Numerical Ranges and Numerical Indices

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Schedule of the talk

1. Basic notation
2. Numerical range of operators
3. Numerical ranges and surjective isometries
4. Numerical index of Banach spaces
5. Banach spaces with numerical index one

Notation

Basic notation

X Banach space.

- $\mathbb{K}$ base field (it may be $\mathbb{R}$ or $\mathbb{C}$),
- S_X unit sphere, B_X unit ball,
- X^* dual space,
- $L(X)$ bounded linear operators,
- $\mathrm{Iso}(X)$ surjective linear isometries,
- $T^* \in L(X^*)$ adjoint operator of $T \in L(X)$,
- $\mathrm{aconv}(B) = \mathrm{co}(\mathbb{T}\, B)$ absolutely convex hull of B,
- $\mathrm{ext}(C)$ extreme points of C,
- *slice* of C:

$$S(C, x^*, \alpha) = \big\{x \in C \ : \ \mathrm{Re}\, x^*(x) > \sup \mathrm{Re}\, x^*(C) - \alpha\big\}$$

where $x^* \in X^*$ and $0 < \alpha < \sup \mathrm{Re}\, x^*(C)$.

Numerical range of operators

2. Numerical range of operators
    - Definitions and first properties

F. F. Bonsall and J. Duncan
Numerical Ranges. Vol I and II.
London Math. Soc. Lecture Note Series, 1971 & 1973.

Numerical range: Hilbert spaces

Hilbert space numerical range (Toeplitz, 1918)

- A $n \times n$ real or complex matrix

$$W(A) = \big\{ (Ax \mid x) \ : \ x \in \mathbb{K}^n, \ (x \mid x) = 1 \big\}.$$

- H real or complex Hilbert space, $T \in L(H)$,

$$W(T) = \big\{ (Tx \mid x) \ : \ x \in H, \ \|x\| = 1 \big\}.$$

Remark

★ Given $T \in L(H)$ we associate

- a sesquilinear form $\varphi_T(x,y) = (Tx \mid y) \qquad (x,y \in H)$,
- a quadratic form $\widehat{\varphi_T}(x) = \varphi_T(x,x) = (Tx \mid x) \qquad (x \in H)$.

★ Then, $W(T) = \widehat{\varphi_T}(S_H)$. Therefore:

- $\widehat{\varphi_T}(B_H) = [0,1]\, W(T)$,
- $\widehat{\varphi_T}(H) = \mathbb{R}^+\, W(T)$.
- But we cannot get $W(T)$ from $\widehat{\varphi_T}(B_H)$!

Numerical range: Hilbert spaces. Properties.

Some properties

H Hilbert space, $T \in L(H)$:

- $W(T)$ is convex.
- $T, S \in L(H)$, $\alpha, \beta \in \mathbb{K}$:
  - $W(\alpha T + \beta S) \subseteq \alpha W(T) + \beta W(S)$;
  - $W(\alpha \mathrm{Id} + S) = \alpha + W(S)$.
- $W(U^*TU) = W(T)$ for every $T \in L(H)$ and every U unitary.
- $\mathrm{Sp}(T) \subseteq \overline{W(T)}$.
- If T is normal, then $\overline{W(T)} = \overline{\mathrm{co}}\, \mathrm{Sp}(T)$.
- In the real case ($\dim(H) > 1$), there is $T \in L(H)$, $T \neq 0$ with $W(T) = \{0\}$.
- In the complex case,

  $$\sup\{|(Tx \mid x)| \ : \ x \in S_H\} \geqslant \frac{1}{2}\,\|T\|.$$

  If T is actually self-adjoint, then

  $$\sup\{|(Tx \mid x)| \ : \ x \in S_H\} = \|T\|.$$

Numerical range: Hilbert spaces. Motivation.

Some reasons to study numerical ranges

- It gives a “picture” of the matrix/operator which allows to “see” many properties (algebraic or geometrical) of the matrix/operator.
- It is a comfortable way to study the spectrum.
- It is useful to estimate spectral radii of small perturbations of matrices.
- It is useful to work with some concepts like hermitian operator, skew-hermitian operator, dissipative operator. . .

Example

Consider $A = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ \varepsilon & 0 \end{pmatrix}$.

- $\mathrm{Sp}(A) = \{0\}$, $\mathrm{Sp}(B) = \{0\}$.
- $\mathrm{Sp}(A+B) = \{\pm\sqrt{M\varepsilon}\} \subseteq W(A+B) \subseteq W(A) + W(B)$,
- so the spectral radius of $A+B$ is bounded above by $\frac{1}{2}(|M| + |\varepsilon|)$.

Numerical range: Banach spaces (I)

Banach spaces numerical range (Bauer 1962; Lumer, 1961)

X Banach space, $T \in L(X)$,

$$V(T) = \left\{x^*(Tx) \ : \ x^* \in S_{X^*},\ x \in S_X,\ x^*(x) = 1\right\}$$

Some properties

X Banach space, $T \in L(X)$.

