(c_0,Y)$ is dense in $\mathcal{L}(c_0,Y)$.

★ Negative results: Gowers' counterexample

Theorem (Gowers, 1990)
ℓ_p does not have property B for any $1<p<\infty$.

The construction
Let X be the space of sequences (a_i) such that

$$\lim_{N\to\infty}\left(\sum_{i=1}^N a_i^* \Big/ \sum_{i=1}^N \tfrac{1}{i}\right)=0$$

(where (a_i^*) is the decreasing rearrangement of $(|a_i|)$), endowed with the norm

$$\|(a_i)\|=\max_{N\in\mathbb{N}}\left(\sum_{i=1}^N a_i^* \Big/ \sum_{i=1}^N \tfrac{1}{i}\right).$$

- X is a Banach space,
- the formal inclusion $T: X\longrightarrow \ell_p$ is bounded,
- for $x_0\in S_X$ there is $n\in\mathbb{N}$ and $\delta>0$ such that $\|x_0\pm\delta e_n\|\leqslant 1$,
- so, if $S\in\mathrm{NA}(X,\ell_p)$, then there is $n\in\mathbb{N}$ such that $S(e_n)=0$.
- Therefore, $\mathrm{dist}(T,\mathrm{NA}(X,\ell_p))\geqslant 1$.

★ Negative results: strictly convex spaces

Theorem (Acosta, 1999)
Every infinite-dimensional strictly convex space fails property B.

The domain space
Fix $w = (w_n) \in \ell_2 \setminus \ell_1$ decreasing, positive, with $w_1 < 1$, and let $Z(w)$ be the Banach space of sequences z of scalars with norm

$$\|z\| := \|(1-w)z\|_\infty + \|wz\|_1 < \infty.$$

Let $X(w) = \overline{\text{lin}}\{e_n : n \in \mathbb{N}\} \subset Z(w)^*$.

- (e_n) is a one-unconditional normalized basis of $X(w)$, $X(w)^* \equiv Z(w)$,
- $B_{X(w)} = \left\{u \in X(w) : \left\|\dfrac{u}{1-w}\right\|_1 \leqslant 1\right\} + \left\{v \in X(w) : \left\|\dfrac{v}{w}\right\|_\infty \leqslant 1\right\}$,
- $B_{X(w)} = \overline{\text{co}}\left\{\theta_m(1-w_m)e_m + \displaystyle\sum_{i=1}^{n}\theta_i w_i e_i \ : m, n \in \mathbb{N}, |\theta_i| = 1\,\forall i\right\}$,
- If $x_0 \in S_{X(w)}$ and $N \in \mathbb{N}$, there is $n \geqslant N$ and $\delta > 0$ such that $\|x_0 \pm \delta e_n\| \leqslant 1$.

★ Negative results: strictly convex spaces II

The domain space (recalling)
Fix $w = (w_n) \in \ell_2 \setminus \ell_1$ decreasing, positive, with $w_1 < 1$, consider $X(w)$:

- $B_{X(w)} = \overline{\text{co}}\{\theta_m(1-w_m)e_m + \sum_{i=1}^{n}\theta_i w_i e_i \ : m, n \in \mathbb{N}, |\theta_i| = 1\,\forall i\}$,
- If $x_0 \in S_{X(w)}$ and $N \in \mathbb{N}$, there is $n \geqslant N$ and $\delta > 0$ such that $\|x_0 \pm \delta e_n\| \leqslant 1$.

The argument
Y infinite-dimensional strictly convex.

- By Dvoretzky-Rogers theorem, there is $(y_n) \subset S_Y$ such that $\sum_{n\geqslant 1} w_n y_n$ converges unconditionally, *so* $\{\sum_{n=1}^{\infty}\theta_n w_n y_n : |\theta_n| \leqslant 1\,\forall n\}$ *is bounded,*
- hence $T(e_n) = y_n$ defines a bounded linear operator on $X(w)$.
- If $S \in \text{NA}(X(w), Y)$, then there exists $n \in \mathbb{N}$ such that $S(e_n) = 0$,
- so $\|T - S\| \geqslant \|T(e_n) - S(e_n)\| = \|y_n\| = 1$. Therefore, Y fails property B.

Consequence
Y separable having property B in every equivalent norm $\implies$ Y is finite-dimensional.
★ What's about the converse?

⋆ Negative results: $L_1(\mu)$ spaces

Theorem (Acosta, 1999)
Every infinite-dimensional $L_1(\mu)$ space fails property B.

The domain space
Fix $w = (w_n) \in \ell_2 \setminus \ell_1$ decreasing, positive, with $w_1 < 1$, consider $X(w)$:

- $B_{X(w)} = \overline{\mathrm{co}}\big\{\theta_m(1-w_m)e_m + \sum_{i=1}^{n} \theta_i w_i e_i \ : \ m,n \in \mathbb{N},\ |\theta_i| = 1\,\forall i\big\}$,
- For $x^* \in \mathrm{NA}(X(w),\mathbb{K})$, $w\chi_{\mathrm{supp}(x^*)} \in \ell_1$.

The argument

- By Dvoretzky-Rogers theorem, there is $(f_n) \subset S_{L_1(\mu)}$ such that $\sum_{n\geqslant 1} w_n f_n$ converges unconditionally, *so* $\big\{\sum_{n=1}^{\infty} \theta_n w_n f_n \ : \ |\theta_n| \leqslant 1\,\forall n\big\}$ *is bounded;*
- so $T(e_n) = f_n$ defines a bounded linear operator on $X(w)$.
- If $S \in \mathrm{NA}(X(w), L_1(\mu))$, then there exists $I \subset \mathbb{N}$ with $w\chi_I \notin \ell_1$ such that

$$\textstyle\sum_{n\in I} w_n \|Se_n\| \leqslant \|S\|.$$

- As $\|Te_n\| = 1\ \forall n$, we have $\|T - S\| \geqslant 1$. Therefore, $L_1(\mu)$ fails property B.

1.5 Some results on classical spaces

⋆ Some classical spaces: positive results

Example (Johnson-Wolfe, 1979)
In the real case, $\mathrm{NA}(C(K_1), C(K_2))$ is dense in $\mathcal{L}(C(K_1), C(K_2))$.

