Numerical index theory

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Mini-course

Kent State University, Spring 2012

Schedule of the talk

1. Basic notation
2. Numerical range of operators
3. Two results on surjective isometries
4. Numerical index of Banach spaces
5. The alternative Daugavet property
6. Lush spaces
7. Slicely countably determined spaces
8. Remarks on the containment of c_0 and ℓ_1
9. Numerical index of L_p-spaces
10. Extremely non-complex Banach spaces

Notation

Basic notation I

- $\mathbb{K}$ base field ($\mathbb{R}$ or $\mathbb{C}$):
  - $\mathbb{T}$ modulus-one scalars,
  - $\operatorname{Re} z$ real part of z ($\operatorname{Re} z = z$ if $\mathbb{K} = \mathbb{R}$).
- H Hilbert space: $(\cdot \mid \cdot)$ denotes the inner product.
- X Banach space:
  - S_X unit sphere, B_X unit ball,
  - X^* dual space,
  - $L(X)$ bounded linear operators,
  - $W(X)$ weakly compact linear operators,
  - $\operatorname{Iso}(X)$ surjective linear isometries,
- X Banach space, $T \in L(X)$:
  - $\operatorname{Sp}(T)$ spectrum of T.
  - $T^* \in L(X^*)$ adjoint operator of T.

Notation

Basic notation (II)

X Banach space, $B \subset X$, C convex subset of X:

- B is *rounded* if $\mathbb{T}B = B$,
- $\operatorname{co}(B)$ convex hull of B,
- $\overline{\operatorname{co}}(B)$ closed convex hull of B,
- $\operatorname{aconv}(B) = \operatorname{co}(\mathbb{T}\, B)$ absolutely convex hull of B,
- $\overline{\operatorname{aconv}}(B) = \overline{\operatorname{co}}(\mathbb{T}\, B)$ absolutely convex hull of B,
- $\operatorname{ext}(C)$ extreme points of C,
- *slice* of C:

$$S(C, x^*, \alpha) = \{x \in C \ : \ \operatorname{Re} x^*(x) > \sup \operatorname{Re} x^*(C) - \alpha\}$$

where $x^* \in X^*$ and $0 < \alpha < \sup \operatorname{Re} x^*(C)$.

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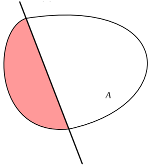

Numerical range of operators

F. F. Bonsall and J. Duncan
Numerical Ranges. Vol I and II.
London Math. Soc. Lecture Note Series, 1971 & 1973.

Numerical range: Hilbert spaces

Hilbert space numerical range (Toeplitz, 1918)

- A $n \times n$ real or complex matrix

$$W(A) = \{(Ax \mid x) \ : \ x \in \mathbb{K}^n, \ (x \mid x) = 1\}.$$

- H real or complex Hilbert space, $T \in L(H)$,

$$W(T) = \{(Tx \mid x) \ : \ x \in H, \ \|x\| = 1\}.$$

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Remark

★ Given $T \in L(H)$ we associate

- a sesquilinear form $\varphi_T(x,y) = (Tx \mid y) \qquad (x,y \in H)$,
- a quadratic form $\widehat{\varphi_T}(x) = \varphi_T(x,x) = (Tx \mid x) \qquad (x \in H)$.

★ Then, $W(T) = \widehat{\varphi_T}(S_H)$.

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★ Then, $W(T) = \widehat{\varphi_T}(S_H)$. Therefore:

- $\widehat{\varphi_T}(B_H) = [0,1]\, W(T)$,
- $\widehat{\varphi_T}(H) = \mathbb{R}^+\, W(T)$.
- But we cannot get $W(T)$ from $\widehat{\varphi_T}(B_H)$!

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- In the complex case,

$$\sup\{|(Tx \mid x)| \ : \ x \in S_H\} \geqslant \frac{1}{2}\|T\|.$$

  If T is actually self-adjoint, then

$$\sup\{|(Tx \mid x)| \ : \ x \in S_H\} = \|T\|.$$

Proving a result

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- For $x, y \in S_H$ fixed, use the polarization formula:

$$(Tx \mid y) = \frac{1}{4}\Big[(T(x+y) \mid x+y) \;-\; (T(x-y) \mid x-y) \\ +\; i\,(T(x+iy) \mid x+iy) \;-\; i\,(T(x-iy) \mid x-iy)\Big].$$

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- By the parallelogram's law:

$$|(Tx \mid y)| \leqslant \frac{1}{4}\,v(T)\left[2\|x\|^2 + 2\|y\|^2 + 2\|x\|^2 + 2\|iy\|^2\right] = 2v(T).$$

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- We just take supremum on $x, y \in S_H$ ✓

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Some reasons to study numerical ranges

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- It is useful to estimate spectral radii of small perturbations of matrices.

Example

Consider $A = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ \varepsilon & 0 \end{pmatrix}$.

- $\mathrm{Sp}(A) = \{0\}$, $\mathrm{Sp}(B) = \{0\}$.
- $\mathrm{Sp}(A+B) = \{\pm\sqrt{M\varepsilon}\} \subseteq W(A+B) \subseteq W(A) + W(B)$,
- so the spectral radius of $A+B$ is bounded above by $\frac{1}{2}(|M| + |\varepsilon|)$.

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- It is a comfortable way to study the spectrum.
- It is useful to estimate spectral radii of small perturbations of matrices.
- It is useful to work with some concepts like hermitian operator, skew-hermitian operator, dissipative operator. . .

Numerical range: Banach spaces (I)

Banach spaces numerical range (Bauer 1962; Lumer, 1961)

X Banach space, $T \in L(X)$,

$$V(T) = \{x^*(Tx) \ : \ x^* \in S_{X^*},\ x \in S_X,\ x^*(x) = 1\}$$

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The numerical range as a derivative

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i.e. $\sup \operatorname{Re} V(T)$ is the derivative of the norm at Id in the direction of T.

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Stronger result (Bollobás, 1970)

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$$V(T) \subseteq V(T^*) \subseteq \overline{V(T)}.$$

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- Then $V(T_{\mathbb{R}}) = \operatorname{Re} V(T)$.

- Consequence:
  X complex, then there is $S \in L(X_{\mathbb{R}})$ with $\|S\| = 1$ and $V(S) = \{0\}$.

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- It allows to carry to the general case the concepts of hermitian operator, skew-hermitian operator, dissipative operators...
- It gives a description of the Lie algebra corresponding to the Lie group of all onto isometries on the space.
- It gives an easy and quantitative proof of the fact that Id is an strongly extreme point of $B_{L(X)}$ (MLUR point).

Numerical radius: definition and properties

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X real or complex Banach space, $T \in L(X)$,

$$\begin{aligned} v(T) &= \sup\left\{|\lambda| \ : \ \lambda \in V(T)\right\} \\ &= \sup\left\{|x^*(Tx)| \ : \ x^* \in S_{X^*},\ x \in S_X,\ x^*(x) = 1\right\} \end{aligned}$$

Numerical radius: definition and properties

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5. In particular, this is the case for $X = C(K)$.

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If $f_0(\xi_0) \sim 1$, then we were done. This our goal.

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and find $\varphi : K \longrightarrow [0,1]$ continuous with $\text{supp}(\varphi) \subset V$ and $\varphi(\xi_0) = 1$.

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If $X = L_1(\mu)$, then $X^* \equiv C(K_\mu)$. Therefore, $v(T) = v(T^*) = \|T^*\| = \|T\|$ ✓

Differences between real and complex spaces

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Example

X complex Banach space, define $T \in L(X_{\mathbb{R}})$ by

$$T(x) = i\,x \qquad (x \in X).$$

- $\|T\| = 1$ and $v(T) = 0$ if viewed in $X_{\mathbb{R}}$.
- $\|T\| = 1$ and $V(T) = \{i\}$, so $v(T) = 1$ if viewed in (complex) X.

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Theorem (Bohnenblust-Karlin, 1955; Glickfeld, 1970)

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$$v(T) \geqslant \frac{1}{\mathrm{e}}\|T\|.$$

The constant $\frac{1}{\mathrm{e}}$ is optimal:

$\exists$ X two-dimensional complex, $\exists$ $T \in L(X)$ with $\|T\| = \mathrm{e}$ and $v(T) = 1$.

Proof of Bohnenblust-Karlin's theorem. Preliminaries

The exponential function

X Banach space, $T \in L(X)$, define $\exp(T) = \displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} T^n$.

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X Banach space, $T, S \in L(X)$.

- $TS = ST \implies \exp(T+S) = \exp(T)\exp(S)$.
- $\exp(T)\exp(-T) = \exp(0) = \mathrm{Id} \implies \exp(T)$ surjective isomorphism.
- $\{\exp(\rho T) : \rho \in \mathbb{R}_0^+\}$ one-parameter semigroup generated by T.
- $\|\exp(T)\| \leqslant e^{\|T\|}$ (we will improve this inequality in the sequel).

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X Banach, $T \in L(X)$, then $\|\exp(\zeta T)\| \leqslant \mathrm{e}^{|\zeta| v(T)}$ for every $\zeta \in \mathbb{K}$.

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  and we are done.

Numerical index: definition and properties

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- Actually, the above inequalities are best possible:

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Numerical index: definition and properties

Numerical index

X real or complex Banach space

$$n(X) = \max\{k \geqslant 0 \ : \ k\,\|T\| \leqslant v(T) \ \forall T \in L(X)\}$$
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- $n(X^*) \leqslant n(X)$.

Numerical index: examples

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6. $n(A(\mathbb{D})) = 1$ and $n\big(H^\infty\big) = 1$.

Surjective isometries

3. Two results on surjective isometries
    - Numerical ranges and isometries
    - Isometries on finite-dimensional spaces
    - Isometries and duality

M. Martín
The group of isometries of a Banach space and duality.
J. Funct. Anal. (2008).

M. Martín, J. Merí, and A. Rodríguez-Palacios.
Finite-dimensional spaces with numerical index zero.
Indiana U. Math. J. (2004).

H. P. Rosenthal
The Lie algebra of a Banach space.
in: *Banach spaces* (Columbia, Mo., 1984), LNM, Springer, 1985.

Semigroups of isometries: motivating example

A motivating example

A real or complex $n \times n$ matrix. TFAE:

- A is skew-adjoint (i.e. $A^* = -A$).
- $B = \exp(\rho A)$ is unitary for every $\rho \in \mathbb{R}$ (i.e. $B^*B = BB^* = \mathrm{Id}$).

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In term of Hilbert spaces

H (n-dimensional) Hilbert space, $T \in L(H)$. TFAE:

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For general Banach spaces

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Theorem (Bonsall-Duncan, 1970's; Rosenthal, 1984)

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- $\|\exp(\rho T)\| \leqslant 1$ for every $\rho \in \mathbb{R}$.
- $\{\exp(\rho T) \ : \ \rho \in \mathbb{R}_0^+\} \subset \operatorname{Iso}(X)$.
- T belongs to the tangent space to $\operatorname{Iso}(X)$ at Id.
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This follows from the exponential formula

$$\sup \operatorname{Re} V(T) = \lim_{\beta \downarrow 0} \frac{\|\operatorname{Id} + \beta\, T\| - 1}{\beta} = \sup_{\alpha > 0} \frac{\log \|\exp(\alpha\, T)\|}{\alpha}.$$

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Remark

If X is complex, there always exists exponential one-parameter semigroups of surjective isometries:

$$t \longmapsto \mathrm{e}^{it}\operatorname{Id} \qquad \text{generator: } i\operatorname{Id}.$$

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Main consequence

If X is a real Banach space such that

$$V(T) = \{0\} \quad \Longrightarrow \quad T = 0,$$

then $\operatorname{Iso}(X)$ is "small":

- it does not contain any exponential one-parameter semigroup,
- the tangent space of $\operatorname{Iso}(X)$ at Id is zero.

