About the mathematical infinity

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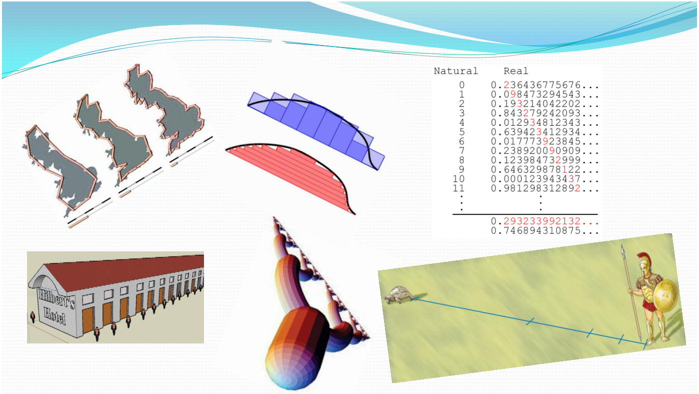


dongguk UNIVERSITY

Seoul (Korea), March 24th 2011

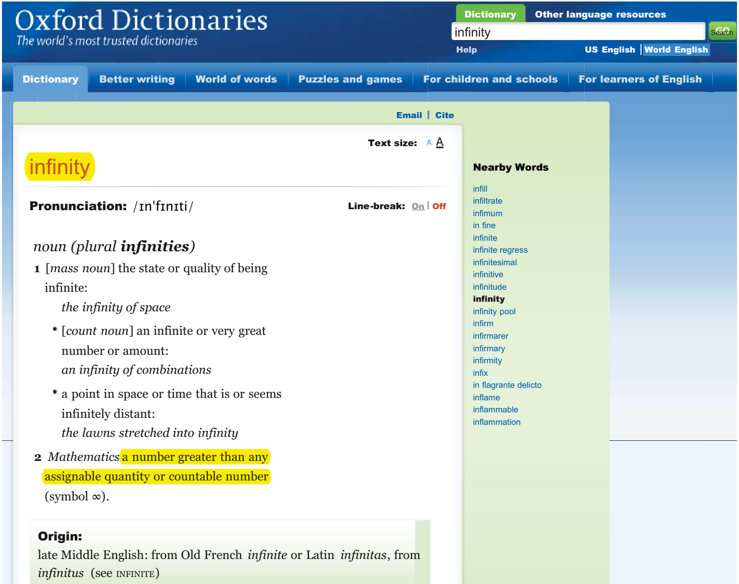
Schedule of the talk

What is infinity?

The infinite! No other question has ever moved so profoundly the spirit of man.

David Hilbert (1862–1943)

Definitions of infinity I



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infinity

Pronunciation: /ɪnˈfɪnɪti/ — Line-break: On | Off

*noun (plural **infinities**)*

1 [*mass noun*] the state or quality of being infinite:

the infinity of space

- [*count noun*] an infinite or very great number or amount:
  an infinity of combinations
- a point in space or time that is or seems infinitely distant:
  the lawns stretched into infinity

2 *Mathematics* a number greater than any assignable quantity or countable number (symbol ∞).

Origin:
late Middle English: from Old French *infinite* or Latin *infinitas*, from *infinitus* (see INFINITE)

Nearby Words

infill
infiltrate
infimum
in fine
infinite
infinite regress
infinitesimal
infinitive
infinitude
infinity
infinity pool
infirm
infirmarer
infirmary
infirmity
infix
in flagrante delicto
inflame
inflammable
inflammation

Definitions of infinity II

Algebra
Applied Mathematics
Calculus and Analysis
Discrete Mathematics
Foundations of Mathematics
Geometry
History and Terminology
Number Theory
Probability and Statistics
Recreational Mathematics
Topology

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Infinity

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Infinity, most often denoted as ∞, is an unbounded quantity that is greater than every real number. The symbol ∞ had been used as an alternative to M (1000) in Roman numerals until 1655, when John Wallis suggested it be used instead for infinity.

Infinity is a very tricky concept to work with, as evidenced by some of the counterintuitive results that follow from Georg Cantor's treatment of infinite sets.