Example (Iwanik, 1979)
$\mathrm{NA}(L_1(\mu), L_1(\nu))$ is dense in $\mathcal{L}(L_1(\mu), L_1(\nu))$.

Theorem (Schachermayer, 1983)
Every weakly compact operator from $C(K)$ can be approximated by (weakly compact) norm attaining operators.

Consequence (Schachermayer, 1983)
$\mathrm{NA}(C(K), L_p(\mu))$ is dense in $\mathcal{L}(C(K), L_p(\mu))$ for $1 \leqslant p < \infty$.

Example (Finet-Payá, 1998)
$\mathrm{NA}(L_1[0,1], L_\infty[0,1])$ is dense in $\mathcal{L}(L_1[0,1], L_\infty[0,1])$.

⋆ Some classical spaces: negative results

Example (Schachermayer, 1983)
$\mathrm{NA}(L_1[0,1], C[0,1])$ is NOT dense in $\mathcal{L}(L_1[0,1], C[0,1])$.

Consequence
$C[0,1]$ does not have property B and it was the first "classical" example.

Example (Aron-Choi-Kim-Lee-M., 2015; M., 2014)

$$\left.\begin{array}{c} Z = C[0,1] \oplus_1 L_1[0,1] \\ \text{or} \\ Z = C[0,1] \oplus_\infty L_1[0,1] \end{array}\right\} \quad\Longrightarrow\quad \mathrm{NA}(Z,Z) \text{ not dense in } \mathcal{L}(Z).$$

1.6 Main open problems

★ Main open problems

The main open problem
★ Do finite-dimensional spaces have Lindenstrauss property B?

(Stunning) open problem
Do finite-dimensional Hilbert spaces have Lindenstrauss property B?

Open problem
Characterize the topological compact spaces K such that $C(K)$ has property B.

Open problem
X Banach space without the RNP, does there exists a renorming of X such that $\mathrm{NA}(X, X)$ is not dense in $\mathcal{L}(X, X)$?

Remark
If $X \simeq Z \oplus Z$, then the above question has a positive answer (use Bourgain-Huff).

2 Norm attaining compact operators

2.1 Posing the problem for compact operators

★ Posing the problem for compact operators

Question
Can every compact operator be approximated by norm-attaining operators?

Observations

- In all the negative examples of the previous section, the authors constructed NON COMPACT operators which cannot be approximated by norm attaining operators.
- Actually, the idea of the proofs is to use that the operator which is not going to be approximated is not compact or, even, it is an isomorphism.
- In most examples, it was even known that compact operators attaining the norm are dense.

Where was it explicitly possed?

- Diestel-Uhl, *Rocky Mount. J. Math.*, 1976.
- Diestel-Uhl, *Vector measures* (monograph), 1977.
- Johnson-Wolfe, *Studia Math.*, 1979.
- Acosta, *RACSAM* (survey), 2006.

★ More observations on compact operators

Question
Can every compact operator be approximated by norm-attaining operators?

Observations

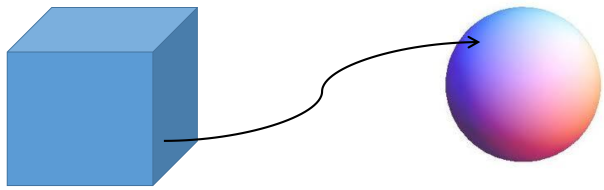
- If X is reflexive, then ALL compact operators from X into Y are norm attaining. (Indeed, compact operators carry weak convergent sequences to norm convergent sequences.)
- It is known from the 1970's that whenever $X = C_0(L)$ or $X = L_1(\mu)$ (and Y arbitrary) or $Y = L_1(\mu)$ or $Y^* \equiv L_1(\mu)$ (and X arbitrary), $\mathrm{NA}(X,Y) \cap \mathcal{K}(X,Y)$ is dense in $\mathcal{K}(X,Y)$.
- On the other hand, for a non reflexive space X and an arbitrary Y, we do not know whether there is any norm attaining operator from X to Y with rank greater than one.
- Actually, we do not know whether there exists a Banach space X such that $\mathrm{NA}(X, \ell_2)$ is contained in the set of rank-one operators.

2.2 The easiest negative example

★ Extending a result by Lindenstrauss

X, Y Banach spaces, $T \in \mathcal{L}(X,Y)$ and $x_0 \in S_X$ with $\|T\| = \|Tx_0\| = 1$.

- If x_0 is not extreme point of B_X, there is $z \in X$ such that $\|x_0 \pm z\| \leqslant 1$, so $\|Tx_0 \pm Tz\| \leqslant 1$.
- If Tx_0 is an extreme point of B_Y, then $Tz = 0$.



Geometrical lemma, Lindenstrauss

X, Y Banach spaces. Suppose that

- for every $x_0 \in S_X$, $\operatorname{lin}\{z \in X : \|x_0 \pm z\| \leqslant 1\}$ has finite codimension,
- Y is strictly convex.

Then, $\mathrm{NA}(X,Y) \subseteq \mathcal{F}(X,Y)$.

First consequence (recalling, Lindenstrauss, 1963)

- $\mathrm{NA}(c_0, Y) \subseteq \mathcal{F}(c_0, Y)$ if Y is strictly convex.
- Therefore, c_0 fails property A.

★ Extending a result by Lindenstrauss (II)

Proposition (extension of Lindenstrauss result)

$X \leqslant c_0$. For every $x_0 \in S_X$, $\operatorname{lin}\{z \in X : \|x_0 \pm z\| \leqslant 1\}$ has finite codimension.

Proof.

- as $x_0 \in c_0$, there exists m such that $|x_0(n)| < 1/2$ for every $n \geqslant m$;
- let $Z = \big\{z \in X : x_0(i) = 0 \text{ for } 1 \leqslant i \leqslant m\big\}$ (finite codimension in X);
- for $z \in Z$ with $\|z\| \leqslant 1/2$, one has $\|x_0 \pm z\| \leqslant 1$.