Isometries in finite-dimensional spaces

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Theorem

X finite-dimensional real space. TFAE:

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- $n(X) = 0$.
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$$\left\| x_0 + e^{i\theta} x_1 \right\| = \|x_0 + x_1\| \qquad (x_0 \in X_0,\ x_1 \in X_1,\ \theta \in \mathbb{R}).$$

(Note that the other 3 cases are included here)

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Question

Can every Banach space X with $n(X) = 0$ be decomposed as in 4 ?

Negative answer

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Infinite-dimensional case

There is an infinite-dimensional real Banach space X with $n(X) = 0$ but X is polyhedral. In particular, X does not contain $\mathbb{C}$ isometrically.

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An easy example is

$$X = \left[\bigoplus_{n \geqslant 2} X_n \right]_{c_0}$$

X_n is the two-dimensional space whose unit ball is the regular polygon of $2n$ vertices.

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Note

Such an example is not possible in the finite-dimensional case.

Quasi affirmative answer

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Finite-dimensional case

X finite-dimensional real space. TFAE:

- $n(X) = 0$.
- $X = X_0 \oplus X_1 \oplus \cdots \oplus X_n$ such that
  - X_0 is a (possible null) real space,
  - $X_1, \ldots, X_n$ are non-null complex spaces,

  there are $\rho_1, \ldots, \rho_n$ **rational** numbers, such that

$$\left\| x_0 + e^{i\rho_1\theta} x_1 + \cdots + e^{i\rho_n\theta} x_n \right\| = \left\| x_0 + x_1 + \cdots + x_n \right\|$$

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Remark

- The theorem is due to Rosenthal, but with real ρ's.
- The fact that the ρ's may be chosen as rational numbers is due to M.–Merí–Rodríguez-Palacios.

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- Use Kronecker's Approximation Theorem to change the eigenvalues of T^2 by rational numbers. ✓

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- But $\left\{ \exp\left(i \dfrac{2\pi k}{\alpha - 1} \right) \ : \ k \in \mathbb{Z} \right\}$ is dense in $\mathbb{T}$, so

$$\|x_0 + (x_1 + x_2)\| = \left\| x_0 + e^{i\rho} (x_1 + x_2) \right\| \quad \forall \rho \in \mathbb{R}$$

and $X = X_0 \oplus Z$ where $Z = X_1 \oplus X_2$ is a complex space

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X real space with $n(X) = 0$.

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- If $\dim(X) = 3$, then $X \equiv \mathbb{R} \oplus \mathbb{C}$ (absolute sum).

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Example

$X = (\mathbb{R}^4, \|\cdot\|)$, $\|(a,b,c,d)\| = \dfrac{1}{4}\displaystyle\int_0^{2\pi} \left|\operatorname{Re}\left(e^{2it}(a+ib) + e^{it}(c+id)\right)\right| \, dt.$

Then $n(X) = 0$ but the unique possible decomposition is $X = \mathbb{C} \oplus \mathbb{C}$ with

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Remark

- $\dim(X) = n \quad \Longrightarrow \quad \dim(\mathcal{Z}(X)) \leqslant \dfrac{n(n-1)}{2}$.
- Equality holds $\quad \Longleftrightarrow \quad$ H Hilbert space.

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An open problem

Given $n \geqslant 3$, which are the possible $\dim\big(\mathcal{Z}(X)\big)$ over all n-dimensional X's?

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- When X is finite-dimensional, $\mathrm{Iso}(X)$ is a Lie-group and $\mathcal{Z}(X)$ is the tangent space (i.e. its Lie-algebra).

Remark

- $\dim(X) = n \quad \Longrightarrow \quad \dim(\mathcal{Z}(X)) \leqslant \dfrac{n(n-1)}{2}$.
- Equality holds $\quad \Longleftrightarrow \quad$ H Hilbert space.

An open problem

Given $n \geqslant 3$, which are the possible $\dim\big(\mathcal{Z}(X)\big)$ over all n-dimensional X's?

Observation (Javier Merí, PhD)

When $\dim(X) = 3$, $\dim(\mathcal{Z}(X))$ cannot be 2.

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- If $\oplus \neq \oplus_2$, then isometries respect summands and $\dim(\mathcal{Z}(X)) = 1$. ✓

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Semigroups of surjective isometries and duality

Semigroups of surjective isometries and duality

Remark

X Banach space.

- $T \in \mathrm{Iso}(X) \implies T^* \in \mathrm{Iso}(X^*)$.
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The answer is yes. This is what we are going to present next.

Semigroups of surjective isometries and duality

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Spaces $C_E(K\|L)$

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- $C_0(K\|L)^\perp \equiv (C_E(K\|L)/C_0(K\|L))^* \equiv E^*$:
- $\Phi : C_E(K\|L) \longrightarrow E$, $\Phi(f) = f|_L$.
  - $\|\Phi\| \leqslant 1$ and $\ker \Phi = C_0(K\|L)$.
  - $\widetilde{\Phi} : C_E(K\|L)/C_0(K\|E) \longrightarrow E$ onto isometry:
  - $\{g \in E \,:\, \|g\| < 1\} \subseteq \Phi\big(\{f \in C_E(K\|L) \,:\, \|f\| < 1\}\big)$. ✓

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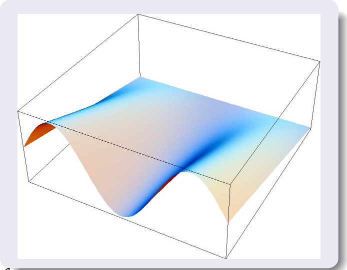

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- Therefore, we may choose $i \in \{1,2\}$ with $\big|[T(f_i)](\xi_0)\big| \sim \|T\|$, but now $|f_i(\xi_0)| = 1$.
- Equivalently, $\big|\delta_{\xi_0}\big(T(f_i)\big)\big| \sim \|T\|$ y $|\delta_{\xi_0}(f_i)| = 1$, so $v(T) \sim \|T\|$. ✓

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Consequence: the example

Take $K = [0,1]$, $L = \Delta$ (Cantor set), $E = \ell_2 \subset C(\Delta)$.

- $\mathrm{Iso}\big(C_{\ell_2}([0,1]\|\Delta)\big)$ has no exponential one-parameter semigroups.
- $C_{\ell_2}([0,1]\|\Delta)^* \equiv \ell_2 \oplus_1 C_0([0,1]\|\Delta)^*$, so taken $S \in \mathrm{Iso}(\ell_2)$

$$\Longrightarrow \quad T = \begin{pmatrix} S & 0 \\ 0 & \mathrm{Id} \end{pmatrix} \in \mathrm{Iso}\big(C_{\ell_2}([0,1]\|\Delta)^*\big)$$

Then, $\mathrm{Iso}\big(C_{\ell_2}([0,1]\|\Delta)^*\big)$ contains infinitely many exponential one-parameter semigroups.

Some comments

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- In $C_{\ell_2}([0,1]\|\Delta)$ there is no $A \in L(X)$ such that the solution to the linear dynamical system

$$x' = A\,x \qquad \big(x : \mathbb{R}_0^+ \longrightarrow C_{\ell_2}([0,1]\|\Delta)\big)$$

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- There are infinitely many such A's in $C_{\ell_2}([0,1]\|\Delta)^*$, in $C_{\ell_2}([0,1]\|\Delta)^{**}\ldots$

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Further results (Koszmider–M.–Merí., 2011)

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- There is X such that
$\operatorname{Iso}(X) = \{-\mathrm{Id}, \mathrm{Id}\}$ and $X^* = \ell_2 \oplus_1 L_1(\nu)$.
- Therefore, there is no semigroups in $\operatorname{Iso}(X)$, but there are infinitely many exponential one-parameter semigroups in $\operatorname{Iso}(X^*)$.

Numerical index of Banach spaces

V. Kadets, M. Martín, and R. Payá.
Recent progress and open questions on the numerical index of Banach spaces.
RACSAM (2006)

Numerical index of Banach spaces: definitions

Numerical radius

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Numerical index (Lumer, 1968)

X Banach space, the numerical index of X is

$$\begin{aligned} n(X) &= \inf\left\{v(T) \ : \ T \in L(X),\ \|T\| = 1\right\} \\ &= \max\left\{k \geqslant 0 \ : \ k\,\|T\| \leqslant v(T) \quad \forall\, T \in L(X)\right\} \\ &= \inf\left\{M \geqslant 0 \ : \ \exists T \in L(X), \|T\| = 1,\ \|\exp(\rho T)\| \leqslant e^{|\rho| M}\ \forall \rho \in \mathbb{R}\right\} \end{aligned}$$

Numerical index of Banach spaces: basic properties

Recalling some basic properties

Numerical index of Banach spaces: basic properties

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- Actually,

$$\{n(X) \ : \ X \text{ complex}, \ \dim(X) = 2\} = [\mathrm{e}^{-1}, 1]$$
$$\{n(X) \ : \ X \text{ real}, \ \dim(X) = 2\} = [0, 1]$$

(Duncan–McGregor–Pryce–White, 1970)

Numerical index of Banach spaces: examples (I)

Some examples

1. H Hilbert space, $\dim(H) > 1$,

$$\begin{array}{ll} n(H) = 0 & \text{if } H \text{ is real} \\ n(H) = 1/2 & \text{if } H \text{ is complex} \end{array}$$

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2. $n\big(L_1(\mu)\big) = 1$ $\quad \mu$ positive measure
   $n\big(C(K)\big) = 1$ $\quad K$ compact Hausdorff space

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3. If A is a C^*-algebra $\Rightarrow \begin{cases} n(A) = 1 & A \text{ commutative} \\ n(A) = 1/2 & A \text{ not commutative} \end{cases}$

(Huruya, 1977; Kaidi–Morales–Rodríguez, 2000)

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4. If A is a function algebra $\Rightarrow$ $n(A) = 1$

   (Werner, 1997)

Numerical index of Banach spaces: some examples (II)

More examples

5. For $n \geqslant 2$, the unit ball of X_n is a $2n$ regular polygon:

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6. Every finite-codimensional subspace of $C[0,1]$ has numerical index 1

(Boyko–Kadets–M.–Werner, 2007)

Numerical index of Banach spaces: some examples (III)

Even more examples

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   - $n\big(\ell_p^{(2)}\big)$ **?**
   - In the real case,

$$\max\left\{\frac{1}{2^{1/p}},\ \frac{1}{2^{1/q}}\right\} M_p \leqslant n\big(\ell_p^{(2)}\big) \leqslant M_p$$

and $M_p = v\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \max\limits_{t\in[0,1]} \dfrac{|t^{p-1} - t|}{1 + t^p}$

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   - In the real case, $n\big(L_p(\mu)\big) \geqslant \dfrac{M_p}{6\, p^{\frac{1}{p}}\, q^{\frac{1}{q}}}$.