Informally, $1/\infty = 0$, a statement that can be made rigorous using the limit concept,

$$\lim_{x\to\infty} \frac{1}{x} = 0.$$

Similarly,

$$\lim_{x\to 0^+} \frac{1}{x} = \infty,$$

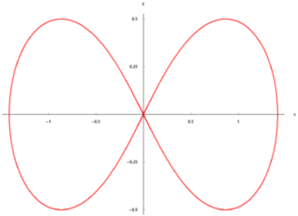
where the notation 0^+ indicates that the limit is taken from the positive side of the real line.

About the symbol ∞

- Unclear origen
- It has the form of the curve *lemniscata*

$$(x^2 + y^2)^2 = x^2 - y^2$$

  which has no beginning or end
- John Wallis (1616–1703) was the first one in using it.
- Maybe the symbol came from the Roman number M (1000) which in Etruscan were similar or from the Greek letter *omega*



Present definition of infinity

- Bernard Bolzano (1781–1848):

  An infinite set is that for which any finite set can be only equivalent to a part and not to the whole of it.

- Richard Dedekind (1831–1916):

  A system S is called infinite when could be put in one-to-one correspondence with one of its proper subset. Otherwise, the system is finite.

- Georg Cantor (1845–1918):

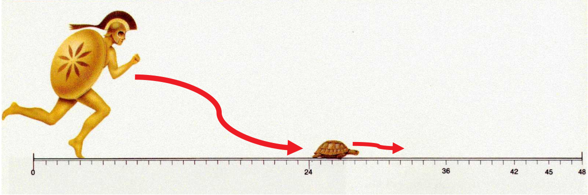
  First systematic study of infinity, arithmetic of infinity, transfinite number... Not all infinities are equal.

The infinite processes and their paradoxes

Achilles and the Tortoise: the paradox

Achilles, the Greek hero, and the Tortoise, where in a race, Achilles gives the Tortoise a head start. Even though Achilles runs faster than the Tortoise, he will never catch her. The argument is as follows: when Achilles reaches the point at which the Tortoise started, the Tortoise is no longer there, having advanced some distance; when Achilles arrives at the point where the Tortoise was when Achilles arrived at the point where the Tortoise started, the Tortoise is no longer there, having advanced some distance; and so on.

Zeno of Elea (490 B.C. – 425 B.C.)



Achilles and the Tortoise II

- Zeno, follower of Parmenides, wanted to prove that *reality is one, change is impossible, and existence is timeless, uniform, necessary, and unchanging*.
- Zeno paradox relays on the idea that infinity can not be achieved:
  - Each movement of Achilles is a positive distance (true),
  - infinity many movements are needed (true),
  - The sum of all of these infinite many distances should be infinity, can not be achieved (false!)
- Aristotle was unable to disprove Zeno's paradoxes.
- To do so, we should know that

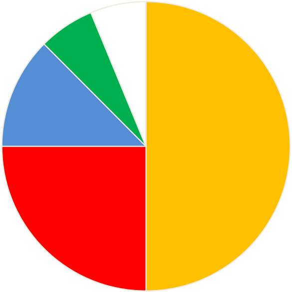
  an "infinite sum" of positive quantities can be finite.

Adding wedges of cheese

Theorem

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\cdots=1$$

Graphically:



Proof:

We take a cheese

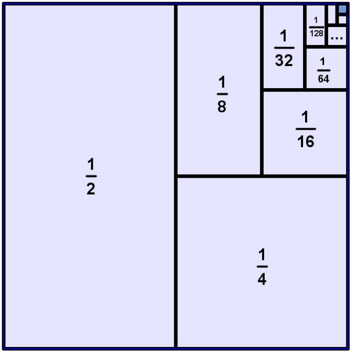
and make wedges...

Adding wedges of cheese

Theorem

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\cdots=1$$

Another proof:



Adding wedges of cheese II

Theorem

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\cdots \text{ is infinity.}$$

Proof:

$$\begin{aligned}
&\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\cdots\\
=&\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\cdots\\
>&\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\cdots\\
=&\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots
\end{aligned}$$

A last cite about Achilles and the Tortoise

Achilles had overtaken the Tortoise, and had seated himself comfortably on its back. So you've got to the end of our race-course? –said the Tortoise. Even though it does consist of an infinite series of distances? I thought some wiseacre or other had proved that the thing couldn't be done?