Main consequence

$X \leqslant c_0$, Y strictly convex. Then $\mathrm{NA}(X,Y) \subseteq \mathcal{F}(X,Y)$.

Question

What's next? How to use this result?

★ Grothendieck's approximation property

Definition (Grothendieck, 1950's)
Z has the approximation property (AP) if for every $K \subset Z$ compact and every $\varepsilon > 0$, there exists $F \in \mathcal{F}(Z)$ such that $\|Fz - z\| < \varepsilon$ for all $z \in K$.

Basic results
X, Y Banach spaces.

- (Grothendieck) Y has AP $\iff$ $\overline{\mathcal{F}(Z,Y)} = \mathcal{K}(Z,Y)$ for all Z.
- (Grothendieck) X^* has AP $\iff$ $\overline{\mathcal{F}(X,Z)} = \mathcal{K}(X,Z)$ for all Z.
- (Grothendieck) X^* AP $\implies$ X AP.
- (Enflo, 1973) There exists $X \leqslant c_0$ without AP.
- (Davie, 1973) There exists $X \leqslant \ell_p$ without AP for $1 \leqslant p < 2$.
- (Szankowski, 1976) There exists $X \leqslant \ell_p$ without AP for $2 < p < \infty$.

★ The first example

Theorem
There exists a **compact** operator which cannot be approximated by norm attaining operators.

Proof:

- consider $X \leqslant c_0$ without AP (Enflo);
- X^* does not has AP $\implies$ there exists Y and $T \in \mathcal{K}(X,Y)$ such that $T \notin \overline{\mathcal{F}(X,Y)}$;
- we may suppose $Y = \overline{T(X)}$, which is separable;
- so Y admits an equivalent strictly convex renorming (Klee);
- we apply the extension of Lindenstrauss result: $\mathrm{NA}(X,Y) \subseteq \mathcal{F}(X,Y)$;
- therefore, $T \notin \overline{\mathrm{NA}(X,Y)}$.

★ Two useful definitions

Definitions
X and Y Banach spaces.

- X has property AK when $\overline{\mathrm{NA}(X,Z) \cap \mathcal{K}(X,Z)} = \mathcal{K}(X,Z) \quad \forall\, Z$;
- Y has property BK when $\overline{\mathrm{NA}(Z,Y) \cap \mathcal{K}(Z,Y)} = \mathcal{K}(Z,Y) \quad \forall\, Z$.

Some basic results

- Finite-dimensional spaces have property AK;
- $Y = \mathbb{K}$ has property BK;
- Real finite-dimensional polyhedral spaces have property BK.

Our negative example (recalling)
There exists $X \leqslant c_0$ failing AK and there exits Y failing BK.

2.3 More negative examples

★ More examples: Domain space

Proposition (what we have proved so far...)
$X \leqslant c_0$ such that X^* fails AP $\implies$ X does not have AK.

Example by Johnson-Schechtman, 2001
Exists X subspace of c_0 **with Schauder basis** such that X^* fails the AP.

Corolary
There exists a Banach space X with Schauder basis failing property AK.

★ More examples: Range space

Strictly convex spaces
Y strictly convex without AP $\implies$ Y fails BK.

Lemma (Grothendieck)
Y has AP iff $\mathcal{F}(X,Y)$ is dense in $\mathcal{K}(X,Y)$ for every $X \leqslant c_0$.

Subspaces of $L_1(\mu)$
$Y \leqslant L_1(\mu)$ (complex case) without AP $\implies$ Y fails BK.

Observation (Globevnik, 1975)
Complex $L_1(\mu)$ spaces are complex strictly convex:

$$f, g \in L_1(\mu),\ \|f\| = 1 \text{ and } \|f + \theta g\| \leqslant 1\, \forall \theta \in B_{\mathbb{C}} \implies g = 0.$$

★ More examples: Domain=Range

Theorem
There exists a Banach space Z and a compact operator from Z to Z which cannot be approximated by norm attaining operators.

Proposition
X and Y Banach spaces, $Z = X \oplus_1 Y$ or $Z = X \oplus_\infty Y$.
$\mathrm{NA}(Z,Z) \cap \mathcal{K}(Z)$ dense in $\mathcal{K}(Z)$ $\implies$ $\mathrm{NA}(X,Y) \cap \mathcal{K}(X,Y)$ dense in $\mathcal{K}(X,Y)$.

Proof. Fix $T_0 \in K(X,Y)$ with $\|T_0\| = 1$ and $0 < \varepsilon < 1/2$.

- Define $S_0 \in K(Z,Z)$ by $S_0(x,y) = (0, T_0(x))$ for every $(x,y) \in X \oplus_\infty Y$, $\|S_0\| = 1$,
- there exists $S \in NA(Z,Z)$ such that $\|S_0 - S\| < \varepsilon$, take $(x_0, y_0) \in S_X \times B_Y$ such that $\|S(x_0,y_0)\| = \|S\|$.
- $\|P_X S\| = \|P_X S - P_X S_0\| \leqslant \|S - S_0\| < \varepsilon$, so $\|P_Y S(x_0,y_0)\| = \|P_Y S\| = \|S\|$.
- Take $x_0^* \in S_{X^*}$ such that $x_0^*(x_0) = 1$ and define the operator $T \in \mathcal{K}(X,Y)$ by
$$T(x) = P_Y S\big(x, x_0^*(x) y_0\big) \qquad \big(x \in X\big).$$
- $\|T\| \leqslant \|P_Y S\|$ and $\|T(x_0)\| = \|P_Y S(x_0,y_0)\| = \|P_Y S\|$, so $T \in NA(X,Y)$.
- On the other hand, for $x \in B_X$,
$$\begin{aligned}\|T_0(x) - T(x)\| &= \big\|P_2 S_0(x, x_0^*(x) y_0)\ -\ P_Y S(x, x_0^*(x) y_0)\big\| \\ &\leqslant \|P_Y S_0 - P_Y S\| \leqslant \|S_0 - S\| < \varepsilon.\end{aligned}$$

2.4 Positive results on property AK

★ Property AK

Definition (recalling)
X Banach space. X has property AK when $\overline{\mathrm{NA}(X,Z) \cap \mathcal{K}(X,Z)} = \mathcal{K}(X,Z) \quad \forall\, Z$.