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Numerical index: open problems on computing

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4. Compute the numerical index of real C^*-algebras.
5. Compute the numerical index of more classical Banach spaces: $C^m[0,1]$, Lip(K), Lorentz spaces, Orlicz spaces. . .

Stability properties

Direct sums of Banach spaces (M.–Payá, 2000)

$$n\Big([\oplus_{\lambda\in\Lambda} X_\lambda]_{c_0}\Big) = n\Big([\oplus_{\lambda\in\Lambda} X_\lambda]_{\ell_1}\Big) = n\Big([\oplus_{\lambda\in\Lambda} X_\lambda]_{\ell_\infty}\Big) = \inf_\lambda n(X_\lambda)$$

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Consequences

- There is a real Banach space X such that

$$v(T) > 0 \quad \text{when } T \neq 0,$$

  but $n(X) = 0$
  (i.e. $v(\cdot)$ is a norm on $L(X)$ which is not equivalent to the operator norm).

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  (i.e. $v(\cdot)$ is a norm on $L(X)$ which is not equivalent to the operator norm).
- For every $t \in [0,1]$, there exist a real X_t isomorphic to c_0 (or ℓ_1 or ℓ_∞) with $n(X_t) = t$.
- For every $t \in [\mathrm{e}^{-1},1]$, there exist a complex Y_t isomorphic to c_0 (or ℓ_1 or ℓ_∞) with $n(Y_t) = t$.

Stability properties (II)

Vector-valued function spaces (López-M.-Merí-Payá-Villena, 2000's)

E Banach space, μ positive σ-finite measure, K compact space. Then

$$n\big(C(K,E)\big) = n\big(C_w(K,E)\big) = n\big(L_1(\mu,E)\big) = n\big(L_\infty(\mu,E)\big) = n(E),$$

and $n\big(C_{w^*}(K,E^*)\big) \leqslant n(E)$

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Tensor products (Lima, 1980)

There is no general formula for $n\big(X\widetilde{\otimes}_\varepsilon Y\big)$ nor for $n\big(X\widetilde{\otimes}_\pi Y\big)$:

- $n\big(\ell_1^{(4)}\widetilde{\otimes}_\pi \ell_1^{(4)}\big) = n\big(\ell_\infty^{(4)}\widetilde{\otimes}_\varepsilon \ell_\infty^{(4)}\big) = 1.$
- $n\big(\ell_1^{(4)}\widetilde{\otimes}_\varepsilon \ell_1^{(4)}\big) = n\big(\ell_\infty^{(4)}\widetilde{\otimes}_\pi \ell_\infty^{(4)}\big) < 1.$

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L_p-spaces (Askoy–Ed-Dari–Khamsi, 2007)

$$n\big(L_p([0,1],E)\big) = n\big(\ell_p(E)\big) = \lim_{m\to\infty} n\big(E \oplus_p \overset{m}{\cdots} \oplus_p E\big).$$

Numerical index and duality

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Proposition

X Banach space, $T \in L(X)$. Then

- $\sup \operatorname{Re} V(T) = \lim\limits_{\alpha \to 0^+} \dfrac{\|\mathrm{Id} + \alpha T\| - 1}{\alpha}$.

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Negative answer (Boyko–Kadets–M.–Werner, 2007)

Consider the space

$$X = \big\{(x,y,z) \in c \oplus_\infty c \oplus_\infty c \ : \ \lim x + \lim y + \lim z = 0\big\}.$$

Then, $n(X) = 1$ but $n(X^*) < 1$.

Numerical index and duality. Proof of main example

$$X = \{(x,y,z) \in c \oplus_\infty c \oplus_\infty c \ : \ \lim x + \lim y + \lim z = 0\}:$$

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$$X^* \equiv \ell_1 \oplus_1 \ell_1 \oplus_1 \ell_1 \oplus_1 Z, \qquad X^{**} \equiv \ell_\infty \oplus_\infty \ell_\infty \oplus_\infty \ell_\infty \oplus_\infty Z^*.$$

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- $A = \{(e_n,0,0,0) \ : \ n \in \mathbb{N}\} \cup \{(0,e_n,0,0) \ : \ n \in \mathbb{N}\} \cup \{(0,0,e_n,0) \ : \ n \in \mathbb{N}\} \subset X^*$.

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- Then $B_{X^*} = \overline{\text{aco}}^{w^*}(A)$ and
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- Then we find $x^{**} \in \text{ext}(B_{X^{**}})$ such that
  $$|x^{**}(T^*(a))| = \|T^*(a)\| > \|T^*\| - \varepsilon.$$
- Since $|x^{**}(a)| = 1$, this gives that $v(T^*) > \|T^*\| - \varepsilon$, so $v(T) = \|T\|$ and $n(X) = 1$. ✓

Numerical index and duality. Proof of main example

$$X = \big\{(x,y,z) \in c \oplus_\infty c \oplus_\infty c \ : \ \lim x + \lim y + \lim z = 0\big\}:$$
$$n(X) = 1 \qquad \text{but} \qquad n(X^*) < 1.$$

Proof

- $c^* = \ell_1 \oplus_1 \mathbb{K}\lim \implies X^* = \big[c^* \oplus_1 c^* \oplus_1 c^*\big]/(\lim,\lim,\lim)$.
- Then, writing $Z = \ell_1^{(3)}/(1,1,1)$, we can identify
$$X^* \equiv \ell_1 \oplus_1 \ell_1 \oplus_1 \ell_1 \oplus_1 Z, \qquad X^{**} \equiv \ell_\infty \oplus_\infty \ell_\infty \oplus_\infty \ell_\infty \oplus_\infty Z^*.$$
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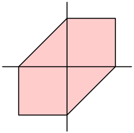

Figure: B_Z

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Example

$X = C_{K(\ell_2)}([0,1]\|\Delta)$. Then $n(X) = 1$ and

$$X^* \equiv K(\ell_2)^* \oplus_1 C_0(K\|\Delta)^* \qquad \text{and} \qquad X^{**} \equiv L(\ell_2) \oplus_\infty C_0(K\|\Delta)^{**}.$$

Therefore, X^{**} is a C^*-algebra, but $n(X^*) = 1/2 < n(X) = 1$.

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★ What about the value $\mathbf{1}$ **?**

Banach spaces with numerical index one

Numerical index 1

Recall that X has numerical index one ($n(X) = 1$) iff

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For Hilbert spaces, the above formula is equivalent to

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Examples

$C(K)$, $L_1(\mu)$, $A(\mathbb{D})$, H^∞, finite-codimensional subspaces of $C[0,1]$...

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A very recent result (Avilés–Kadets–M.–Merí–Shepelska)

If X is real, $\dim(X) = \infty$ and $n(X) = 1$, then $X^* \supset \ell_1$.

More details on this later on.

Proving the 1999 results (I)

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$$\begin{aligned} 1 - |x_0^*(x_0)| &\leqslant |tx^*(sx) - x_0^*(x_0)| \leqslant \\ &\leqslant |tx^*(sx) - tx^*(x_0)| + |tx^*(x_0) - x_0^*(x_0)| < 2\varepsilon. \checkmark \end{aligned}$$

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- (Krein-Milman theorem): every $y^* \in \operatorname{ext}(B_{Y^*})$ has an extension which belongs to $\operatorname{ext}(B_{X^*})$.

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Proposition

X real, $A \subset S_X$ infinite with $|x^*(a)| = 1$ $\forall x^* \in \operatorname{ext}(B_{X^*})$, $\forall a \in A$.
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- $\{a_n\}$ separates points of Y^* $\Longrightarrow$ E_k finite, so $\operatorname{ext}(B_{Y^*})$ countable.
- (Fonf): $Y \supseteq c_0$. So, $X \supseteq c_0$. ✓

Proving the 1999 results (III)

Lemma

X Banach space, $n(X) = 1$
$\Longrightarrow$ $|x_0^*(x_0)| = 1$ for all $x_0^* \in \operatorname{ext}(B_{X^*})$ and all denting point x_0 of B_X.

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X real, RNP, $\dim(X) = \infty$, and $n(X) = 1 \Longrightarrow X \supseteq \ell_1$.

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X Banach space, $n(X)=1$
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- Therefore, $X\supseteq c_0$ or $X\supseteq \ell_1$.
- If X RNP, then $X\not\supseteq c_0$. ✓

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Corollary

X real, $\dim(X) = \infty$, $n(X) = 1$.

- X is not reflexive.
- X^{**}/X is non-separable.

Isomorphic properties (positive results)

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A renorming result (Boyko–Kadets–M.–Merí, 2009)

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Negative result (Bourgain–Delbaen, 1980)

There is X such that $X^* \simeq \ell_1$ and X has the RNP. Then, X can not be renormed with numerical index 1 (in such a case, $X \supset \ell_1$ **!**)

Isometric properties: finite-dimensional spaces

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Finite-dimensional spaces (McGregor, 1971; Lima, 1978)

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Question

What is the situation in the infinite-dimensional case **?**

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- But, since $T_n \longrightarrow T$ and $T^2 = 0$, then $\left[T_n^{**}\right]^2 \longrightarrow 0$ **!!** ✓

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$X = C(\mathbb{T})/A(\mathbb{D})$. $X^* = H^1$ is smooth $\implies$ $n(X) < 1$ & $n(H^1) < 1$.

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Example without completeness

- There is X (non-complete) strictly convex with $X^* \equiv L_1(\mu)$, so $n(X) = 1$.
- $\widetilde{X}$ completion of X. For $F \subseteq S_{\widetilde{X}}$ maximal face, $B_{\widetilde{X}} = \overline{\text{aconv}}(F)$.

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Open question

Is there X with $n(X) = 1$ which is smooth or strictly convex **?**

Asymptotic behavior of the set of spaces with numerical index one

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Theorem (Oikhberg, 2005)

There is a universal constant c such that

$$\operatorname{dist}\big(X, \ell_2^{(m)}\big) \geqslant c\; m^{\frac{1}{4}}$$

for every $m \in \mathbb{N}$ and every m-dimensional X with $n(X) = 1$.

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Old examples

$$\operatorname{dist}\big(\ell_1^{(m)}, \ell_2^{(m)}\big) = \operatorname{dist}\big(\ell_\infty^{(m)}, \ell_2^{(m)}\big) = m^{\frac{1}{2}}$$

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$$\operatorname{dist}\big(X, \ell_2^{(m)}\big) \geqslant c\, m^{\frac{1}{4}}$$

for every $m \in \mathbb{N}$ and every m-dimensional X with $n(X) = 1$.

Old examples

$$\operatorname{dist}\big(\ell_1^{(m)}, \ell_2^{(m)}\big) = \operatorname{dist}\big(\ell_\infty^{(m)}, \ell_2^{(m)}\big) = m^{\frac{1}{2}}$$

Open questions

- Is there a universal constant $\widetilde{c}$ such that

  $$\operatorname{dist}\big(X, \ell_2^{(m)}\big) \geqslant \widetilde{c}\, m^{\frac{1}{2}}$$

  for every $m \in \mathbb{N}$ and every m-dimensional X's with $n(X) = 1$ **?**
- What is the diameter of the set of all m-dimensional X's with $n(X) = 1$ **?**

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- The weakest property is called lushness.