- $V(T)$ is connected but not necessarily convex.
- $T, S \in L(X)$, $\alpha, \beta \in \mathbb{K}$:
  - $V(\alpha T + \beta S) \subseteq \alpha V(T) + \beta V(S)$;
  - $V(\alpha \mathrm{Id} + S) = \alpha + V(S)$.
- $\mathrm{Sp}(T) \subseteq \overline{V(T)}$.
- Actually, $\overline{\mathrm{co}}(\mathrm{Sp}(T)) \subseteq \overline{V(T)}$.
- $\overline{\mathrm{co}}\,\mathrm{Sp}(T) = \bigcap \left\{\overline{V_p(T)}, :\ p \text{ equivalent norm}\right\}$
  where $V_p(T)$ is the numerical range of T in the Banach space (X, p).
- $V(U^{-1}TU) = V(T)$ for every $T \in L(X)$ and every $U \in \mathrm{Iso}(X)$.
- $V(T) \subseteq V(T^*) \subseteq \overline{V(T)}$.

Numerical range: Banach spaces (II)

Some motivation for the numerical range

- It allows to carry from Hilbert spaces to Banach spaces the concepts of hermitian operator, skew-hermitian operator, dissipative operators. . .
- It gives a description of the Lie algebra corresponding to the Lie group of all onto isometries on the space.
- It gives an easy and quantitative proof of the fact that Id is an strongly extreme point of $B_{L(X)}$ (MLUR point).

Numerical radius: definition and properties

Numerical radius

X real or complex Banach space, $T \in L(X)$,

$$\begin{aligned} v(T) &= \sup\left\{|\lambda| \ : \ \lambda \in V(T)\right\} \\ &= \sup\left\{|x^*(Tx)| \ : \ x^* \in S_{X^*},\ x \in S_X,\ x^*(x) = 1\right\} \end{aligned}$$

Elementary properties

X Banach space, $T \in L(X)$

- $v(\cdot)$ is a seminorm, i.e.
  - $v(T+S) \leqslant v(T) + v(S)$ for every $T, S \in L(X)$.
  - $v(\lambda\, T) = |\lambda|\, v(T)$ for every $\lambda \in \mathbb{K}$, $T \in L(X)$.
- $\sup|\mathrm{Sp}(T)| \leqslant v(T)$.
- $v(U^{-1}TU) = v(T)$ for every $U \in \mathrm{Iso}(X)$.
- $v(T^*) = v(T)$.

Numerical radius: examples

Some examples

1. H real Hilbert space $\dim(H) > 1$
   $\Longrightarrow$ exist $T \in L(X)$ with $v(T) = 0$ and $\|T\| = 1$.
2. H complex Hilbert space $\dim(H) > 1$
   - $v(T) \geqslant \frac{1}{2}\|T\|$,
   - the constant $\dfrac{1}{2}$ is optimal.
3. $X = L_1(\mu) \implies v(T) = \|T\|$ for every $T \in L(X)$.
4. $X^* \equiv L_1(\mu) \implies v(T) = \|T\|$ for every $T \in L(X)$.
5. In particular, this is the case for $X = C(K)$.

Numerical radius: real and complex spaces

Example

The numerical range depends on the base field: X complex Banach space, $X_{\mathbb{R}}$ real space underlying X, define $T \in L(X_{\mathbb{R}})$ by

$$T(x) = i\,x \qquad (x \in X).$$

- $\|T\| = 1$ and $v(T) = 0$ if viewed in $X_{\mathbb{R}}$.
- $\|T\| = 1$ and $V(T) = \{i\}$, so $v(T) = 1$ if viewed in (complex) X.

Theorem (Bohnenblust-Karlin; Glickfeld)

X complex Banach space, $T \in L(X)$:

$$v(T) \geqslant \frac{1}{\mathrm{e}}\|T\|.$$

The constant $\dfrac{1}{\mathrm{e}}$ is optimal:

$\exists$ X two-dimensional complex, $\exists$ $T \in L(X)$ with $\|T\| = \mathrm{e}$ and $v(T) = 1$.

Numerical ranges and surjective isometries

3. Numerical ranges and surjective isometries
   - Relationship with semigroups of operators
     - Finite-dimensional spaces
     - Isometries and duality

M. Martín
The group of isometries of a Banach space and duality.
J. Funct. Anal. (2008).

M. Martín, J. Merí, and A. Rodríguez-Palacios.
Finite-dimensional spaces with numerical index zero.
Indiana U. Math. J. (2004).

H. P. Rosenthal
The Lie algebra of a Banach space.
in: *Banach spaces* (Columbia, Mo., 1984), LNM, Springer, 1985.

Relationship with semigroups of operators

A motivating example

A real or complex $n \times n$ matrix. TFAE:

- A is skew-adjoint (i.e. $A^* = -A$).
- $\operatorname{Re}(Ax \mid x) = 0$ for every $x \in H$.
- $B = \exp(\rho A)$ is unitary for every $\rho \in \mathbb{R}$ (i.e. $B^*B = BB^* = \mathrm{Id}$).

In term of Hilbert spaces

H (n-dimensional) Hilbert space, $T \in L(H)$. TFAE:

- $\operatorname{Re} W(T) = \{0\}$.
- $\exp(\rho T) \in \operatorname{Iso}(H)$ for every $\rho \in \mathbb{R}$.