First positive examples

- (Lindenstrauss-Schachermayer) Property α implies property AK;
- (Godun-Troyanski) so every separable Banach space can be renormed to have property AK;
- (Bourgain) RNP implies property AK (in every equivalent norm);
- Property AK is stable by ℓ_1-sums.

Negative examples
Every subspace of c_0 whose dual fails AP;

Question
Are there more positive examples?

★ Leading open problem

Problem
X^* AP $\implies$ X AK?

Observation
Known positive results on property AK are partial answers to the above question, as strong forms of the AP for the dual are involved.

Old known examples

- (Diestel-Uhl, 1976) $L_1(\mu)$ has AK;
- (Johnson-Wolfe, 1979) $C_0(L)$ has AK.

Our next aim is to prove these results and some more.

An interesting new example
If X^* has AP and X has property A $\implies$ X has property AK.

★ Positive results on property AK

Problem
X^* AP $\implies$ X AK?

Partial answer:

(Johnson-Wolfe) With a strong approximation property of the dual...
Suppose there exists a net of contractive projections $(P_\alpha)_\alpha$ in X with finite rank such that $\lim_\alpha P_\alpha^* = \mathrm{Id}_{X^*}$ in SOT. Then, X has AK.

Proof. Fix $T \in \mathcal{K}(X,Y)$.

- $TP_\alpha(B_X) = T(B_{P_\alpha(X)})$ (we need $P_\alpha^2 = P_\alpha$ and $\|P_\alpha\| = 1$).

- Then, TP_α attains the norm.
- As T^* is compact, $P_\alpha^* T^* \longrightarrow T^*$ in norm, so $TP_\alpha \longrightarrow T$ in norm.

Consequences

- (Diestel-Uhl) $L_1(\mu)$ has AK.
- (Johnson-Wolfe) $C_0(L)$ has AK.
- X with monotone and shrinking basis $\implies$ X has AK.
- X with monotone unconditional basis, $X \not\supseteq \ell_1 \implies X$ has AK.
- $X^* \equiv \ell_1 \implies X$ has AK (using a result by Gasparis).
- $X \leqslant c_0$ with monotone basis $\implies$ X has AK (using a result by Godefroy–Saphar).

2.5 Positive results on property BK

★ Property BK

Definition (recalling)
Y Banach space. Y has property BK when $\overline{\mathrm{NA}(Z,Y)\cap\mathcal{K}(Z,Y)} = \mathcal{K}(Z,Y) \quad \forall Z$.

First positive examples

- (Lindenstrauss) Property β implies property BK;
- (Partington) so every Banach space can be renormed to have property BK.
- (Cascales-Guirao-Kadets) $A(\mathbb{D})$ has BK (actually, every uniform algebra).
- Property BK is stable by c_0- and ℓ_∞-sums.

Negative examples

- Every strictly convex space without AP;
- every subspace of the complex $L_1(\mu)$ spaces without AP.

Question
Are there more positive examples?

★ Positive results on property BK I

Main open question
AP $\implies$ BK?

A partial answer (Johnson-Wolfe)

- If Y is polyhedral (real) and has AP $\implies$ Y has BK.
- X (complex) space with AP such that the norm of every finite-dimensional subspace can be calculated as the maximum of a finite set of functionals $\implies$ Y has BK.

Example (Johnson-Wolfe)
$Y \leqslant c_0$ (real or complex) with AP $\implies$ Y has BK.

A somehow reciprocal to the problem. . .
Y separable with BK for every equivalent norm $\implies$ Y has AP.

★ Positive results on property BK II

Main open question
AP $\implies$ BK?

Another partial answer (Johnson-Wolfe)
Y Banach space. Suppose there exists a uniformly bounded net of projections $(Q_\alpha)_\alpha$ in Y such that $\lim_\alpha Q_\alpha = \mathrm{Id}_Y$ in SOT and $Q_\alpha(Y)$ has property BK. Then, Y has property BK.

Proof. X Banach space, $T \in \mathcal{K}(X, Y)$.

- $Q_\alpha T$ converges in norm to T (by compactness of T),
- $Q_\alpha T$ arrives to $Q_\alpha(X)$, which has property BK,
- so each $Q_\alpha T$ can be approximated by norm-attaining compact operators.

Examples (Johnson-Wolfe)

- Y predual of $L_1(\mu)$ (real or complex) $\implies$ Y has BK;
- in particular, real or complex $C_0(L)$ spaces have property BK;
- real $L_1(\mu)$ spaces have property BK.

2.6 Open Problems

★ Some open problems

Main open problem
★ Can every finite-rank operator be approximated by norm-attaining operators ?

Open problem
X Banach space, does there exist a norm-attaining rank-two operator from X to a Hilbert space?

Another main open problem
★ X^* AP $\implies$ X AK?

Open problem
$X \leqslant c_0$ with the metric AP, does it have AK?

Open problem
X such that $X^* \equiv L_1(\mu)$, does X have AK?

Open problem
Y subspace of the real $L_1(\mu)$ without the AP, does Y fail property BK?

3 Numerical radius attaining operators

3.1 Numerical range and numerical radius

★ Numerical range: Hilbert spaces

Hilbert space numerical range (Toeplitz, 1918)

- A $n \times n$ real or complex matrix
$$W(A) = \big\{(Ax \mid x) \ : \ x \in \mathbb{K}^n, \ (x \mid x) = 1\big\}.$$
- H real or complex Hilbert space, $T \in \mathcal{L}(H)$,
$$W(T) = \big\{(Tx \mid x) \ : \ x \in H, \ \|x\| = 1\big\}.$$

Some properties

H Hilbert space, $T \in \mathcal{L}(H)$:

- $W(T)$ is convex.
- In the complex case, $\overline{W(T)}$ contains the spectrum of T.
- If T is normal, then $\overline{W(T)} = \overline{\mathrm{co}}\,\mathrm{Sp}(T)$.