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Relationship between the properties

- One of the key ideas to get interesting results for Banach spaces with numerical index 1 is to study when the three properties below are equivalent.

lushness $\underset{\not\Longleftarrow}{\Longrightarrow}$ Numerical index 1 $\underset{\not\Longleftarrow}{\Longrightarrow}$ ADP

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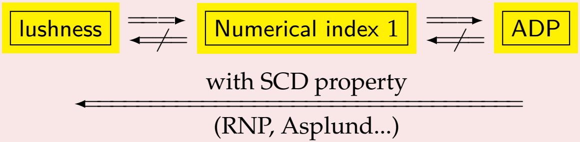

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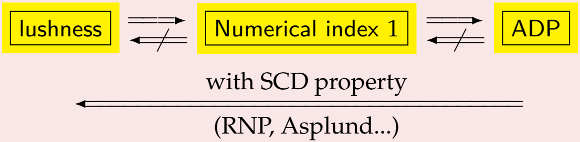

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The oldest open problem

Calculate the numerical index of "classical" spaces.

- In particular, calculate $n(L_p(\mu))$.

The alternative Daugavet property

M. Martín and T. Oikberg
An alternative Daugavet property
J. Math. Anal. Appl. (2004)

M. Martín
The alternative Daugavet property of C^*-algebras and JB^*-triples
Math. Nachr. (2008)

The Daugavet property: motivation

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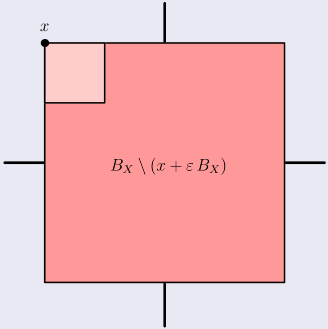
- $x \in S_X$ is a denting point of B_X if for every $\varepsilon > 0$ one has

$$x \notin \overline{\text{co}}\big(B_X \setminus (x + \varepsilon B_X)\big).$$



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- $C[0,1]$ and $L_1[0,1]$ have an extremely opposite property: for every $x \in S_X$ and every $\varepsilon > 0$

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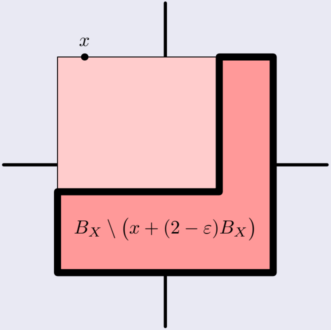

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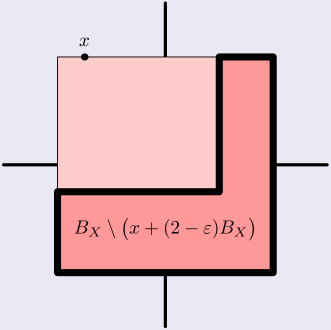
- This geometric property is equivalent to a property of operators on the space.



The Daugavet property: definition

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Classical examples

1. **Daugavet, 1963:**
   Every compact operator on $C[0,1]$ satisfies (DE).
2. **Lozanoskii, 1966:**
   Every compact operator on $L_1[0,1]$ satisfies (DE).
3. **Abramovich, Holub, and more, 80's:**
   $X = C(K)$, K perfect compact space
   or $X = L_1(\mu)$, μ atomless measure
   $\Longrightarrow$ every weakly compact $T \in L(X)$ satisfies (DE).

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The Daugavet property

A Banach space X is said to have the Daugavet property iff every rank-one operator on X satisfies (DE).

★ Then, every weakly compact operator on X satisfies (DE).

(Kadets–Shvidkoy–Sirotkin–Werner, 1997 & 2000)

The Daugavet property: geometric characterizations

Theorem [KSSW]

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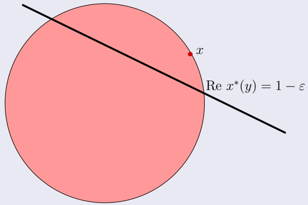

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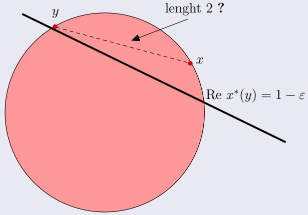

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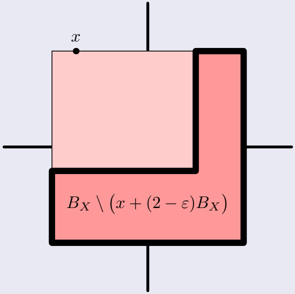

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The DPr, the ADP and numerical index 1

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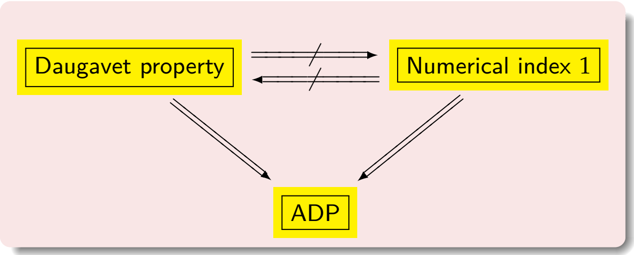

Examples

- $C\big([0,1],K(\ell_2)\big)$ has DPr, but has not numerical index 1
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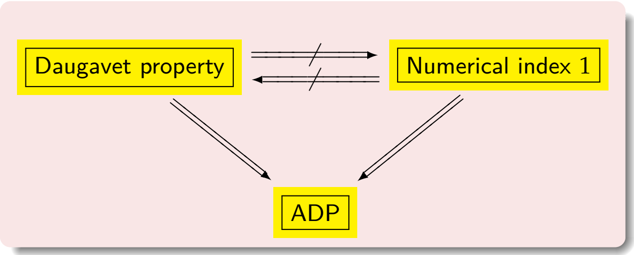

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Remarks

- For RNP or Asplund spaces, $\boxed{\text{ADP}} \Longrightarrow \boxed{\text{numerical index 1}}$.
- Every Banach space with the ADP can be renormed still having the ADP but failing the Daugavet property.

Geometric characterizations of the ADP

Theorem

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Every rank-one operator $T \in L(X)$ (equivalently, every weakly compact operator) satisfies

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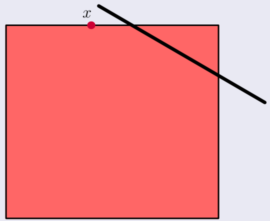

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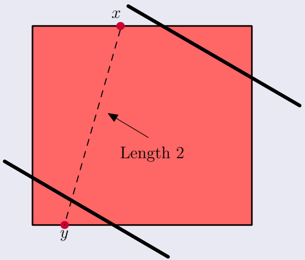

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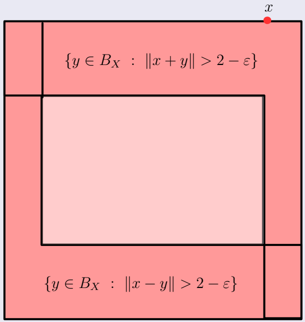

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V_* has numerical index 1 iff:

- V is commutative, or
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Let V_* be the predual of the von Neumann algebra V.

The Daugavet property of V_* is equivalent to:

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- $|v(v_*)| = 1$ for $v \in \operatorname{ext}(B_V)$ and $v_* \in \operatorname{ext}(B_{V_*})$, or
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- $\exists$ a commutative ideal Y such that X/Y has the Daugavet property.

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If X is separable, $X \supset c_0$, then X can be renormed with the ADP.

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Open question

Is there X with the ADP which is smooth or strictly convex **?**

Lush spaces

6 Lush spaces

- Definition and examples
- Lush renorming
- Reformulations of lushness and applications
- Lushness is not equivalent to numerical index one

K. Boyko, V. Kadets, M. Martín, and J. Merí.
Properties of lush spaces and applications to Banach spaces with numerical index 1.
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K. Boyko, V. Kadets, M. Martín, and D. Werner.
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V. Kadets, M. Martín, J. Merí, and R. Payá.
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Illinois J. Math. (to appear).

V. Kadets, M. Martín, J. Merí, and V. Shepelska.
Lushness, numerical index one and duality.
J. Math. Anal. Appl. (2009).

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X is lush if given $x, y \in S_X$, $\varepsilon > 0$, there is $x^* \in S_{X^*}$ such that

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- Then $\max_{\omega \in \mathbb{T}} \|\mathrm{Id} + \omega\, T\| \sim 1 + \|T\| \implies v(T) \sim \|T\|$. ✓

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When we may get a lush equivalent norm?

Proposition

X s
T is

Open problems

- Find more sufficient conditions to get equivalent lush norms.

Re

7. Y if $c_0 \subseteq Y \subseteq \ell_\infty$ (canonical copies).

Theorem

X separable, $X \supseteq c_0 \implies X$ admits an equivalent lush norm.

Corollary

Every closed subspace of c_0 admits an equivalent lush norm.

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- When $X \supseteq \ell_\infty$ **?**

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Even more examples of lush spaces

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Observation

X Banach space. Consider the following assertions.

(a) Exists $A \subset B_{X^*}$ norming, $|x^{**}(a^*)| = 1$ $\forall a^* \in A$ and $\forall x^{**} \in \operatorname{ext}(B_{X^{**}})$.

(b) For $x \in S_X$ and $\varepsilon > 0$, exists $x^* \in S_{X^*}$ such that

$$x \in S(B_X, x^*, \varepsilon) \qquad \text{and} \qquad B_X = \overline{\operatorname{aconv}}\big(S(B_X, x^*, \varepsilon)\big).$$

$$\boxed{\text{(a)}} \Longrightarrow \boxed{\text{(b)}} \Longrightarrow \boxed{\text{lushness}}$$

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Definition (Werner, 1997)

X is nicely embedded in $C_b(\Omega)$ if exists $J : X \longrightarrow C_b(\Omega)$ linear isometry with

(N1) $\|J^*\delta_s\| = 1$ $\forall s \in \Omega$,

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Even more examples of lush spaces

8. Nicely embedded Banach spaces (they fulfil (a)).
9. In particular, function algebras (as $A(\mathbb{D})$ and H^∞).

Some reformulations of lushness

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- X is lush,
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- There is $G \subseteq S_{X^*}$ norming such that

$$B_X = \overline{\operatorname{aconv}}(S(B_X, x^*, \varepsilon)) \qquad (\varepsilon > 0,\ \ x^* \in G).$$

- Therefore, $|x^{**}(x^*)| = 1$ for every $x^{**} \in \operatorname{ext}(B_{X^{**}})$ and every $x^* \in G$.
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We almost returned to the almost-CL-space definition **!!**

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Consequence

$X \subseteq C[0,1]$ strictly convex or smooth $\implies$ $C[0,1]/X$ contains $C[0,1]$.

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X lush separable, $\dim(X) = \infty \implies$ there is $G \in S_{X^*}$ infinite such that

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- There is $E\subseteq X$ separable and lush.
- Then $E^*\supseteq c_0$ or $E^*\supseteq \ell_1 \implies E^*\supseteq \ell_1$.
- By "lifting" property of $\ell_1 \implies X^*\supseteq \ell_1$. ✓

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What happens if just $n(X) = 1$ **?** The same, we will prove later.