For general Banach spaces

X Banach space, $T \in L(X)$. TFAE:

- $\operatorname{Re} V(T) = \{0\}$.
- $\exp(\rho T) \in \operatorname{Iso}(X)$ for every $\rho \in \mathbb{R}$.

Characterizing uniformly continuous semigroups of operators

Theorem (Bonsall-Duncan, 1970's; Rosenthal, 1984)

X real or complex Banach space, $T \in L(X)$. TFAE:

- $\operatorname{Re} V(T) = \{0\}$ (T is skew-hermitian).
- $\|\exp(\rho T)\| \leqslant 1$ for every $\rho \in \mathbb{R}$.
- $\{\exp(\rho T) \ : \ \rho \in \mathbb{R}_0^+\} \subset \operatorname{Iso}(X)$.
- T belongs to the tangent space to $\operatorname{Iso}(X)$ at Id.
- $\displaystyle\lim_{\rho\to 0} \frac{\|\operatorname{Id} + \rho\, T\| - 1}{\rho} = 0$.

Main consequence

If X is a real Banach space such that

$$v(T) = 0 \quad \Longrightarrow \quad T = 0,$$

then $\operatorname{Iso}(X)$ is "small":

- it does not contain any uniformly continuous one-parameter semigroups,
- the tangent space of $\operatorname{Iso}(X)$ at Id is zero.

Isometries on finite-dimensional spaces

Theorem (Rosenthal, 1984)

X real finite-dimensional Banach space. TFAE:

- $\mathrm{Iso}(X)$ is infinite.
- There is $T \in L(X)$, $T \neq 0$, with $V(T) = \{0\}$.

Theorem (Rosenthal, 1984; M.–Merí–Rodríguez-Palacios, 2004)

X finite-dimensional real space. TFAE:

- $\mathrm{Iso}(X)$ is infinite.
- $X = X_0 \oplus X_1 \oplus \cdots \oplus X_n$ such that
  - X_0 is a (possible null) real space,
  - $X_1, \ldots, X_n$ are non-null complex spaces,

  there are $\rho_1, \ldots, \rho_n$ **rational** numbers, such that

$$\left\| x_0 + \mathrm{e}^{i\,\rho_1\,\theta}\, x_1 + \cdots + \mathrm{e}^{i\,\rho_n\,\theta}\, x_n \right\| = \left\| x_0 + x_1 + \cdots + x_n \right\|$$

  for every $x_i \in X_i$ and every $\theta \in \mathbb{R}$.

Isometries on finite-dimensional spaces II

Remark

- The theorem is due to Rosenthal, but with real ρ's.
- The fact that the ρ's may be chosen as rational numbers is due to M.–Merí–Rodríguez-Palacios.

Corollary

X real space with infinitely many isometries.

- If $\dim(X) = 2$, then $X \equiv \mathbb{C}$.
- If $\dim(X) = 3$, then $X \equiv \mathbb{R} \oplus \mathbb{C}$ (absolute sum).

Example

$X = (\mathbb{R}^4, \|\cdot\|)$, $\|(a,b,c,d)\| = \dfrac{1}{4}\displaystyle\int_0^{2\pi} \left|\operatorname{Re}\left(e^{2it}(a+ib) + e^{it}(c+id)\right)\right| \, dt.$

Then, $\operatorname{Iso}(X)$ is infinite but the unique possible decomposition is $X = \mathbb{C} \oplus \mathbb{C}$ with

$$\left\| e^{it}x_1 + e^{2it}x_2 \right\| = \|x_1 + x_2\|.$$

The Lie-algebra of a Banach space

Lie-algebra

X real Banach space, $\mathcal{Z}(X) = \big\{T \in L(X) \ : \ V(T) = \{0\}\big\}$.

- When X is finite-dimensional, $\operatorname{Iso}(X)$ is a Lie-group and $\mathcal{Z}(X)$ is the tangent space (i.e. its Lie-algebra).

Remark

If $\dim(X) = n$, then

$$0 \leqslant \dim(\mathcal{Z}(X)) \leqslant \frac{n(n-1)}{2}.$$

An open problem

Given $n \geqslant 3$, which are the possible $\dim\big(\mathcal{Z}(X)\big)$ over all n-dimensional X's?

Observation (Javier Merí, PhD)

When $\dim(X) = 3$, $\dim(\mathcal{Z}(X))$ cannot be 2.

Semigroups of surjective isometries and duality

Remark

X Banach space.

- $T \in \mathrm{Iso}(X) \implies T^* \in \mathrm{Iso}(X^*)$.
- $\mathrm{Iso}(X^*)$ can be bigger than $\mathrm{Iso}(X)$.

The problem

- How much bigger can be $\mathrm{Iso}(X^*)$ than $\mathrm{Iso}(X)$?
- Is it possible that $\mathcal{Z}\big(\mathrm{Iso}(X^*)\big)$ is big while $\mathcal{Z}\big(\mathrm{Iso}(X)\big)$ is trivial?