Lewis Carroll, What the Tortoise Said to Achilles, 1894

The legend of the invention of Chess

Once upon a time, the Indian king Iadava was really sad after the loosing of his son in a battle. But one day it came to the sad palace, a young Brahmin named Sessa and asked the guards to see the king saying that he had invented a game especially for him in order to cheer his hours of solitude, the Chaturanga, Chess' antecesor.
Pleased with the beautiful game invented for him, Iadava told Sessa – Ask me what you want to and I will give you immediately.–

The legend of the invention of Chess. II

Sessa kindly explained to Iadava –I would like to be paid with rice in the following way– I receive a grain for the first square, two grains for the second square, four grains for the third... and so on until the sixty-fourth square.
By hearing such a humble request, Iadava began to laugh nonstop. After a while, he ordered that Sessa would be given what he had requested immediately.
It was clear soon that the request was less humble and more complicated than thought: When moving on through the squares, the quantity of rice couldn't be managed.

The computation of the amount of rice

The bookkeepers of the kingdom were able to calculate the exact quantity of grains needed:

$$18446744073709551615 \simeq 18 * 10^{18} \quad \textit{grains of rice}$$

But, all rice produced in the kingdom during 100 years was not enough to pay!

Let us calculate the amount of rice (called X):

$$X = 1 + 2 + 2^2 + 2^3 + \cdots + 2^{63}$$
$$X - 1 = 2 + 2^2 + 2^3 + \cdots + 2^{63} = 2(1 + 2 + 2^2 + 2^3 + \cdots + 2^{62})$$
$$X - 1 = 2(X - 2^{63})$$

Therefore, $X = 2^{64} - 1 \simeq 18 * 10^{18}$.

The end of the history

From this point, there are several different ends of the history:

- **FIRST:** The king ordered to decapitate Sessa.
- **SECOND:** Iadava appointed Sessa to first vizier for life.
- **THREE:** The king, wishful to fulfil his impossible promise, ask a mathematician who gave the following solution:
  - They offer to Sessa to consider an infinite board
  - and made the following calculation for X (the amount of rice to be given to Sessa)

$$\begin{aligned} X &= 1 + 2 + 2^2 + 2^3 + 2^4 + 2^5 + \cdots \\ X - 1 &= 2 + 2^2 + 2^3 + 2^4 + 2^5 + \cdots \\ X - 1 &= 2\left(1 + 2 + 2^2 + 2^3 + 2^4 + \cdots\right) \\ X - 1 &= 2X \end{aligned}$$

  - Therefore, $X = -1!$ and the king asked Sessa to give a grain of rice to him :-)

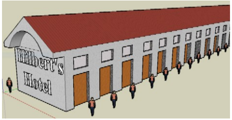
How many infinities do we have?

3. How many infinities do we have?
    - Hilbert's hotel
    - Cantor and the continuum

Hilbert's hotel

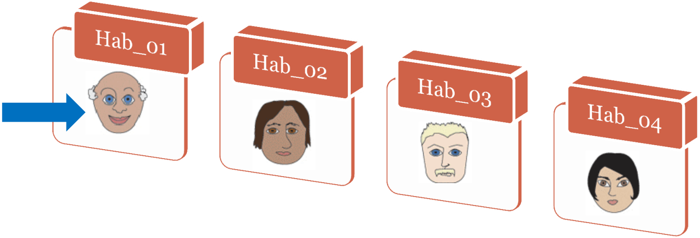
- This is an history invented by David Hilbert (1862–1943) to show that many infinities are the same.
- Once upon a time in which a hotel with infinitely many rooms was built.
- Its slogan is

  "We are always full, but we always have a room for you".



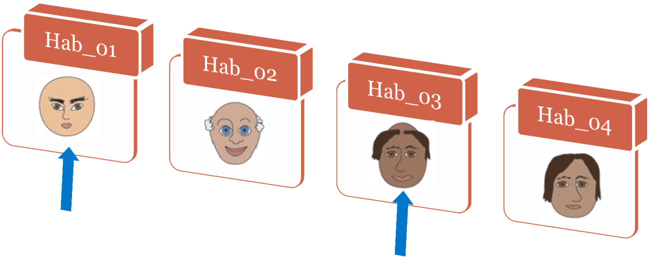
Infinity plus one equals infinity

- The hotel is full but we want to put up another guest.
- Is it possible?
- Of course:
  - we move each guest to the next room,
  - this gives one free room,
  - which will be occupied by the new guest.
- Iterating the process, we may put up any finite quantity of new guests.