Lushness is not equivalent to numerical index one

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Slicely countably determined spaces

7. Slicely countably determined spaces
   - Slicely Countably Determined sets and spaces
   - Applications to numerical index 1 spaces
   - SCD operators
   - Open questions

A. Avilés, V. Kadets, M. Martín, J. Merí, and V. Shepelska
Slicely Countably Determined Banach spaces
Trans. Amer. Math. Soc. (2010)

SCD sets: Definitions and preliminary remarks

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X Banach space, $A \subset X$ bounded and convex.

SCD sets

A is Slicely Countably Determined (SCD) if there is a sequence $\{S_n : n \in \mathbb{N}\}$ of slices of A satisfying one of the following equivalent conditions:

SCD sets: Definitions and preliminary remarks

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Remarks

- A is SCD iff $\overline{A}$ is SCD.
- If A is SCD, then it is separable.

SCD sets: Elementary examples I

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Example

A separable and $A = \overline{\mathrm{conv}}(\mathsf{dent}(A)) \Longrightarrow A$ is SCD.

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In particular, A RNP separable $\implies A$ SCD.

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Corollary

- If X is separable LUR $\implies B_X$ is SCD.
- So, every separable space can be renormed such that $B_{(X,|\cdot|)}$ is SCD.

SCD sets: Elementary examples II

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Negative example

If X has the Daugavet property $\Longrightarrow$ B_X is not SCD.
Therefore, $B_{C[0,1]}$, $B_{L_1[0,1]}$ are not SCD.

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- For every $n, m \in \mathbb{N}$, consider $S_{n,m} = S(A, x_n^*, 1/m)$.
- It is easy to show that any slice of A contains one of the $S_{n,m}$. ✓

Negative example

If X has the Daugavet property $\Longrightarrow$ B_X is not SCD.
Therefore, $B_{C[0,1]}$, $B_{L_1[0,1]}$ are not SCD.

Proof.

- Fix $x_0 \in B_X$ and $\{S_n\}$ sequence of slices of B_X.

SCD sets: Elementary examples II

Example

If X^* is separable $\Longrightarrow$ A is SCD.

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Proof.

- Fix $x_0 \in B_X$ and $\{S_n\}$ sequence of slices of B_X.
- By [KSSW] there is a sequence $(x_n) \subset B_X$ such that
  - $x_n \in S_n$ for every $n \in \mathbb{N}$,
  - $(x_n)_{n \geqslant 0}$ is equivalent to the basis of ℓ_1,
  - so $x_0 \notin \overline{\text{lin}}\{x_n : n \in \mathbb{N}\}$. ✓

SCD sets: Further examples I

SCD sets: Further examples I

Convex combination of slices

$W = \sum_{k=1}^{m} \lambda_k S_k \subset A$ where $\lambda_k \geqslant 0$, $\sum \lambda_k = 1$, S_k slices.

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In the definition of SCD we can use a sequence $\{S_n : n \in \mathbb{N}\}$ of convex combination of slices.

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A has small combinations of slices iff every slice of A contains convex combinations of slices of A with arbitrary small diameter.

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Particular case

A strongly regular + separable $\Longrightarrow$ A is SCD.

SCD sets: Further examples II

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Bourgain's lemma

Every relative weak open subset of A contains a convex combination of slices.

SCD sets: Further examples II

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Corollary

In the definition of SCD we can use a sequence $\{S_n : n \in \mathbb{N}\}$ of relative weak open subsets.

SCD sets: Further examples II

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In the definition of SCD we can use a sequence $\{S_n : n \in \mathbb{N}\}$ of relative weak open subsets.

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A π-base of the weak topology of A is a family $\{V_i : i \in I\}$ of weak open sets of A such that every weak open subset of A contains one of the V_i's.

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Proposition

If $(A, \sigma(X, X^*))$ has a countable π-base $\Longrightarrow$ A is SCD.

SCD sets: Further examples III

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Theorem

A separable without ℓ_1-sequences $\Longrightarrow (A, \sigma(X, X^*))$ has a countable π-base.

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Remark

- Every subspace of a SCD space is SCD.
- This is false for quotients.

SCD spaces: stability properties

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Theorem

$Z \subset X$. If Z and X/Z are SCD $\Longrightarrow X$ is SCD.

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Corollary

$X_1, \ldots, X_m$ SCD $\Longrightarrow X_1 \oplus \cdots \oplus X_m$ SCD.

SCD spaces: stability properties II

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$X_1, X_2, \ldots$ SCD, E with unconditional basis.

- $E \not\supset c_0 \implies [\bigoplus_{n\in\mathbb{N}} X_n]_E$ SCD.
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Examples

1. $c_0(\ell_1)$ and $\ell_1(c_0)$ are SCD.
2. $c_0 \otimes_\varepsilon c_0$, $c_0 \otimes_\pi c_0$, $c_0 \otimes_\varepsilon \ell_1$, $c_0 \otimes_\pi \ell_1$, $\ell_1 \otimes_\varepsilon \ell_1$, and $\ell_1 \otimes_\pi \ell_1$ are SCD.
3. $K(c_0)$ and $K(c_0, \ell_1)$ are SCD.
4. $\ell_2 \otimes_\varepsilon \ell_2 \equiv K(\ell_2)$ and $\ell_2 \oplus_\pi \ell_2 \equiv \mathcal{L}_1(\ell_2)$ are SCD

The DPr, the ADP and numerical index 1

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Recalling the properties

1. **Kadets-Shvidkoy-Sirotkin-Werner, 1997:**
   X has the Daugavet property (DPr) if

   $$\|\mathrm{Id} + T\| = 1 + \|T\| \qquad \text{(DE)}$$

   for every rank-one $T \in L(X)$.
   ★Then every weakly compact T also satisfies (DE).

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   ★Equivalently, $v(T) = \|T\|$ for EVERY $T \in L(X)$.

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3. **M.-Oikhberg, 2004:** X has the alternative Daugavet property (ADP) if every rank-one $T \in L(X)$ satisfies (aDE).
   ★Then every weakly compact T also satisfies (aDE).

Relations between these properties

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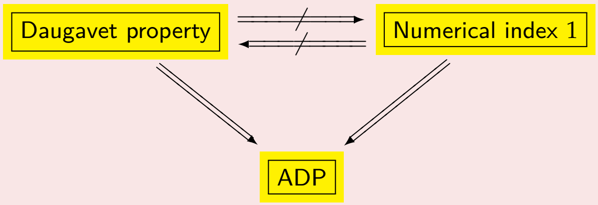

Examples

- $C([0,1],K(\ell_2))$ has DPr, but has not numerical index 1
- c_0 has numerical index 1, but has not DPr
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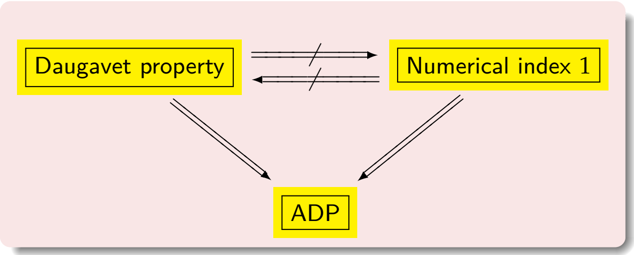

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Remarks

- For RNP or Asplund spaces, $\boxed{\text{ADP}} \Longrightarrow \boxed{\text{numerical index 1}}$.
- Every Banach space with the ADP can be renormed still having the ADP but failing the Daugavet property.

ADP + SCD $\Longrightarrow$ numerical index 1

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Characterizations of the ADP

X Banach space. TFAE:

- X has ADP (i.e. $\max_{\theta\in\mathbb{T}} \|\mathrm{Id} + \theta\, T\| = 1 + \|T\|$ for all T rank-one).

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Theorem

X ADP + B_X SCD $\Longrightarrow$ given $x \in S_X$ and $\varepsilon > 0$, there is $y^* \in S_{X^*}$ such that
$$x \in S(B_X, y^*, \varepsilon) \quad \text{and} \quad B_X = \overline{\mathrm{conv}}\big(\mathbb{T}\, S(B_X, y^*, \varepsilon)\big).$$

★ This implies lushness and so, numerical index 1.

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Open question

X real, $\dim(X) = \infty$, $n(X) = 1$ $\Longrightarrow$ $X \supset c_0$ or $X \supset \ell_1$ **?**

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Examples

T is an SCD-operator when $T(B_X)$ is separable and

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Main corollary

X ADP + T does not fix copies of ℓ_1 $\implies \max_{\theta\in\mathbb{T}} \|\mathrm{Id} + \theta\, T\| = 1 + \|T\|$.

SCD operators

SCD operator

$T \in L(X)$ is an SCD-operator if $T(B_X)$ is an SCD-set.

Examples

T is an SCD-operator when $T(B_X)$ is separable and

1. $T(B_X)$ is RPN,
2. $T(B_X)$ has no ℓ_1 sequences,
3. T does not fix copies of ℓ_1

Theorem

- X ADP + T SCD-operator $\implies \max_{\theta \in \mathbb{T}} \|\mathrm{Id} + \theta T\| = 1 + \|T\|$.
- X DPr + T

Remark

Separability is not needed !

Main corollary

X ADP + T does not fix copies of $\ell_1 \implies \max_{\theta \in \mathbb{T}} \|\mathrm{Id} + \theta T\| = 1 + \|T\|$.

Open questions

On SCD-sets

- Find more sufficient conditions for a set to be SCD.
- For instance, if X has 1-symmetric basis, is B_X an SCD-set **?**
- Is SCD equivalent to the existence of a countable π-base for the weak topology **?**

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- Find more sufficient conditions for a set to be SCD.
- For instance, if X has 1-symmetric basis, is B_X an SCD-set **?**
- Is SCD equivalent to the existence of a countable π-base for the weak topology **?**

On SCD-spaces

- E with unconditional basis. Is E SCD **?**
- X, Y SCD. Are $X \otimes_\varepsilon Y$ and $X \otimes_\pi Y$ SCD **?**

On the containment of c_0 or ℓ_1

8 Remarks on the containment of c_0 and ℓ_1

A. Avilés, V. Kadets, M. Martín, J. Merí, and V. Shepelska.
Slicely countably determined Banach spaces.
Trans. Amer. Math. Soc. (2010).

V. Kadets, M. Martín, J. Merí, and R. Payá.
Smoothness and convexity for Banach spaces with numerical index 1.
Illinois J. Math. (2009).

Containment of c_0 or ℓ_1

Open question (Godefroy, private communication)

X real, $\dim(X) = \infty$, $n(X) = 1 \implies X \supset c_0$ or $X \supset \ell_1$ **?**

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- (KMMP 2009): In the separable case, lushness implies $|x^{**}(x^*)|=1$ for every $x^{**}\in \operatorname{ext}(B_{X^{**}})$ and every $x^*\in G$, G norming for X.