The answer is yes. This is what we are going to present next.

Semigroups of surjective isometries and duality

The construction (M., 2008)

$E \subset C(\Delta)$ separable Banach space. We consider the Banach space

$$C_E([0,1]\|\Delta) = \{f \in C[0,1] \ : \ f|_\Delta \in E\}\,.$$

Then, every $T \in L\big(C_E([0,1]\|\Delta)\big)$ satisfies $\sup|V(T)| = \|T\|$ and

$$C_E([0,1]\|\Delta)^* \equiv E^* \oplus_1 L_1(\mu).$$

The main consequence

Take $E = \ell_2$ (real). Then

- $\operatorname{Iso}\big(C_{\ell_2}([0,1]\|\Delta)\big)$ is "small" (there is no uniformly continuous semigroups).
- Since $C_{\ell_2}([0,1]\|\Delta)^* \equiv \ell_2 \oplus_1 L_1(\mu)$, given $S \in \operatorname{Iso}(\ell_2)$, the operator

$$T = \begin{pmatrix} S & 0 \\ 0 & \mathrm{Id} \end{pmatrix} \in \operatorname{Iso}\big(C_{\ell_2}([0,1]\|\Delta)^*\big)\,.$$

- Therefore, $\operatorname{Iso}\big(C_{\ell_2}([0,1]\|\Delta)^*\big)$ contains infinitely many uniformly continuous semigroups of isometries.

Some comments

In terms of linear dynamical systems

- In $C_{\ell_2}([0,1]\|\Delta)$ there is no $A \in L(X)$ such that the solution to the linear dynamical system
$$x' = A\,x \qquad \left(x : \mathbb{R}_0^+ \longrightarrow C_{\ell_2}([0,1]\|\Delta)\right)$$
(which is $x(t) = \exp(t\,A)(x(0)))$ is given by a semigroup of isometries.
- There are infinitely many such A's in $C_{\ell_2}([0,1]\|\Delta)^*$, in $C_{\ell_2}([0,1]\|\Delta)^{**}\ldots$

Further results (Koszmider–M.–Merí., 2009)

- There are unbounded As on $C_{\ell_2}([0,1]\|\Delta)$ such that the solution to the linear dynamical system
$$x'(t) = A\,x(t)$$
is a one-parameter C_0 semigroup of isometries.
- There is $\mathcal{X}$ such that
$\operatorname{Iso}(\mathcal{X}) = \{-\mathrm{Id}, \mathrm{Id}\}$ and $\mathcal{X}^* = \ell_2 \oplus_1 L_1(\nu)$.
- Therefore, there is no semigroups in $\operatorname{Iso}(\mathcal{X})$, but there are infinitely many exponential one-parameter semigroups in $\operatorname{Iso}(\mathcal{X}^*)$.

Numerical index of Banach spaces

④ Numerical index of Banach spaces
  - Basic definitions and examples
  - Stability properties
  - Duality
  - The isomorphic point of view

V. Kadets, M. Martín, and R. Payá.
Recent progress and open questions on the numerical index of Banach spaces.
RACSAM (2006)

Numerical index of Banach spaces: definitions

Numerical radius

X Banach space, $T \in L(X)$. The numerical radius of T is

$$v(T) = \sup\left\{|x^*(Tx)| \ : \ x^* \in S_{X^*},\ x \in S_X,\ x^*(x) = 1\right\}$$

Remark

The numerical radius is a continuous seminorm in $L(X)$. Actually, $v(\cdot) \leqslant \|\cdot\|$

Numerical index (Lumer, 1968)

X Banach space, the numerical index of X is

$$\begin{aligned} n(X) &= \inf\left\{ v(T) \ : \ T \in L(X),\ \|T\| = 1\right\} \\ &= \max\left\{ k \geqslant 0 \ : \ k\,\|T\| \leqslant v(T) \ \ \forall\ T \in L(X)\right\} \\ &= \inf\left\{M \geqslant 0 \ : \ \exists T \in L(X),\, \|T\| = 1,\ \|\exp(\rho T)\| \leqslant e^{\rho M}\ \forall \rho \in \mathbb{R}\right\} \end{aligned}$$

Numerical index of Banach spaces: basic properties

Some basic properties

- $n(X) = 1$ iff v and $\|\cdot\|$ coincide.
- $n(X) = 0$ iff v is not an equivalent norm in $L(X)$
- X complex $\Rightarrow$ $n(X) \geqslant 1/\mathrm{e}$.