★ Numerical range: Banach spaces

Banach space numerical range (Bauer 1962; Lumer, 1961)

X Banach space, $T \in \mathcal{L}(X)$,
$$V(T) = \big\{x^*(Tx) \ : \ x^* \in S_{X^*}, \, x \in S_X, \, x^*(x) = 1\big\}$$

Some properties

X Banach space, $T \in \mathcal{L}(X)$:

- $V(T)$ is connected (not necessarily convex).
- In the complex case, $\overline{V(T)}$ contains the spectrum of T.
- In fact,
$$\overline{\mathrm{co}}\,\mathrm{Sp}(T) = \bigcap \overline{\mathrm{co}}\,V(T),$$
the intersection taken over all numerical ranges $V(T)$ corresponding to equivalent norms on X.

★ Some motivations for the numerical range

For Hilbert spaces

- It is a comfortable way to study the spectrum.
- It is useful to work with some concept like hermitian operator, skew-hermitian operator, dissipative operator...
- It is useful to estimate spectral radii of small perturbations of matrices.

For Banach spaces

- It allows to carry to the general case the concepts of hermitian operator, skew-hermitian operator, dissipative operators…
- It gives a description of the Lie algebra corresponding to the Lie group of all onto isometries on the space.
- It gives an easy and quantitative proof of the fact that Id is an strongly extreme point of $B_{\mathcal{L}(X)}$ (MLUR point).

★ Numerical radius

Numerical radius
X Banach space, $T \in \mathcal{L}(X)$. The numerical radius of T is

$$v(T) = \sup\left\{|x^*(Tx)| \;:\; x^* \in S_{X^*},\; x \in S_X,\; x^*(x) = 1\right\}.$$

★ Notation: $\Pi(X) = \left\{(x, x^*) \;:\; x^* \in S_{X^*},\, x \in S_X,\, x^*(x) = 1\right\}$.
With this notation, $v(T) = \sup\left\{|x^*(Tx)| \;:\; (x, x^*) \in \Pi(X)\right\}$.

Remark
The numerical radius is a continuous seminorm in $\mathcal{L}(X)$. Actually, $v(\cdot) \leqslant \|\cdot\|$.

Numerical radius attaining operators
$T \in \mathcal{L}(X)$ attains its numerical radius when

$$\exists\ (x, x^*) \in \Pi(X) \;:\; |x^*T(x)| \;=\; v(T)$$

★ $\mathrm{NRA}(X) = \{T \in \mathcal{L}(X) \;:\; T \text{ attains its numerical radius}\}$

3.2 Known results on numerical radius attaining operators

★ Numerical radius attaining operators: first results

Numerical radius attaining operators
X Banach space, $T \in \mathcal{L}(X)$ attains its numerical radius when

$$\exists\ (x, x^*) \in \Pi(X) \;:\; |x^*T(x)| \;=\; \sup\left\{|y^*(Ty)| \;:\; (y, y^*) \in \Pi(X)\right\}.$$

Some examples

- If $\dim(X) < \infty$, then $\mathrm{NRA}(X) = \mathcal{L}(X)$ ($\Pi(X)$ is compact).
- Even in $X = \ell_2$ there are (diagonal) operators which do not attain their numerical radius.
- Suppose $v(T) = \|T\|$:
  - $T \in \mathrm{NRA}(X) \implies T \in \mathrm{NA}(X, X)$,
  - $T \in \mathrm{NA}(X, X) \not\Longrightarrow T \in \mathrm{NRA}(X, X)$.

Main problem here
When is $\mathrm{NRA}(X)$ dense in $\mathcal{L}(X)$**?**

The study of this problem was initiated in the PhD dissertation of B. Sims of 1972, where some positive results were given.

★ Some positive results

Proposition (Berg-Sims, 1984)
X uniformly convex $\implies$ $\mathrm{NRA}(X)$ dense in $\mathcal{L}(X)$.

Proposition (Acosta-Payá, 1989)
For every Banach space X, $\{T \in \mathcal{L}(X) : T^{**} \in \mathrm{NRA}(X^{**})\}$ is dense.

Theorem (Acosta-Payá, 1993)
If X has the RNP, then $\mathrm{NRA}(X)$ is dense in $\mathcal{L}(X)$.

Examples (Cardasi, 1985)
$C(K)$ and $L_1(\mu)$ (real case) satisfy the density of numerical radius attaining operators.

Proposition (Acosta, 1991 & 1993)
Property α and property β (real case) implies the density of numerical radius attaining operators.

- Consequence: every real space can be renormed to get the density of numerical radius attaining operators.

★ Some negative results

Example (Payá, 1992)
There is a Banach space Z for which $\mathrm{NRA}(Z)$ is not dense in $\mathcal{L}(Z)$.

- $Z = c_0 \oplus_\infty Y$, where Y is a concrete strictly convex renorming of c_0.

Example (Acosta-Aguirre-Payá, 1992)
For $Z = G \oplus_\infty \ell_2$ (G from Gowers' counterexample), $\mathrm{NRA}(Z)$ is not dense in $\mathcal{L}(Z)$.

Example (Kim-Lee-M., 2016?)
For $Z = c_0 \oplus_1 Y$ ($Y \simeq c_0$ strictly convex), $\mathrm{NRA}(Z)$ is not dense in $\mathcal{L}(Z)$.

- $\mathrm{NRA}(c_0 \oplus_1 Y)$ dense in $\mathcal{L}(c_0 \oplus_1 Y) \implies \mathrm{NA}(c_0, Y)$ dense in $\mathcal{L}(c_0, Y)$.

Example (Capel-M.-Merí, preprint)
For $Z = L_1[0,1] \oplus_1 C[0,1]$ and $Z = L_1[0,1] \oplus_\infty C[0,1]$, $\overline{\mathrm{NRA}(Z)} \neq \mathcal{L}(Z)$.

- $v(T) = \|T\|$ for every $T \in \mathcal{L}(Z)$, and $\mathrm{NA}(Z, Z)$ is not dense in $\mathcal{L}(Z)$.

None of these examples produce a **compact** operator outside $\overline{\mathrm{NRA}(Z)}$.