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- (LMP 1999): This gives $X^*\supseteq c_0$ or $X^*\supseteq \ell_1 \implies X^*\supseteq \ell_1$ ✓

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★ Equivalent reformulation of the problem:

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★ Equivalent reformulation of the problem:

Equivalent open problem

X real separable, $X \not\supset \ell_1$, exists $G \subseteq S_{X^*}$ norming with

$$B_X = \overline{\operatorname{aconv}}\left(\{x \in B_X \ : \ x^*(x) = 1\}\right) \qquad (x^* \in G).$$

Does $X \supseteq c_0$ **?**

Numerical index of L_p-spaces

9 Numerical index of L_p-spaces
- The 2000's results on the numerical index on L_p-spaces
- The new results on the numerical index of L_p-spaces

M. Martín, and J. Merí.
A note on the numerical index of the L_p-space of dimension two.
Linear Mult. Algebra (2009)

M. Martín, J. Merí, and M. Popov.
On the numerical index of real $L_p(\mu)$-spaces.
Israel J. Math. (2011)

M. Martín, J. Merí, and M. Popov.
On the numerical radius of opearators on Lebesgue spaces.
J. Funct. Anal. (2011)

M. Martín, J. Merí, M. Popov, and B. Randrianantoanina.
Numerical index of absolute sums of Banach spaces.
J. Math. Anal. Appl. (2011)

Numerical index of L_P-spaces

Known results on the numerical index of L_p-spaces

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1. $n(\ell_p) \leqslant n\big(\ell_p^{(m+1)}\big) \leqslant n\big(\ell_p^{(m)}\big)$ for $m \in \mathbb{N}$.
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3. In the real case,

$$\max\left\{\frac{1}{2^{1/p}}, \frac{1}{2^{1/q}}\right\} v\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \leqslant n\big(\ell_p^{(2)}\big) \leqslant v\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

   and $v\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \max\limits_{t\in[0,1]} \dfrac{|t^{p-1} - t|}{1 + t^p}$

   (M.–Merí, 2009)

Ideas behind the proofs I

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The numerical index decreases with the dimension

$$n(\ell_p) \leqslant n\big(\ell_p^{(m+1)}\big) \leqslant n\big(\ell_p^{(m)}\big) \text{ for } m \in \mathbb{N}.$$

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Proposition (M.–Payá, 2000)

$Z = U \oplus V$ with absolute sum (i.e. $\|u+v\| = f(\|u\|, \|v\|)$ for $u \in U,\ v \in V$).
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Proof of the decreasing

- $\ell_p^{(m)}$ is an absolute summand in both $\ell_p^{(m+1)}$ and in ℓ_p.

Ideas behind the proofs II

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One inequality

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- $E \equiv E(X)$ so $n(E) \leqslant n(\ell_p^{(m)})$.

Ideas behind the proofs III

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The reversed inequality

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Ideas behind the proofs III

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Proposition (M.–Merí–Popov–Randrianantoanina, 2011)

Z Banach space, $\{Z_i\}_{i\in I}$ increasing family of one-complemented subspaces whose union is dense. Then, $\Longrightarrow n(Z) \geqslant \limsup_{i\in I} n(Z_i)$.

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Corollary

Z Banach space with monotone basis (e_m), $Z_m = \text{span}\{e_k : 1 \leqslant k \leqslant m\}$.
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- $E = L_p[0,1]$, (e_m) Haar system $\Longrightarrow$ $Z_m \equiv \ell_p^{(m)}$ for $m = 2^k$ ($k \in \mathbb{N}$).

Ideas behind the proofs IV

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The two-dimensional case

In the real case,

$$\max\left\{\frac{1}{2^{1/p}},\ \frac{1}{2^{1/q}}\right\} M_p \leqslant n\left(\ell_p^{(2)}\right) \leqslant M_p \quad \text{where} \quad M_p = \max_{t\in[0,1]} \frac{|t^{p-1}-t|}{1+t^p}$$

Ideas behind the proofs IV

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Proposition (Duncan-McGregor-Pryce-White, 1970)

$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ operator in $\ell_p^{(2)}$. Then

$$v(T) = \max\left\{\max_{t\in[0,1]} \frac{|a+d\,t^p| + |b\,t + c\,t^{p-1}|}{1+t^p},\ \max_{t\in[0,1]} \frac{|d+a\,t^p| + |c\,t + b\,t^{p-1}|}{1+t^p}\right\}.$$

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Proof of the result

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$$v(T) = \max\left\{ \max_{t\in[0,1]} \frac{|a+d\,t^p| + \left|b\,t + c\,t^{p-1}\right|}{1+t^p},\ \max_{t\in[0,1]} \frac{|d+a\,t^p| + \left|c\,t + b\,t^{p-1}\right|}{1+t^p} \right\}.$$

Proof of the result

- $n(\ell_p^{(2)}) \leqslant M_p$ since $\left\| \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\| = 1$ and $v\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = M_p$.

Ideas behind the proofs IV

The two-dimensional case

In the real case,

$$\max\left\{\frac{1}{2^{1/p}},\ \frac{1}{2^{1/q}}\right\} M_p \leqslant n\big(\ell_p^{(2)}\big) \leqslant M_p \quad \text{where} \quad M_p = \max_{t\in[0,1]} \frac{|t^{p-1}-t|}{1+t^p}$$

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- We compare $v(T)$ with M_p, but we use $\|T\|_1$ and $\|T\|_\infty$ instead of $\|T\|_p$.

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The 2010's results

- We left the finite-dimensional approach and introduce the absolute numerical radius.

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The 2010's results

- We left the finite-dimensional approach and introduce the absolute numerical radius.
- This allows to show that $n(L_p[0,1]) > 0$ in the real case.

The absolute numerical radius in L_p

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The numerical radius in L_p

- For $x \in L_p(\mu)$, write $x^\# = |x|^{p-1}\operatorname{sign}(\overline{x})$.
- It is the unique element in $L_q(\mu)$ such that

$$\|x\|_p^p = \|x^\#\|_q^q \quad \text{and} \quad \int x\, x^\#\, d\mu = \|x\|_p\, \|x^\#\|_q = \|x\|_p^p.$$

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$$\begin{aligned} v(T) &= \sup\left\{ \left| \int x^\# T x \, d\mu \right| \,:\, x \in L_p(\mu),\ \|x\|_p = 1 \right\} \\ &= \sup\left\{ \left| \int |x|^{p-1}\,\mathrm{sign}(\overline{x})\, T x \, d\mu \right| \,:\, x \in L_p(\mu),\ \|x\|_p = 1 \right\} \end{aligned}$$

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Absolute numerical radius

For $T \in L\big(L_p(\mu)\big)$,
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Obvious remark

$$v(T) \leqslant |v|(T) \leqslant \|T\| \text{ for every } T \in L\big(L_p(\mu)\big).$$

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Giving an estimation of $n(L_p(\mu))$

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Roadmap

We would like to give an estimation of $n(L_p(\mu))$ in two steps:

- First, we study the relationship between $v(T)$ and $|v|(T)$ for all operators T.
- Second, we study the relationship between $|v|(T)$ and $\|T\|$ for all operators T. Here, we actually calculate $|n|(L_p(\mu))$.

Relating the numerical radius and the absolute numerical radius

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The constant

Write

$$M_p = \max_{t\in[0,1]} \frac{|t^{p-1} - t|}{1 + t^p} = v \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

the numerical radius taken in the real ℓ_p^2.

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★ We may use M_p to relate v and $|v|$:

Theorem (M.–Merí–Popov, 2011)

In the real case,

$$v(T) \geqslant \frac{M_p}{6} \, |v|(T)$$

for every $T \in L(L_p(\mu))$.

Calculating $|n|(L_p(\mu))$ I

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Set $\kappa_p := \max_{\tau>0} \dfrac{\tau^{p-1}}{1+\tau^p} = \max_{\lambda\in[0,1]} \lambda^{\frac{1}{q}}(1-\lambda)^{\frac{1}{p}} = \dfrac{1}{p^{1/p}q^{1/q}}$.

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The best possibility for $|n|(L_p(\mu))$

If $\dim(L_p(\mu)) \geqslant 2$, then there is a (positive) operator $T \in L(L_p(\mu))$ with

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- For ℓ_p: consider the extension by zero of the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.
- For $L_p[0,1]$:

$$Tf = 2\left[\int_0^{1/2} f(s)\,ds\right]\chi_{[\frac{1}{2},1]} \qquad \big(f \in L_p[0,1]\big).$$

Calculating $|n|(L_p(\mu))$ II

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$$y = x \vee \tau Tx \qquad \text{and} \qquad A = \{\omega \in \Omega \,:\, x(\omega) \geqslant \tau(Tx)(\omega)\},$$

and observe that

$$\|y\|^p = \int_A x^p \, d\mu + \int_{\Omega \setminus A} (\tau Tx)^p \, d\mu \leqslant 1 + \tau^p \qquad \text{and} \qquad y^\# = x^{p-1} \vee (\tau Tx)^{p-1}.$$

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- Taking supremum on $\tau > 0$ and $\varepsilon > 0$, we get $|v|(T) \geqslant \kappa_p$.

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- If $T \in L(L_p[0,1])$ is rank-one $\implies v(T) \geqslant \kappa_p^2 \|T\|$.
- If $T \in L(L_p[0,1])$ is **compact**, then

$$v(T) \geqslant \kappa_p^2 \|T\| \text{ (complex case)}, \quad v(T) \geqslant \max_{\tau>0} \frac{\kappa_p \tau^{p-1} - \tau}{1+\tau^p} \|T\| \text{ (real case)}.$$

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Extremely non-complex Banach spaces

10 Extremely non-complex Banach spaces
- Motivation
- Extremely non-complex Banach spaces
- Surjective isometries

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Isometries and duality. Reminder

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Example (produced with numerical ranges)

There is a Banach space X such that

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★ In terms of linear dynamical systems:

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$$x' = A\,x \qquad \big(x : \mathbb{R}_0^+ \longrightarrow X\big)$$

  is given by a semigroup of isometries.
- There are infinitely many such A's on X^*

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  is given by a semigroup of isometries.
- There are infinitely many such A's on X^*
- But there are unbounded As on X such that the solution of the linear dynamical system is a one-parameter C_0 semigroup of isometries.

Isometries and duality. Reminder

Example (produced with numerical ranges)

There is a Banach space X such that

- $\mathrm{Iso}(X)$ has no exponential one-parameter semigroups.
- $\mathrm{Iso}(X^*)$ contains infinitely many exponential one-parameter semigroups.

★ In terms of linear dynamical systems:

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We would like to find $\mathcal{X}$ such that

- $\mathrm{Iso}(\mathcal{X})$ has no C_0 semigroup of isometries.
- $\mathrm{Iso}(\mathcal{X}^*)$ has exponential semigroup of isometries

Numerical range of unbounded operators

Numerical range of unbounded operators (1960's)

X Banach space, $T : D(T) \longrightarrow X$ linear,

$$V(T) = \{x^*(Tx) \ : \ x^* \in X^*,\ x \in D(T),\ x^*(x) = \|x^*\| = \|x\| = 1\}.$$

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Teorema (Stone, 1932)

H Hilbert space, A densely defined operator. TFAE:

- A generates an strongly continuous one-parameter semigroup of unitary operators (onto isometries).
- $A^* = -A$.
- $\mathrm{Re}(Ax \mid x) = 0$ for every $x \in D(A)$.

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Which Banach spaces have unbounded operators with numerical range zero?

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- In $C_0(\mathbb{R})$, $\Phi(t)(f)(s) = f(t+s)$ is an strongly continuous one-parameter semigroup of isometries (generated by the derivative).
- In $C_E([0,1]\|\Delta)$ there are also strongly continuous one-parameter semigroup of isometries.

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- In $C_E([0,1]\|\Delta)$ there are also strongly continuous one-parameter semigroup of isometries.

Consequence

We have to completely change our approach to the problem.