  (Bohnenblust–Karlin, 1955; Glickfeld, 1970)
- Actually,

$$\{n(X) \ : \ X \text{ complex}, \ \dim(X) = 2\} = [\mathrm{e}^{-1}, 1]$$
$$\{n(X) \ : \ X \text{ real}, \ \dim(X) = 2\} = [0, 1]$$

(Duncan–McGregor–Pryce–White, 1970)

Numerical index of Banach spaces: examples (I)

Some examples

1. H Hilbert space, $\dim(H) > 1$,

$$\begin{aligned} n(H) &= 0 && \text{if } H \text{ is real} \\ n(H) &= 1/2 && \text{if } H \text{ is complex} \end{aligned}$$

2. $n\big(L_1(\mu)\big) = 1$ $\quad\mu$ positive measure
   $n\big(C(K)\big) = 1$ $\quad K$ compact Hausdorff space

   (Duncan et al., 1970)

3. If A is a C^*-algebra $\Rightarrow \begin{cases} n(A) = 1 & A \text{ commutative} \\ n(A) = 1/2 & A \text{ not commutative} \end{cases}$

   (Huruya, 1977; Kaidi–Morales–Rodríguez, 2000)

4. If A is a function algebra $\Rightarrow$ $n(A) = 1$

   (Werner, 1997)

Numerical index of Banach spaces: some examples (II)

More examples

5. For $n \geqslant 2$, the unit ball of X_n is a $2n$ regular polygon:

$$n(X_n) = \begin{cases} \tan\left(\frac{\pi}{2n}\right) & \text{if } n \text{ is even,} \\ \\ \sin\left(\frac{\pi}{2n}\right) & \text{if } n \text{ is odd.} \end{cases}$$

(M.–Merí, 2007)

6. Every finite-codimensional subspace of $C[0,1]$ has numerical index 1

(Boyko–Kadets–M.–Werner, 2007)

Numerical index of Banach spaces: some examples (III)

Even more examples

7. Numerical index of L_p-spaces, $1 < p < \infty$:
   - $n\big(L_p[0,1]\big) = n(\ell_p) = \lim\limits_{m\to\infty} n\big(\ell_p^{(m)}\big)$.

     (Ed-Dari, 2005 & Ed-Dari-Khamsi, 2006)
   - $n\big(\ell_p^{(2)}\big)$ **?**
   - In the real case,

$$\max\left\{\frac{1}{2^{1/p}},\ \frac{1}{2^{1/q}}\right\} M_p \leqslant n\big(\ell_p^{(2)}\big) \leqslant M_p$$

and $M_p = v\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \max\limits_{t\in[0,1]} \dfrac{|t^{p-1} - t|}{1 + t^p}$

(M.–Merí, 2009)

   - In the real case, $n\big(L_p(\mu)\big) \geqslant \dfrac{M_p}{8\mathrm{e}}$.
   - In particular, $n\big(L_p(\mu)\big) > 0$ for $p \neq 2$.

(M.–Merí–Popov, 2009)

Numerical index: open problems on computing

Open problems

1. Compute $n\big(L_p[0,1]\big)$ for $1 < p < \infty$, $p \neq 2$.
2. Is $n\big(\ell_p^{(2)}\big) = M_p$ (real case) **?**
3. Is $n\big(\ell_p^{(2)}\big) = p^{-\frac{1}{p}}\, q^{-\frac{1}{q}}$ (complex case) **?**
4. Compute the numerical index of real C^*-algebras.
5. Compute the numerical index of more classical Banach spaces: $C^m[0,1]$, $\mathrm{Lip}(K)$, Lorentz spaces, Orlicz spaces. . .

Stability properties

Direct sums of Banach spaces (M.–Payá, 2000)

$$n\Big(\big[\oplus_{\lambda\in\Lambda} X_\lambda\big]_{c_0}\Big) = n\Big(\big[\oplus_{\lambda\in\Lambda} X_\lambda\big]_{\ell_1}\Big) = n\Big(\big[\oplus_{\lambda\in\Lambda} X_\lambda\big]_{\ell_\infty}\Big) = \inf_\lambda n(X_\lambda)$$

Vector-valued function spaces (López-M.-Merí-Payá-Villena, 2000's)

E Banach space, μ positive σ-finite measure, K compact space. Then

$$n\big(C(K,E)\big) = n\big(C_w(K,E)\big) = n\big(L_1(\mu,E)\big) = n\big(L_\infty(\mu,E)\big) = n(E),$$

and $n\big(C_{w^*}(K,E^*)\big) \leqslant n(E)$

L_p-spaces (Askoy–Ed-Dari–Khamsi, 2007)

$$n\big(L_p([0,1],E)\big) = n\big(\ell_p(E)\big) = \lim_{m\to\infty} n\big(E \oplus_p \overset{m}{\cdots} \oplus_p E\big).$$

Numerical index and duality

Proposition

X Banach space, $T \in L(X)$. Then

- $\sup \operatorname{Re} V(T) = \lim\limits_{\alpha \to 0^+} \dfrac{\|\mathrm{Id} + \alpha T\| - 1}{\alpha}$.
- Then, $v(T^*) = v(T)$ for every $T \in L(X)$.
- Therefore, $n(X^*) \leqslant n(X)$.

(Duncan–McGregor–Pryce–White, 1970)

Question (From the 1970's)

Is $n(X) = n(X^*)$ **?**

Negative answer (Boyko–Kadets–M.–Werner, 2007)

Consider the space

$$X = \big\{(x,y,z) \in c \oplus_\infty c \oplus_\infty c \ : \ \lim x + \lim y + \lim z = 0\big\}.$$

Then, $n(X) = 1$ but $n(X^*) < 1$.

Numerical index and duality (II)

The above example can be squeezed to get more counterexamples.