3.3 The counterexample

★ The counterexample

Example
Given $1 < p < 2$, there are a subspace X of c_0 and a quotient Y of ℓ_p such that $\mathcal{K}(X \oplus_\infty Y)$ is not contained in the closure of $\mathrm{NRA}(X \oplus_\infty Y)$.

The proof needs five steps:

- use that the norm of Y^* is smooth enough (lemma 1);
- use that X is strongly flat (lemma 2);
- calculate numerical radius of operators on ℓ_∞-sums (lemma 3);
- glue these three results and use numerical radius attaining operators (proposition ★)
- use the AP and finish the proof (proof of the example).

★ Step 1: using the smoothness of Y^*

Smoothness and duality mapping

Let Z be a Banach space.

- The norm of Z is smooth if it is Gâteaux differentiable at every $z \in Z \setminus \{0\}$.
- The normalized duality mapping $J_Z : Z \longrightarrow 2^{Z^*}$ of Z is given by

$$J(z) = \{z^* \in Z^* \ : \ z^*(z) = \|z^*\|^2 = \|z\|^2\} \qquad (z \in Z).$$

- If the norm of Z is smooth, J is single-valued and the map $\widetilde{J}_Z : Z \setminus \{0\} \longrightarrow S_{Z^*}$ given by

$$\widetilde{J}_Z(z) = J\left(\frac{z}{\|z\|}\right) = \frac{J(z)}{\|J(z)\|} \qquad (z \in Z \setminus \{0\})$$

  is well defined.
- $\widetilde{J}_Z(z)$ can be alternatively defined as the unique $z^* \in S_{Z^*}$ such that $z^*(z) = \|z\|$.
- If the norm of Z is C^2-smooth, then $\widetilde{J}_Z$ is Fréchet differentiable on $Z \setminus \{0\}$.

★ Step 1: using the smoothness of Y^* II

Smoothness and pre-duality mapping

Let Y be a reflexive Banach space whose dual norm is C^2-smooth. Then $\widetilde{J}_{Y^*} : Y^* \setminus \{0\} \longrightarrow S_Y$ is Fréchet differentiable.

- $\widetilde{J}_{Y^*}(y^*)$ is the unique $y \in S_Y$ such that $y^*(y) = \|y^*\|$.

Lemma 1

Y (reflexive) space such that the norm of Y^* is C^2-smooth on $Y^* \setminus \{0\}$, X Banach space. Suppose that $A \in \mathcal{L}(Y)$, $B \in \mathcal{L}(X, Y)$, and $(y_0, y_0^*) \in \Pi(Y)$ satisfy that

$$|y^*(Ay)| + \|B^*y^*\| \leqslant |y_0^*(Ay_0)| + \|B^*y_0^*\|$$

for all $(y, y^*) \in \Pi(Y)$. Then,

$$\lim_{t \to 0} \frac{\|B^*y_0^* + tB^*h^*\| + \|B^*y_0^* - tB^*h^*\| - 2\|B^*y_0^*\|}{t} = 0$$

for every $h^* \in S_{Y^*}$.

Proof. Observe first that the assumption on Y implies reflexivity. Therefore, we may and do identify Y^{**} with Y and consider the normalized duality mapping $\widetilde{J}_{Y^*} : Y^* \setminus \{0\} \longrightarrow S_Y$ and observe that it is Fréchet differentiable by the hypothesis on Y. Hence, the function $F : Y^* \setminus \{0\} \longrightarrow \mathbb{R}$ given by

$$F(y^*) = \left|y^*\left[A\big(\widetilde{J}_{Y^*}(y^*)\big)\right]\right| \qquad \big(y^* \in Y^* \setminus \{0\}\big)$$

is Fréchet differentiable at every $y^* \in Y^* \setminus \{0\}$ for which $F(y^*) \neq 0$. Next, we fix $h^* \in S_{Y^*}$ and for $0 \leqslant t < 1$ we define:

$$y_t^* = y_0^* + th^*, \qquad \phi(t) = \|y_t^*\|, \qquad F_1(t) = F(y_t^*), \qquad \text{and} \qquad F_2(t) = \|B^*y_t^*\|.$$

On the one hand, F_2 is right-differentiable at the origin as it is a convex function. On the other hand, if we assume that $0 \neq |y_0^*(Ay_0)| = F(y_0^*) = F_1(0)$ (observe that $y_0 = \widetilde{J}_{Y^*}(y_0^*)$ by smoothness), we get that F_1 is differentiable at the origin.

Now, by using the inequality in the hypothesis for

$$y^* = \phi(t)^{-1} y_t^* \qquad \text{and} \qquad y = \widetilde{J}_{Y^*}(\phi(t)^{-1} y_t^*) = \widetilde{J}_{Y^*}(y_t^*),$$

we obtain that

$$F_1(t) + F_2(t) \leqslant \phi(t)[F_1(0) + F_2(0)] \qquad (0 \leqslant t < 1), \tag{1}$$

which gives

$$\frac{F_1(t) - F_1(0)}{t} + \frac{F_2(t) - F_2(0)}{t} \leqslant \frac{\phi(t) - 1}{t}[F_1(0) + F_2(0)] \qquad (0 < t < 1).$$

Taking right-derivatives, we obtain

$$F_1'(0) + \partial_+ F_2(0) \leqslant \phi'(0)[F_1(0) + F_2(0)]$$

(where $\partial_+ F_2(0)$ is the right-derivative of F_2 at 0) or, equivalently,

$$D_F(y_0^*)(h^*) + \lim_{t\to 0^+} \frac{\|B^* y_0^* + tB^* h^*\| - \|B^* y_0^*\|}{t} \leqslant D_{\|\cdot\|_{Y^*}}(y_0^*)(h^*)\big[F(y_0^*) + \|B^*(y_0^*)\|\big].$$

If we repeat the above argument for $-h^*$, we get the analogous inequality

$$D_F(y_0^*)(-h^*) + \lim_{t\to 0^+} \frac{\|B^* y_0^* - tB^* h^*\| - \|B^* y_0^*\|}{t} \leqslant D_{\|\cdot\|_{Y^*}}(y_0^*)(-h^*)\big[F(y_0^*) + \|B^*(y_0^*)\|\big].$$

Adding the above two equations, taking into account that both F and the norm of Y^* are Fréchet differentiable, we obtain

$$\lim_{t\to 0^+} \frac{\|B^* y_0^* + tB^* h^*\| + \|B^* y_0^* - tB^* h^*\| - 2\|B^* y_0^*\|}{t} \leqslant 0 \tag{2}$$

as desired. We recall that we required that $|y_0^*(Ay_0)| \neq 0$ to use the differentiability of F_1. If, otherwise, we have $|y_0^*(Ay_0)| = 0$, observe that inequality (1) implies

$$F_2(t) \leqslant \phi(t) F_2(0),$$

and we can repeat the arguments above without the use of F_1.