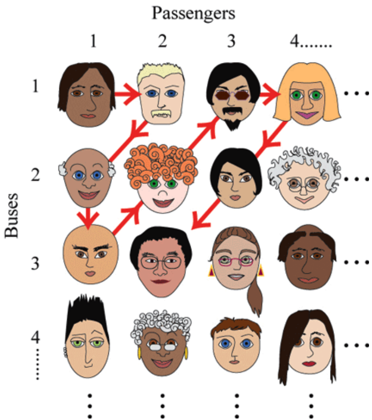
Infinity plus infinity equals infinity

- Imagine that the hotel is still full but a bus with infinitely many new guests arrives.
- Is is possible to put up them? Again Yes:
  - there are infinitely many even rooms,
  - we move the guest in room n to the room $2n$,
  - all odd rooms are free, and they are infinitely many,
  - we put up the new guests in the odd rooms.
- The same can be done with any finitely many number of busses.



Even harder!

- The hotel is full and infinitely many busses arrive, each one with infinitely many new guests.
- Is it possible to put up them?
- Yes again:
  - we liberalize the infinitely many odd rooms,
  - now it is only needed to order the infinitely many passenger of the infinitely many busses,
  - this is a bad idea. . .
  - better like that.



Are equal all the infinities?

- We have seem that "many" infinities are surprisingly equal.
- Nevertheless, Georg Cantor (1845–1918) proved that not all infinites are equals and created an algebra of the infinite sets.
- To present Cantor's example, let us recall some sets of numbers:

  - $\mathbb{N} = \{1, 2, 3, 4, \dots\}$ positive integers;
  - $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ integers;
  - $\mathbb{Q} = \{p/q \ : \ p \in \mathbb{Z}, \ q \in \mathbb{N}\}$ rational numbers;
  - $\mathbb{R}$ real numbers.

- Hilbert's hotel tale shows that $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ are equivalent infinite sets, all of them can be counted, be possed in an infinite list.
- What's the matter with $\mathbb{R}$?

Cantor's example

- Cantor proved that it is not possible to count $\mathbb{R}$:
- Let us consider any infinite list of number between 0 and 1:

$$\begin{array}{lcl}
1 & \longrightarrow & 0,\boxed{1}2567894\dots \\
2 & \longrightarrow & 0,8\boxed{3}809823\dots \\
3 & \longrightarrow & 0,99\boxed{9}90023\dots \\
4 & \longrightarrow & 0,000\boxed{1}2785\dots \\
\dots & &
\end{array}$$

- Write a number changing the boxed digit for any other digit between 0 and 8:

$$0,3870\dots$$

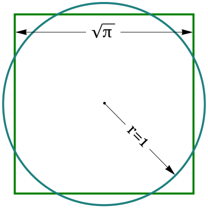
- This number is not in the list!
- So, no list may contain all real numbers between 0 and 1 and, therefore, $\mathbb{R}$ **is not** equivalent to $\mathbb{N}$.

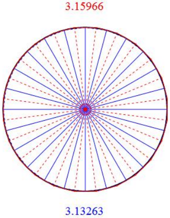
How to measure areas?

4. How to measure areas?
   - The Ancient Greek mathematics
   - Newton and circle's quadrature
   - Measure in the 20th Century
   - 20th Century paradox

The quadratures in Greek mathematics

- Quadrature problems in Ancient Greek mathematics consist in, given a geometric figure, find an square with the same area, following elementary processes.
- The most famous quadrature problem is that of circle quadrature.
- Ancient Greek mathematicians knew how to quadrate triangles and, therefore, any polygon.
- From here, the invented a process called exhaustion to get the quadrature of figures which can be approximated by polygons.