Example 1

- Exists X real with $n(X) = 1$ and $n(X^*) = 0$.
- Exists X complex with $n(X) = 1$ and $n(X^*) = 1/\mathrm{e}$.

Example 2

- Given $t \in]0,1]$, exists X real with $n(X) = t$ and $n(X^*) = 0$.
- Given $t \in]1/\mathrm{e},1]$, exists X complex with $n(X) = t$ and $n(X^*) = 1/\mathrm{e}$.

Numerical index and duality (III)

Some positive partial answers

One has $n(X) = n(X^*)$ when

- X is reflexive (evident).
- X is a C^*-algebra or a von Neumann predual (1970's – 2000's).
- X is L-embedded in X^{**} (M., 2009).
- If X has RNP and $n(X) = 1$, then $n(X^*) = 1$ (M., 2002).
- If X is M-embedded in X^{**} and $n(X) = 1$
  $\Longrightarrow$ $n(Y) = 1$ for $X \subseteq Y \subseteq X^{**}$.

Example

$X = C_{K(\ell_2)}([0,1]\|\Delta)$. Then $n(X) = 1$ and

$$X^* \equiv K(\ell_2)^* \oplus_1 C_0(K\|\Delta)^* \qquad \text{and} \qquad X^{**} \equiv L(\ell_2) \oplus_\infty C_0(K\|\Delta)^{**}.$$

Therefore, X^{**} is a C^*-algebra, but $n(X^*) = 1/2 < n(X) = 1$.

Numerical index and duality: open problems

Main question

Find isometric or isomorphic properties assuring that $n(X) = n(X^*)$.

Question 1

If Z has a unique predual X, does $n(X) = n(X^*)$ **?**

Question 2

Z dual space, does there exists a predual X such that $n(X) = n(X^*)$ **?**

Question 4

If X has the RNP, does $n(X) = n(X^*)$ **?**

The isomorphic point of view

Renorming and numerical index (Finet–M.–Payá, 2003)

$(X, \|\cdot\|)$ (separable or reflexive) Banach space. Then

- Real case:
$$[0,1[\subseteq \{n(X,|\cdot|) \ : \ |\cdot| \simeq \|\cdot\|\}$$
- Complex case:
$$[\mathrm{e}^{-1},1[\subseteq \{n(X,|\cdot|) \ : \ |\cdot| \simeq \|\cdot\|\}$$

Open question

The result is known to be true when X has a long biorthogonal system.
Is it true in general **?**

Remark

In some sense, any other value of $n(X)$ but $\mathbf{1}$ is isomorphically trivial.
★ What about the value $\mathbf{1}$ **?**

Banach spaces with numerical index one

5. Banach spaces with numerical index one
   - How to deal with numerical index 1 property?
   - The old approach: working with weaker properties
   - The new approach: stronger properties
   - The link: slicely countably determined Banach spaces

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RACSAM (2006)

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Banach spaces with numerical index one

Numerical index 1

Recall that X has numerical index one ($n(X) = 1$) iff

$$\|T\| = \sup\{|x^*(Tx)| \ : \ x \in S_X, \ x^* \in S_{X^*}, \ x^*(x) = 1\}$$

(i.e. $v(T) = \|T\|$) for every $T \in L(X)$.

Examples

$C(K)$, $L_1(\mu)$, $A(\mathbb{D})$, H^∞, finite-codimensional subspaces of $C[0,1]$...

Leading questions (still unsolved)

Let X be an infinite-dimensional Banach space with $n(X) = 1$.

- Can X be smooth or strictly convex **?**
- Does X contain c_0 or ℓ_1 **?**

What we are going to show

Let X be an infinite-dimensional real Banach space with $n(X) = 1$.

- X^* cannot be smooth nor strictly convex.
- $X^* \supseteq \ell_1$.

How to deal with numerical index 1 property?

One the one hand: weaker properties

- In a general Banach space, we only can construct compact (actually, nuclear) operators.
- Actually, we only may easily calculate the norm of rank-one operators.
- All the results given before for Banach spaces in which we use numerical index 1 only need
  $$v(T) = \|T\| \text{ for every rank-one operator } T.$$
- This is called the alternative Daugavet property (ADP).

One the other hand: stronger properties

- We do not know any operator-free characterization of Banach spaces with numerical index 1.
- When we know that a Banach space has numerical index 1 (or that it can be renormed with numerical index 1), we actually prove more.
- There are some sufficient geometrical conditions.
- The weakest property is called lushness.

How to deal with numerical index 1 property?