Next, observe that the function $\dfrac{\|B^* y_0^* + tB^* h^*\| + \|B^* y_0^* - tB^* h^*\| - 2\|B^* y_0^*\|}{t}$ is non-negative for every $t > 0$ by the convexity of the norm, and so the limit in (2) is actually equal to zero. Finally, as changing t by $-t$ in this limit just changes the sign of the function and the limit is zero, we may replace right-limit by regular limit, getting the statement of the proposition. □

★ Step 2: using that X is strongly flat

Strongly flat

X Banach space, $x_0 \in S_X$.

- $\operatorname{Flat}(x_0) = \{x \in X : \|x_0 \pm x\| \leqslant 1\}$;
- X is strongly flat if $\operatorname{codim}\big(\overline{\operatorname{lin}} \operatorname{Flat}(x_0)\big) < \infty$.

Lemma 2

X strongly flat Banach space, Y Banach space. Suppose that for $B \in \mathcal{L}(X,Y)$ there is $y_0^* \in S_{Y^*}$ such that

$$\lim_{t\to 0^+} \frac{\|B^* y_0^* + tB^* h^*\| + \|B^* y_0^* - tB^* h^*\| - 2\|B^* y_0^*\|}{t} \leqslant 0$$

for every $h^* \in S_{Y^*}$ and that $B^* y_0^*$ attains its norm on X. Then, B has finite-rank.

Proof. Write $x_0^* = B^* y_0^*$. As x_0^* attains its norm, we may take $x_0 \in S_X$ such that $\operatorname{Re} x_0^*(x_0) = \|x_0^*\|$. We claim that $Bz = 0$ for every $x \in \operatorname{Flat}(x_0)$, and this finishes the proof by the hypothesis on X. Therefore, let us prove the claim. Fixed $x \in \operatorname{Flat}(x_0)$, for each $h^* \in S_{Y^*}$, we write $x^* = \theta B^* h^*$, where θ is a modulus-one scalar satisfying that $\operatorname{Re} x^*(x) = |x^*(x)|$. Next, given $\varepsilon > 0$, we use the inequality in the hypothesis to find $r > 0$ such that

$$\|x_0^* + tx^*\| + \|x_0^* - tx^*\| < 2\|x_0^*\| + t\varepsilon$$

for every $t \in (0, r)$. Now, as $\|x_0 \pm x\| \leqslant 1$, we get that

$$\begin{aligned} 2\|x_0^*\| + t\varepsilon &> \|x_0^* + tx^*\| + \|x_0 - tx^*\| \\ &\geqslant \operatorname{Re}\Big([x_0^* + tx^*](x_0 + x) + [x_0^* - tx^*](x_0 - x)\Big) \\ &= 2\|x_0^*\| + 2t \operatorname{Re} x^*(x) = 2\|x_0^*\| + 2t|x^*(x)|. \end{aligned}$$

This gives that $2|x^*(x)| < \varepsilon$, and the arbitrariness of ε implies that

$$0 = |x^*(x)| = \big|[B^*h^*](x)\big| = |h^*(Bx)|.$$

Since this is true for every $h^* \in S_{Y^*}$, we get that $Bx = 0$, as claimed. □

★ Step 3: numerical radius and ℓ_∞-sums

Lemma 3 (Payá, 1992)
X, Y Banach spaces, $Z = X \oplus_\infty Y$ and P_X, P_Y natural projections. For $T \in \mathcal{L}(Z)$, we have

1. $v(T) = \max\{v(P_X T), v(P_Y T)\}$;
2. $T \in \operatorname{NRA}(Z)$ and $v(P_Y T) > v(P_X T) \implies P_Y T \in \operatorname{NRA}(Z)$;
3. $v(P_Y T) = \sup\{|y^*(P_Y T(y + x))| \,:\, (y, y^*) \in \Pi(Y),\, x \in B_X\}$;
4. $P_Y T \in \operatorname{NRA}(Z) \iff$ the supremum above is attained.

★ Step 4: gluing the thee results and using $\operatorname{NRA}(Z)$

Proposition ★
Y such that the norm of Y^* is C^2-smooth on $Y^* \setminus \{0\}$, X strongly flat, $Z = X \oplus_\infty Y$. For $A \in \mathcal{L}(Y)$ and $B \in \mathcal{L}(X, Y)$, define $T \in \mathcal{L}(Z)$ by

$$T(x + y) = A(y) + B(x) \qquad \big(x \in X,\ y \in Y\big).$$

If $T \in \operatorname{NRA}(Z)$, then B is of finite-rank.

Proof. Consider the projection P_Y from Z onto Y. It is clear that $P_Y T = T$ and Lemma 3 provides the existence of $(y_0, y_0^*) \in \Pi(Y)$ and $x_0 \in B_X$ such that

$$|y^*(Ay + Bx)| \leqslant |y_0^*(Ay_0 + Bx_0)|$$

for every $(y, y^*) \in \Pi(Y)$ and every $x \in B_X$. By rotating x, we actually get

$$|y^*(Ay)| + |y^*(Bx)| \leqslant |y_0^*(Ay_0)| + |y_0^*(Bx_0)|$$

or, equivalently,

$$|y^*(Ay)| + |[B^*y^*](x)| \leqslant |y_0^*(Ay_0)| + |[B^*y_0^*](x_0)|. \tag{3}$$