Complex structures

Definition

X has complex structure if there is $T \in L(X)$ such that $T^2 = -\mathrm{Id}$.

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Some remarks

- This gives a structure of vector space over $\mathbb{C}$:

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$$|||x||| = \max\{\|e^{i\theta}x\| \ :\ \theta \in [0,2\pi]\} \qquad (x \in X)$$

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one gets that $(X, |||\cdot|||)$ is a complex Banach space.
- If T is an isometry, then actually the given norm of X is complex.
- Conversely, if X is a complex Banach space, then

$$T(x) = i\,x \qquad \big(x \in X\big)$$

satisfies $T^2 = -\mathrm{Id}$ and T is an isometry.

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1. If $\dim(X) < \infty$, X has complex structure iff $\dim(X)$ is even.
2. If $X \simeq Z \oplus Z$ (in particular, $X \simeq X^2$), then X has complex structure.
3. There are infinite-dimensional Banach spaces without complex structure:
   - **Dieudonné, 1952:** the James' space $\mathcal{J}$ (since $\mathcal{J}^{**} \equiv \mathcal{J} \oplus \mathbb{R}$).
   - **Szarek, 1986:** uniformly convex examples.
   - **Gowers-Maurey, 1993:** their H.I. space.
   - **Ferenczi-Medina Galego, 2007:** there are odd and even infinite-dimensional spaces X.
     - X is even if admits a complex structure but its hyperplanes does not.
     - X is odd if its hyperplanes are even (and so X does not admit a complex structure).

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Definition

X is extremely non-complex if $\operatorname{dist}(T^2, -\mathrm{Id})$ is the maximum possible, i.e.

$$\|\mathrm{Id} + T^2\| = 1 + \|T^2\| \qquad (T \in L(X))$$

The Daugavet equation

What Daugavet did in 1963

The norm equality

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Classical examples

1. **Daugavet, 1963:**
   Every compact operator on $C[0,1]$ satisfies (DE).
2. **Lozanoskii, 1966:**
   Every compact operator on $L_1[0,1]$ satisfies (DE).
3. **Abramovich, Holub, and more, 80's:**
   $X = C(K)$, K perfect compact space
   or $X = L_1(\mu)$, μ atomless measure
   $\Longrightarrow$ every weakly compact $T \in L(X)$ satisfies (DE).

The Daugavet property

The Daugavet property (Kadets–Shvidkoy–Sirotkin–Werner, 1997)

A Banach space X is said to have the Daugavet property iff every rank-one operator on X satisfies (DE).

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Let X be a Banach space with the Daugavet property. Then

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Let X be a Banach space with the Daugavet property. Then

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- X does not embed into a Banach space with unconditional basis.

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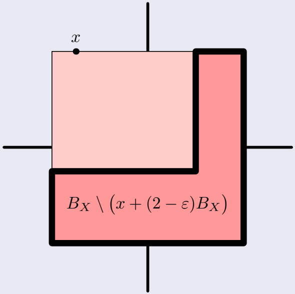
- Every weakly compact operator on X satisfies (DE).
- X contains ℓ_1.
- X does not embed into a Banach space with unconditional basis.
- **Geometric characterization**: X has the Daugavet property iff for each $x \in S_X$

$$\overline{\text{co}}\left(B_X \setminus \big(x + (2-\varepsilon)B_X\big)\right) = B_X\,.$$

(Kadets–Shvidkoy–Sirotkin–Werner, 1997 & 2000)



The Daugavet property II

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More examples

The following spaces have the Daugavet property.

- **Wojtaszczyk, 1992:**
  The disk algebra and H^∞.
- **Werner, 1997:**
  "Nonatomic" function algebras.
- **Oikhberg, 2005:**
  Non-atomic C^*-algebras and preduals of non-atomic von Neumann algebras.
- **Becerra–M., 2005**:
  Non-atomic JB^*-triples and their preduals.
- **Becerra–M., 2006:**
  Preduals of $L_1(\mu)$ without Fréchet-smooth points.
- **Ivankhno, Kadets, Werner, 2007:**
  $\mathrm{Lip}(K)$ when $K \subseteq \mathbb{R}^n$ is compact and convex.

Daugavet–type inequalities

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Some examples

- **Benyamini–Lin, 1985:**
  For every $1 < p < \infty$, $p \neq 2$, there exists $\psi_p : (0,\infty) \longrightarrow (0,\infty)$ such that
  $$\|\mathrm{Id} + T\| \geqslant 1 + \psi_p(\|T\|)$$
  for every compact operator T on $L_p[0,1]$.

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  If ψ_p is the best possible function above, then
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- **Oikhberg, 2005:**
  If $K(\ell_2) \subseteq X \subseteq L(\ell_2)$, then
  $$\|\mathrm{Id} + T\| \geqslant 1 + \frac{1}{8\sqrt{2}}\|T\|$$
  for every compact T on X.

Norm equalities for operators

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Motivating question

Are there other norm equalities which could define interesting properties of Banach spaces **?**

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Concretely

We looked for non-trivial norm equalities of the forms

$$\|\mathrm{Id}+T\| = f(\|T\|) \quad \text{or} \quad \|g(T)\| = f(\|T\|) \quad \text{or} \quad \|\mathrm{Id}+g(T)\| = f(\|g(T)\|)$$

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Solution

We proved that there are few possibilities. . .

Equalities of the form $\|\mathrm{Id} + T\| = f(\|T\|)$

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Proposition

X real or complex, $f : \mathbb{R}_0^+ \longrightarrow \mathbb{R}$ arbitrary, $a, b \in \mathbb{K}$. If the norm equality

$$\|a\,\mathrm{Id} + b\,T\| = f(\|T\|)$$

holds for every rank-one operator $T \in L(X)$, then

$$f(t) = |a| + |b|\,t \qquad \big(t \in \mathbb{R}_0^+\big).$$

If $a \neq 0$, $b \neq 0$, then X has the Daugavet property.

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Then, we have to look for Daugavet-type equalities in which $\mathrm{Id} + T$ is replaced by something different.

Proof

We have...

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$f(t) = |a| + |b|\,t \qquad (t \in \mathbb{R}_0^+)$.

- Trivial if $a \cdot b = 0$. Suppose $a \neq 0$ and $b \neq 0$ and write $\omega_0 = \frac{\overline{b}}{|b|}\frac{a}{|a|} \in \mathbb{T}$.

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- Since $\|T_t\| = t$, we have

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- It follows that

$$|a| + |b|\,t \geqslant f(t) = \|a\mathrm{Id} + b\,T_t\| \geqslant \big\|[a\mathrm{Id} + b\,T_t](x_0)\big\|$$

$$= \|a\,x_0 + b\,\omega_0 t\,x_0\| = |a + b\,\omega_0 t|\,\|x_0\| = \left|a + b\frac{\overline{b}}{|b|}\frac{a}{|a|}\,t\right| = |a| + |$$

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- Finally, for rank-one $T \in L(X)$, write $S = \frac{a}{b}\,T$ and observe

$$|a|\big(1 + \|T\|\big) = |a| + |b|\,\|S\| = \|a\mathrm{Id} + b\,S\| = |a|\,\|\mathrm{Id} + T\|.\checkmark$$

Equalities of the form $\|g(T)\| = f(\|T\|)$

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Theorem

X real or complex with $\dim(X) \geqslant 2$.
Suppose that the norm equality

$$\|g(T)\| = f(\|T\|)$$

holds for every rank-one operator $T \in L(X)$, where

- $g : \mathbb{K} \longrightarrow \mathbb{K}$ is analytic,
- $f : \mathbb{R}_0^+ \longrightarrow \mathbb{R}$ is arbitrary.

Then, there are $a, b \in \mathbb{K}$ such that

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Corollary

Only three norm equalities of the form

$$\|g(T)\| = f(\|T\|)$$

are possible:

- $b = 0$: $\|a\,\mathrm{Id}\| = |a|$,
- $a = 0$: $\|b\,T\| = |b|\,\|T\|$,

  (trivial cases)
- $a \neq 0$, $b \neq 0$: $\|a\,\mathrm{Id} + b\,T\| = |a| + |b|\,\|T\|$,

  (Daugavet property)

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$$x_0^*(x_0) = 0 \qquad \text{and} \qquad x_1^*(x_1) = 1,$$

and define the operators $T_0 = x_0^* \otimes x_0$ and $T_1 = x_1^* \otimes x_1$.

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  and define the operators $T_0 = x_0^* \otimes x_0$ and $T_1 = x_1^* \otimes x_1$.
- Then $g(\lambda T_0) = a_0 \mathrm{Id} + a_1 \lambda\, T_0$ and $g(\lambda T_1) = a_0 \mathrm{Id} + \widetilde{g}(\lambda)\, T_1$ $(\lambda \in \mathbb{C})$.

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- Take $x_0, x_1 \in S_X$ and $x_0^*, x_1^* \in S_{X^*}$ such that

$$x_0^*(x_0) = 0 \qquad \text{and} \qquad x_1^*(x_1) = 1,$$

  and define the operators $T_0 = x_0^* \otimes x_0$ and $T_1 = x_1^* \otimes x_1$.
- Then $g(\lambda T_0) = a_0 \mathrm{Id} + a_1 \lambda\, T_0$ and $g(\lambda T_1) = a_0 \mathrm{Id} + \widetilde{g}(\lambda)\, T_1$ $(\lambda \in \mathbb{C})$.
- Therefore, for $\lambda \in \mathbb{C}$ we have

$$\|a_0 \mathrm{Id} + \widetilde{g}(\lambda) T_1\| = \|g(\lambda T_1)\| = f(|\lambda|) = \|g(\lambda T_0)\| = \|a_0 \mathrm{Id} + a_1 \lambda T_0\|.$$

Proof (complex case)

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$\|g(T)\| = f(\|T\|)\ \forall T \in L(X)$ rank-one

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- We use the triangle inequality to get

$$|\widetilde{g}(\lambda)| \leqslant 2|a_0| + |a_1||\lambda| \qquad (\lambda \in \mathbb{C}),$$

- and so $\widetilde{g}$ is a degree-one polynomial by Cauchy inequalities. ✓

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Remark

If X has the Daugavet property and g is analytic, then

$$\|\mathrm{Id} + g(T)\| = |1 + g(0)| - |g(0)| + \|g(T)\|$$

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- Our aim here is not to show that g has a suitable form,
- but it is to see that for every g another simpler equation can be found.
- From now on, we have to separate the complex and the real case.

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- COMPLEX CASE:

Proposition

X complex, $\dim(X) \geqslant 2$. Suppose that

$$\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$$

for every rank-one $T \in L(X)$, where

- $g : \mathbb{C} \longrightarrow \mathbb{C}$ analytic non-constant,
- $f : \mathbb{R}_0^+ \longrightarrow \mathbb{R}$ *continuous*.

Then

$$\big\|(1 + g(0))\mathrm{Id} + T\big\| = |1 + g(0)| - |g(0)| + \big\|g(0)\mathrm{Id} + T\big\|$$

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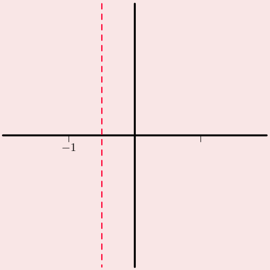
We obtain two different cases:

- $|1 + g(0)| - |g(0)| \neq 0$ or
- $|1 + g(0)| - |g(0)| = 0$.