Relationship between the properties

- One of the key ideas to get interesting results for Banach spaces with numerical index 1 is to study when the three properties below are equivalent.
- A very interesting property appears: the slicely countably determination.

$$\begin{tikzcd}
\boxed{\text{lushness}} \arrow[r, Rightarrow, shift left] & \boxed{\text{Numerical index 1}} \arrow[l, Rightarrow, shift left, "/" marking] \arrow[r, Rightarrow, shift left] & \boxed{\text{ADP}} \arrow[l, Rightarrow, shift left, "/" marking] \arrow[ll, Rightarrow, bend left, "\text{with SCD property}"', "\text{(RNP, Asplund...)}"]
\end{tikzcd}$$

5 Banach spaces with numerical index one

- How to deal with numerical index 1 property?
- The old approach: working with weaker properties
- The new approach: stronger properties
- The link: slicely countably determined Banach spaces

Isomorphic properties (prohibitive results)

Question

Does every Banach space admit an equivalent norm with numerical index 1 **?**

Negative answer (López–M.–Payá, 1999)

Not every real Banach space can be renormed to have numerical index 1.
Concretely:

- If X is real, reflexive, and $\dim(X) = \infty$, then $n(X) < 1$.
- Actually, if X is real, X^{**}/X separable and $n(X) = 1$, then X is finite-dimensional.
- Moreover, if X is real, RNP, $\dim(X) = \infty$, and $n(X) = 1$, then $X \supset \ell_1$.

Proving the 1999 results (I)

Lemma

X Banach space, $n(X) = 1$
$\Longrightarrow$ $|x_0^*(x_0)| = 1$ for all $x_0^* \in \operatorname{ext}(B_{X^*})$ and all denting point x_0 of B_X.

Proof:

- Fix $\varepsilon > 0$. As x_0 denting point, $\exists y^* \in S_{X^*}$ and $\alpha > 0$ such that
$$\|z - x_0\| < \varepsilon \quad \text{whenever } z \in B_{X^*} \text{ satisfies } \operatorname{Re} y^*(z) > 1 - \alpha.$$
- (Choquet's lemma): $x_0^* \in \operatorname{ext}(B_{X^*})$, $\exists y \in S_X$ and $\beta > 0$ such that
$$|z^*(x_0) - x_0^*(x_0)| < \varepsilon \quad \text{whenever } z^* \in B_{X^*} \text{ satisfies } \operatorname{Re} z^*(y) > 1 - \beta.$$
- Let $T = y^* \otimes y \in L(X)$. $\|T\| = 1 \implies v(T) = 1$.
- We may find $x \in S_X$, $x^* \in S_{X^*}$, such that
$$x^*(x) = 1 \qquad \text{and} \qquad |x^*(Tx)| = |y^*(x)||x^*(y)| > 1 - \min\{\alpha, \beta\}.$$
- By choosing suitable $s, t \in \mathbb{T}$ we have
$$\operatorname{Re} y^*(sx) = |y^*(x)| > 1 - \alpha \qquad \& \qquad \operatorname{Re} tx^*(y) = |x^*(y)| > 1 - \beta.$$
- It follows that $\|sx - x_0\| < \varepsilon$ and $|tx^*(x_0) - x_0^*(x_0)| < \varepsilon$, and so
$$\begin{aligned} 1 - |x_0^*(x_0)| &\leqslant |tx^*(sx) - x_0^*(x_0)| \leqslant \\ &\leqslant |tx^*(sx) - tx^*(x_0)| + |tx^*(x_0) - x_0^*(x_0)| < 2\varepsilon. \checkmark \end{aligned}$$

Proving the 1999 results (II)

Proposition

X real, $A \subset S_X$ infinite with $|x^*(a)| = 1\ \forall x^* \in \operatorname{ext}(B_{X^*})$, $\forall a \in A$.
$\Longrightarrow\ X \supseteq c_0$ or $X \supseteq \ell_1$.

Proof:

- $X \supseteq \ell_1$ ✓
- (Rosenthal ℓ_1-theorem): Otherwise, $\exists\ \{a_n\} \subseteq A$ non-trivial weak Cauchy.
- Consider Y the closed linear span of $\{a_n\ :\ n \in \mathbb{N}\}$.
- $\|a_n - a_m\| = 2$ if $n \neq m \implies \dim(Y) = \infty$.
- (Krein-Milman theorem): every $y^* \in \operatorname{ext}(B_{Y^*})$ has an extension which belongs to $\operatorname{ext}(B_{X^*})$.
- So, $|y^*(a_n)| = 1\ \forall y^* \in \operatorname{ext}(B_{Y^*})$, $\forall n \in \mathbb{N}$.
- $\{a_n\}$ weak Cauchy $\implies$ $\{y^*(a_n)\}$ is eventually 1 or -1.
- Then $\operatorname{ext}(B_{Y^*}) = \bigcup_{k\in\mathbb{N}} (E_k \cup -E_k)$ where

$$E_k = \{y^* \in \operatorname{ext}(B_{Y^*})\ :\ y^*(a_n) = 1 \text{ for } n \geqslant k\}.$$

- $\{a_n\}$ separates points of Y^* $\implies$ E_k finite, so $\operatorname{ext}(B_{Y^*})$ countable.
- (Fonf): $Y \supseteq c_0$. So, $X \supseteq c_0$. ✓

Proving the 1999 results (III)

Lemma

X Banach space, $n(X) = 1$
$\implies |x_0^*(x_0)| = 1$ for all $x_0^* \in \operatorname{ext}(B_{X^*})$ and all denting point x_0 of B_X.