By taking supremum on $x \in B_X$, we obtain

$$|y^*(Ay)| + \|B^*y^*\| \leqslant |y_0^*(Ay_0)| + \|B^*y_0^*\|$$

for all $(y, y^*) \in \Pi(Y)$. As the norm of Y^* is C^2 smooth at $Y^* \setminus \{0\}$, it follows from Lemma 1 that

$$\lim_{t \to 0} \frac{\|B^* y_0^* + tB^* h^*\| + \|B^* y_0^* - tB^* h^*\| - 2\|B^* y_0^*\|}{t} = 0$$

for every $h^* \in S_{Y^*}$. On the other hand, when we take $(y, y^*) = (y_0, y_0^*)$ in equation (3), we obtain

$$\big|[B^* y_0^*](x)\big| \leqslant \big|[B^* y_0^*](x_0)\big|$$

for every $x \in B_X$, meaning that the functional $B^* y_0^* \in X^*$ attains its norm at x_0. These two facts and the assumption on X allow us to apply Lemma 2 to get that B is of finite-rank. □

★ Step 5: The AP and the proof of the example

Example

Given $1 < p < 2$, there are a subspace X of c_0 and a quotient Y of ℓ_p such that $\mathcal{K}(X \oplus_\infty Y)$ is not contained in the closure of $\mathrm{NRA}(X \oplus_\infty Y)$.

- Take Y quotient of ℓ_p without the AP;
- consider $X \leqslant c_0$ such that exists $S \in \mathcal{K}(X, Y) \setminus \overline{\mathcal{F}(X, Y)}$;
- define $T \in \mathcal{K}(Z)$ by $T(x + y) = Sx$;
- work with Proposition ★ to get that $T \notin \overline{\mathrm{NRA}(Z)}$.

Rest of the proof. Suppose, for the sake of contradiction, that there is a sequence $\{T_n\}$ in $\mathrm{NRA}(Z)$ converging to T in norm. We clearly have that $P_Y T = T$ and $P_X T = 0$. We get that $\{P_Y T_n\} \longrightarrow T$, $\{P_X T_n\} \longrightarrow 0$, so $\{v(P_Y T_n)\} \longrightarrow v(T) = v(P_Y T)$ and $\{v(P_X T_n)\} \longrightarrow 0$. It follows from Lemma 3 that $v(T) = \|S\| > 0$ and that $P_Y T_n \in \mathrm{NRA}(Z)$ for every n large enough. Therefore, removing some terms of the sequence $\{T_n\}$ and replacing T_n by $P_Y T_n$, there is no restriction in assuming that

$$T_n \in \mathrm{NRA}(Z), \qquad P_Y T_n = T_n \qquad \forall n \in \mathbb{N}, \qquad \text{and} \qquad \|T_n - T\| \longrightarrow 0.$$

Now, observe that for each $n \in \mathbb{N}$ there are operators $A_n \in L(Y)$ and $B_n \in L(X, Y)$ such that

$$T_n(y + x) = A_n(y) + B_n(x) \qquad (y \in Y,\, x \in X).$$

The norm of Y^* is C^2-smooth and X is strongly flat, so we can use Proposition ★ with T_n to conclude that B_n is a finite-rank operator for every $n \in \mathbb{N}$. But this leads to a contradiction because $\{B_n\}$ converges in norm to S, finishing thus the proof. □

3.4 Positive results

★ Some positive results I

The positive results to get density of numerical radius attaining operators also works for compact operators:

Positive results

X Banach space satisfying one of the following conditions:

- X has RNP,
- X has property α,
- X is real and has property β.

Then $\mathrm{NRA}(X) \cap \mathcal{K}(X)$ is dense in $\mathcal{K}(X)$.

In all the proofs, every operator is perturbed by a compact operator to get a numerical radius attaining one.

★ Some positive results II: CL-spaces

Definition (Fullerton, 1961)
A Banach space X is a CL-space if B_X is the absolutely convex hull of every maximal convex subset of S_X.

Examples
Real or complex $C(K)$ spaces and real $L_1(\mu)$ spaces are CL-spaces.

Theorem (Acosta, 1990)
X CL-space. Then:

- For every $T \in \mathcal{L}(X)$, $v(T) = \|T\|$;
- $T \in \mathrm{NA}(X, X) \iff T \in \mathrm{NRA}(X)$.

Main consequence
$X = C(K)$ (real or complex) or $X = L_1(\mu)$ (real) $\Longrightarrow$ $\overline{\mathrm{NRA}(X) \cap \mathcal{K}(X)} = \mathcal{K}(X)$.

Another consequence
$X = C[0,1] \oplus_1 L_1[0,1]$ (real) or $X = C[0,1] \oplus_\infty L_1[0,1]$ (real)
$\Longrightarrow$ $\mathrm{NRA}(X) \cap \mathcal{K}(X)$ dense in $\mathcal{K}(X)$.

★ Recall that $\mathrm{NRA}(X)$ is NOT dense in $\mathcal{L}(X)$.

3.5 Open problems

★ Open problems

Open problem
X Banach space without the RNP, does there exists a renorming of X such that $\mathrm{NRA}(X)$ is not dense in $\mathcal{L}(X)$**?**

Open problem
X Banach space without the RNP, does there exists a renorming of X such that $\mathrm{NRA}(X) \cap \mathcal{K}(X)$ is not dense in $\mathcal{K}(X)$**?**

Open problem
Do we have $\overline{\mathrm{NRA}(X) \cap \mathcal{K}(X)} = \mathcal{K}(X)$ for X such that $X^* \equiv L_1(\mu)$**?**

Open problem
Suppose that $v(T) = \|T\|$ for every $T \in \mathcal{L}(X)$ and $\mathrm{NA}(X, X)$ is dense in $\mathcal{L}(X)$. Does $\mathrm{NRA}(X)$ have to be dense in $\mathcal{L}(X)$**?**

Open problem
Suppose that $v(T) = \|T\|$ for every $T \in \mathcal{K}(X)$ and $\mathrm{NA}(X, X) \cap \mathcal{K}(X)$ is dense in $\mathcal{K}(X)$. Does $\mathrm{NRA}(X) \cap \mathcal{K}(X)$ have to be dense in $\mathcal{K}(X)$**?**