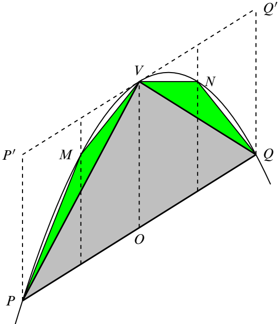


Quadrature of the parabola by Archimedes

- A remarkable example is that of quadrature of the parabola by Archimedes of Syracuse (287 B.C. – 212 B.C.):
- *The area of the parabolic segment is equal to the four thirds of the area of the inscribed triangle.*
    - $S \rightsquigarrow$ area of the inscribed triangle;
    - $\text{area}(\triangle VNQ) + \text{area}(\triangle VMP) = \frac{1}{4}\text{area}(\triangle PVQ)$;
    - We iterate the process and get

$$S + \frac{1}{4}S + \frac{1}{4^2}S + \frac{1}{4^3}S + \cdots$$

    - The sum of this infinite series is $\frac{4}{3}S$ and it is "proved" that it fills the area of the parabolic segment.
    - This proof is dark and tedious since they do not know the concept of infinity.



Newton and circle's quadrature

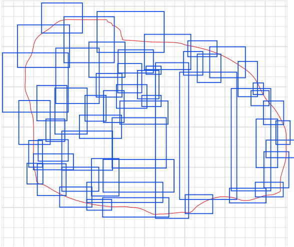
- Isaac Newton (1643–1727) solved the circle's quadrature problem (but not in the sense of the Ancient Greeks).
- He proved the Fundamental Theorem of Calculus, which relates areas and tangents.
- To quadrate the circle, he calculated the area of the region ABC writing the function y as an infinite sum (Newton's binomial).
- We get that

$$\pi = \frac{3\sqrt{3}}{4} + 24\left(\frac{2}{3\cdot 2^3} - \frac{1}{5\cdot 2^5} - \frac{1}{28\cdot 2^7} - \frac{1}{72\cdot 2^9} - \cdots\right)$$

  and 22 terms gives 16 decimal digits of π.
- In 1882, Ferdinand Lindemann (1852–1939) proved the impossibility of circle's quadrature in the sense of the Ancient Greeks.

How is the measure theory nowadays?

- The modern measure theory is due to Henri Lebesgue (1875 – 1941).
- Shortly, it consists in covering with an infinite quantity of rectangles;
- we add the areas of all rectangles (infinite sum);
- and take the best of the possibilities.
- It is a double infinite process:
  - each covering has infinitely many rectagles,
  - there are infinitely many covering, so
  - to get the best possibility needs an infinite process.

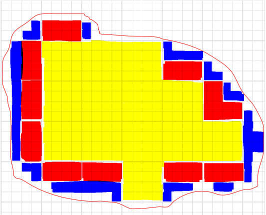


Is it possible to measure from inside?

- For a "reasonable" set, we may get its area counting the rectangles which are contained.
- Shortly, the process consists in taking measures using an each time more accurate "millimeter paper":
  - mark the squares of side 1 inside the picture: **13**;
  - mark the squares of side $1/2$ inside the rest: **21**;
  - mark the squares of side $1/4$ inside the rest: **54**;
  - getting an approximation of the area

$$\text{Area} \simeq 13 + 21 \cdot \frac{1}{2} + 54 \cdot \frac{1}{4} = 37\mathrm{u}^2.$$

  - We may continue the process till infinity, getting in most cases, the area of the picture.



Lebesgue's measure theory

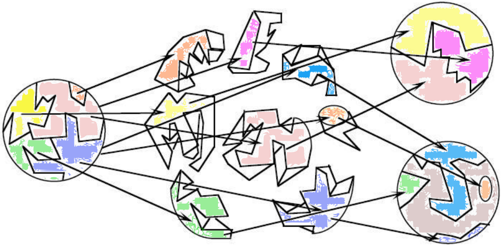
- Lebesgue's ideas caused a revolution in Mathematics.
- The innovation is the idea that each covering contains infinitely many rectangles, and this is the key to get a solid theory.
- All the Mathematical Analysis of the 20th and 21st centuries needs Lebesgue's ideas.
- Other scientific field like quantum physics cannot be even thought without this language.
- Even though Lebesgue measure theory is complete, it is not possible to measure all sets. This **cannot** be done by **any** other way of measure.

Banach-Tarski paradox

Banach–Tarski (1924)

An sphere can be divided into finitely many pieces in such a way that they can be moved and grouped producing two sphere of the same size than the original one.

- Pieces are not deformed in any matter.
- When grouping the pieces, there are no overlappings nor holes.
- It is impossible that the volumen of all pieces can be meassured.