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Theorem

If $\operatorname{Re} g(0) \neq -1/2$ and

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Example

If $X = C[0,1] \oplus_2 C[0,1]$, then

- $\left\|\mathrm{Id} + \mathrm{e}^{i\theta}\, T\right\| = \|\mathrm{Id} + T\|$
  for every $\theta \in \mathbb{R}$, rank-one $T \in L(X)$.
- X does not have the Daugavet property.

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- The proofs are not valid (we use Picard's Theorem).

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- They work when g is onto.
- But we do not know what is the situation when g is not onto, even in the easiest examples:
  - $\|\mathrm{Id} + T^2\| = 1 + \|T^2\|$,
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- $\|\mathrm{Id} - T\| = \|\mathrm{Id} + T\|$ for every rank-one $T \in L(X)$.
- X does not have the Daugavet property.

The question

Godefroy, private communication

Is there any real Banach space X (with $\dim(X) > 1$) such that

$$\|\mathrm{Id} + T^2\| = 1 + \|T^2\|$$

for every operator $T \in L(X)$ **?**

In other words, are there extremely non-complex spaces other than $\mathbb{R}$ **?**

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Some examples

1. If $\dim(X) < \infty$, X has complex structure iff $\dim(X)$ is even.
2. **Dieudonné, 1952:** the James' space $\mathcal{J}$ (since $\mathcal{J}^{**} \equiv \mathcal{J} \oplus \mathbb{R}$).
3. **Szarek, 1986:** uniformly convex examples.
4. **Gowers-Maurey, 1993:** their H.I. space.
5. **Ferenczi-Medina Galego, 2007:** there are odd and even infinite-dimensional spaces X.
   - X is even if admits a complex structure but its hyperplanes does not.
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(Un)fortunately. . .

This did not work and we moved to $C(K)$ spaces.

The first example: weak multiplications

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Let K be a compact space. $T \in L\big(C(K)\big)$ is a weak multiplication if

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K perfect, $T = g\,\mathrm{Id} + S \in L\big(C(K)\big)$ weak multiplication
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- $\|g^2\,\mathrm{Id} + S\| = \|\mathrm{Id} + S + (g^2\,\mathrm{Id} - \mathrm{Id})\| \geqslant \|\mathrm{Id} + S\| - \|g^2\,\mathrm{Id} - \mathrm{Id}\|$
  $= 1 + \|S\| - \big(1 - \min g^2(K)\big) = \|S\| + \min g^2(K)$.

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Proof

Just think that the set of operators satisfying (DE) is closed.

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Proof

If $\|u + v\| = \|u\| + \|v\| \quad \Longrightarrow \quad \|\alpha\, u + \beta\, v\| = \alpha\|u\| + \beta\|v\|$ for $\alpha, \beta \in \mathbb{R}_0^+$.

The first example: weak multiplications. II

Weak multiplication

Let K be a compact space. $T \in L\big(C(K)\big)$ is a weak multiplication if

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There are perfect compact spaces K such that all operators on $C(K)$ are weak multiplications.

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Consequence

Therefore, there are extremely non-complex $C(K)$ spaces.

More examples: weak multipliers

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Let K be a compact space. $T \in L\big(C(K)\big)$ is a weak multiplier if

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More examples: weak multipliers

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Let K be a compact space. $T \in L\big(C(K)\big)$ is a weak multiplier if

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Corollary

There are infinitely many non-isomorphic extremely non-complex Banach spaces.

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There is a family $(K_i)_{i\in I}$ of pairwise disjoint perfect and totally disconnected compact spaces such that

- every operator on $C(K_i)$ is a weak multiplier,
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Theorem

There are some compactifications $\widetilde{K}$ of the above family $(K_i)_{i\in I}$ such that the corresponding $C(\widetilde{K})$'s are extremely non-complex.

Further examples II

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Main consequence

There are perfect compact spaces K_1, K_2 such that:

- $C(K_1)$ and $C(K_2)$ are extremely non-complex,
- $C(K_1)$ contains a complemented copy of $C(\Delta)$.
- $C(K_2)$ contains a 1-complemented isometric copy of ℓ_∞.

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Observation

- $C(K_1)$ and $C(K_2)$ have operators which are not weak multipliers.
- They are not indecomposable spaces.

Related open questions

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Find topological consequences on K when $C(K)$ is extremely non-complex.
For instance:
If $C(K)$ is extremely non-complex and $\psi : K \longrightarrow K$ is continuous, are there an open subset U of K such that $\psi|_U = \mathrm{id}$ and $\psi(K \setminus U)$ is finite **?**

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- We will show latter than $\varphi : K \longrightarrow K$ homeomorphism $\implies \varphi = \mathrm{id}$.

Extremely non-complex Banach spaces

Definition

X is extremely non-complex if $\operatorname{dist}(T^2, -\mathrm{Id})$ is the maximum possible, i.e.

$$\|\mathrm{Id} + T^2\| = 1 + \|T^2\| \qquad \big(T \in L(X)\big)$$

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Examples

There are several extremely non-complex $C(K)$ spaces:

- If $T = g\mathrm{Id} + S$ for every $T \in L(C(K))$ (K Koszmider).
- If $T^* = g\mathrm{Id} + S$ for every $T \in L(C(K))$ (K weak Koszmider).
- One $C(K)$ containing a complemented copy of $C(\Delta)$.
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Isometries on extremely non-complex spaces. I

Theorem

X extremely non-complex.

- $T \in \mathrm{Iso}(X) \implies T^2 = \mathrm{Id}.$
- $T_1, T_2 \in \mathrm{Iso}(X) \implies T_1 T_2 = T_2 T_1.$
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- Take $S = \frac{1}{\sqrt{2}}\left(T - T^{-1}\right) \implies S^2 = \frac{1}{2}T^2 - \mathrm{Id} + \frac{1}{2}T^{-2}.$

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- Then $\mathrm{Id} = \frac{1}{2}T^2 + \frac{1}{2}T^{-2}.$
- Since Id is an extreme point of $B_{L(X)} \implies T^2 = T^{-2} = \mathrm{Id}.$ ✓

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$$\mathrm{Id} = (T_1 T_2)(T_1 T_2)$$
$$\implies T_1 T_2 = T_1 (T_1 T_2 T_1 T_2) T_2 = (T_1 T_1) T_2 T_1 (T_2 T_2) = T_2 T_1. \checkmark$$

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$\Phi(t) = \Phi(t/2 + t/2) = \Phi(t/2)^2 = \mathrm{Id}$. ✓

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- $\mathrm{Iso}(X)$ identifies with the set $\mathrm{Unc}(X)$ of unconditional projections on X:

$$P \in \mathrm{Unc}(X) \iff P^2 = P,\ 2P - \mathrm{Id} \in \mathrm{Iso}(X)$$
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- $\mathrm{Iso}(X) \equiv \mathrm{Unc}(X)$ is a Boolean algebra
  $\iff P_1P_2 \in \mathrm{Unc}(X)$ when $P_1, P_2 \in \mathrm{Unc}(X)$
  $\iff \left\| \frac{1}{2}\left(\mathrm{Id} + T_1 + T_2 - T_1T_2\right) \right\| = 1$ for every $T_1, T_2 \in \mathrm{Iso}(X)$.

Extremely non-complex $C_E(K\|L)$ spaces.

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$C_{\ell_2}(K\|L)$ is not isomorphic to a $C(K')$ space since $\ell_2 \xhookrightarrow{\text{comp}} C_{\ell_2}(K\|L)^*$.

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But we are able to give a better result...

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Theorem (Banach-Stone like)

$C_E(K\|L)$ extremely non-complex, $T \in \mathrm{Iso}(C_E(K\|L))$
$\Longrightarrow$ exists $\theta : K \setminus L \longrightarrow \{-1,1\}$ continuous such that

$$[T(f)](x) = \theta(x)f(x) \qquad \big(x \in K \setminus L,\ f \in C_E(K\|L)\big)$$

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- $\phi(x) = x$ for all $x \in D_0$.

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- $\phi^2 = \mathrm{id}$, $\theta(x)\,\theta(\phi(x)) = 1$, ϕ homeomorphism.
- $\phi(x) = x$ for all $x \in D_0$.
- $D_0 = K \setminus L$.

Isometries on extremely non-complex $C_E(K\|L)$ spaces

Theorem (Banach-Stone like)

$C_E(K\|L)$ extremely non-complex, $T \in \mathrm{Iso}(C_E(K\|L))$
$\Longrightarrow$ exists $\theta : K \setminus L \longrightarrow \{-1,1\}$ continuous such that

$$[T(f)](x) = \theta(x) f(x) \qquad \big(x \in K \setminus L,\ f \in C_E(K\|L)\big)$$

Sketch of the proof.

- $D_0 = \{x \in K \setminus L \,:\, \exists y \in K \setminus L,\, \theta_0 \in \{-1,1\}$ with $T^*(\delta_x) = \theta_0 \delta_y\}$ dense in K.
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- θ is continuous. $\checkmark$

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Consequences: cases $E = C(L)$ and $E = 0$

- $C(K)$ extremely non-complex, $\varphi : K \longrightarrow K$ homeomorphism $\Longrightarrow$ $\varphi = \mathrm{id}$
- $C_0(K \setminus L) \equiv C_0(K\|L)$ extremely non-complex, $\varphi : K \setminus L \longrightarrow K \setminus L$ homeomorphism $\Longrightarrow$ $\varphi = \mathrm{id}$
- In both cases, the group of surjective isometries identifies with a Boolean algebra of clopen sets.

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Consequences: general case

- If for every $x \in L$, there is $f \in E$ with $f(x) \neq 0$
  $\Longrightarrow$ θ extends to the whole K and

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for every surjective isometry T.

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- But for $E = \ell_2$, $0 \in \overline{\mathrm{ext}\,(B_{E^*})}^{w^*}$.

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Consequence: connected case

If K and $K \setminus L$ are connected, then

$$\mathrm{Iso}\big(C_E(K\|L)\big) = \{-\mathrm{Id}, +\mathrm{Id}\}$$

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Koszmider, 2004

$\exists\ \mathcal{K}$ weak Koszmider space such that $\mathcal{K} \setminus F$ is connected if $|F| < \infty$.

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Proof.

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- $\mathcal{K} \setminus \mathcal{L}$ connected $\implies \mathrm{Iso}(X) = \{-\mathrm{Id}, +\mathrm{Id}\}$.
- $X^* = \ell_2 \oplus_1 C_0(\mathcal{K} \| \mathcal{L})^*$, so $\mathrm{Iso}(\ell_2) \subset \mathrm{Iso}(X^*)$. ✓

Open questions on extremely non-complex Banach spaces

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X extremely non complex

- Does X have the Daugavet property **?**

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- If $Y \leqslant X$ is 1-codimensional, is Y extremely non complex **?**
- Is it possible that $X \simeq Z \oplus Z \oplus Z$ **?**


That's all Folks!

Schedule of the talk

1. Basic notation
2. Numerical range of operators
3. Two results on surjective isometries
4. Numerical index of Banach spaces
5. The alternative Daugavet property
6. Lush spaces
7. Slicely countably determined spaces
8. Remarks on the containment of c_0 and ℓ_1
9. Numerical index of L_p-spaces
10. Extremely non-complex Banach spaces