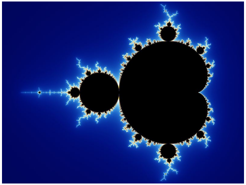


England coast length

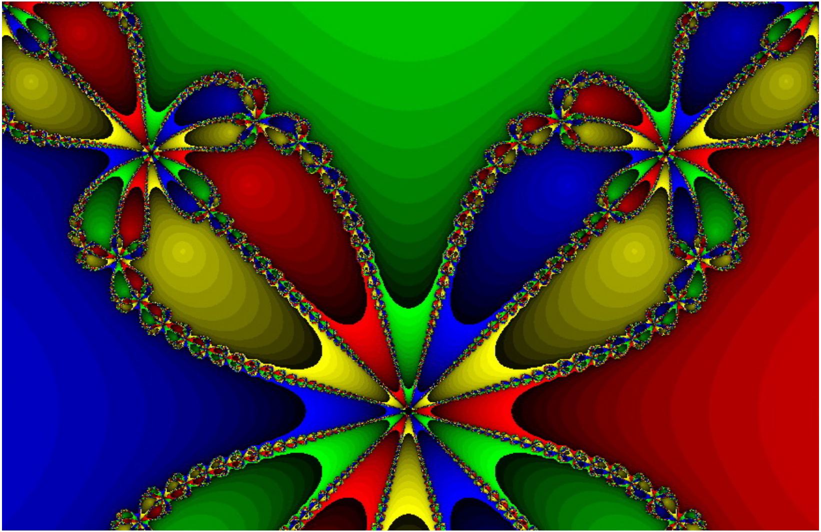
- At the end of the 19th Century, English parliament sent a young cartographer to measure the length of the England coast.
- It took 10 years to calculate, using a 50 meters tape, that the length is 2000 km.
- Requested to realize a more accurate measure, the not so young cartographer, got after 10 years and using a 10 meters tape, that the length of the cost is ¡2800 km!
- The experienced cartographer was sent again to measure the length of the cost with a 1 meter tape getting after 10 years that this length is of 3600 km.
- The poor cartographer was sentenced to eternity to measure the length of the coast getting each time a different value!

An example of fractal structure

- It looks like the English coast is a curve with infinite length, but this make no sense if it were a "reasonable curve".
- A good model for this phenomenon is due to Benoît Mandelbrot (1924 – 2010).
- The model for this coast is a Fractal:
  - self-similar structure
  - whose dimension is not an integer number.
- In the case of the English coast, its dimension is statistically estimated as 1.25.
- Mandelbrot set is an example of fractal:



Some nice examples of fractals



Some nice examples of fractals

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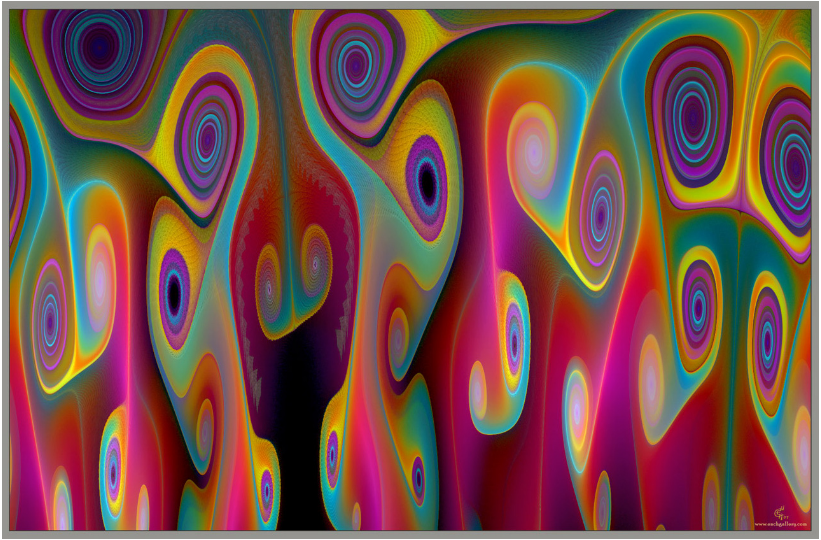

Some nice examples of fractals

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