Proposition

X real, $A \subset S_X$ infinite with $|x^*(a)| = 1 \ \forall x^* \in \operatorname{ext}(B_{X^*})$, $\forall a \in A$.
$\implies X \supseteq c_0$ or $X \supseteq \ell_1$.

Main consequence

X real, RNP, $\dim(X) = \infty$, and $n(X) = 1 \implies X \supseteq \ell_1$.

Corollary

X real, $\dim(X) = \infty$, $n(X) = 1$.

- X is not reflexive.
- X^{**}/X is non-separable.

Isometric properties: finite-dimensional spaces

Finite-dimensional spaces (McGregor, 1971; Lima, 1978)

X real or complex finite-dimensional space. TFAE:

- $n(X) = 1$.
- $|x^*(x)| = 1$ for every $x^* \in \operatorname{ext}(B_{X^*})$, $x \in \operatorname{ext}(B_X)$.
- $B_X = \operatorname{aconv}(F)$ for every maximal convex subset F of S_X (X is a CL-space).

Remark

This shows a rough behavior of the norm of a finite-dimensional space with numerical index 1:

- The space is not smooth.
- The space is not strictly convex.

Question

What is the situation in the infinite-dimensional case **?**

Isometric properties: infinite-dimensional spaces

Theorem (Kadets–M.–Merí–Payá, 2009)

X infinite-dimensional Banach space, $n(X)=1$. Then

- X^* is neither smooth nor strictly convex.
- The norm of X cannot be Fréchet-smooth.
- There is no WLUR points in S_X.

Corollary

$X=C(\mathbb{T})/A(\mathbb{D})$. $X^*=H^1$ is smooth $\implies$ $n(X)<1$ & $n(H^1)<1$.

Example without completeness

- There is X (non-complete) strictly convex with $X^*\equiv L_1(\mu)$, so $n(X)=1$.
- $\widetilde{X}$ completion of X. For $F\subseteq S_{\widetilde{X}}$ maximal face, $B_{\widetilde{X}}=\overline{\operatorname{aconv}}(F)$.

Open question

Is there X with $n(X)=1$ which is smooth or strictly convex **?**

5 Banach spaces with numerical index one

- How to deal with numerical index 1 property?
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K. Boyko, V. Kadets, M. Martín, and J. Merí.
Properties of lush spaces and applications to Banach spaces with numerical index 1.
Studia Math. (2009).

V. Kadets, M. Martín, J. Merí, and R. Payá.
Smoothness and convexity for Banach spaces with numerical index 1.
Illinois J. Math. (to appear).

Sufficient conditions for numerical index one

Some sufficient conditions

Let X be a Banach space. Consider:

(a) **Lindenstrauss, 1964:** X has the 3.2.I.P. if the intersection of every family of three mutually intersecting balls is not empty.

(b) **Fullerton, 1961:** X is a CL-space if B_X is the absolutely convex hull of every maximal face of S_X.

(c) **Lima, 1978:** X is an almost-CL-space if B_X is the closed absolutely convex hull of every maximal face of S_X.

$$\boxed{\text{(a)}} \;\underset{\not\Longleftarrow}{\Longrightarrow}\; \boxed{\text{(b)}} \;\underset{\not\Longleftarrow}{\Longrightarrow}\; \boxed{\text{(c)}} \;\underset{\not\Longleftarrow}{\Longrightarrow}\; \boxed{n(X)=1}$$

Observation

Showing that (c) $\Longrightarrow$ $n(X)=1$, one realizes that (c) is too much.

Lushness (Boyko–Kadets–M.–Werner, 2007)

X is lush if given $x,y\in S_X$, $\varepsilon>0$, there is $x^*\in S_{X^*}$ such that

$$x\in S(B_X,x^*,\varepsilon) \quad \text{and} \quad \operatorname{dist}\big(y,\operatorname{aconv}\big(S(B_X,x^*,\varepsilon)\big)\big)<\varepsilon.$$

Definition and first property

Lushness (Boyko–Kadets–M.–Werner, 2007)

X is lush if given $x, y \in S_X$, $\varepsilon > 0$, there is $x^* \in S_{X^*}$ such that

$$x \in S(B_X, x^*, \varepsilon) \quad \text{and} \quad \operatorname{dist}\big(y, \operatorname{aconv}\big(S(B_X, x^*, \varepsilon)\big)\big) < \varepsilon.$$

Theorem

X lush $\implies n(X) = 1$.

Proof.

- $T \in L(X)$ with $\|T\| = 1$, $\varepsilon > 0$. Find $y_0 \in S_X$ which $\|Ty_0\| > 1 - \varepsilon$.
- Use lushness for $x_0 = Ty_0/\|Ty_0\|$ and y_0 to get $x^* \in S_{X^*}$ and

$$v = \sum_{i=1}^{n} \lambda_i \theta_i x_i \quad \text{where} \ \ x_i \in S(B_X, x^*, \varepsilon),\ \lambda_i \in [0,1],\ \sum \lambda_i = 1,\ \theta_i \in \mathbb{T},$$

with $\operatorname{Re} x^*(x_0) > 1 - \varepsilon$ and $\|v - y_0\| < \varepsilon$.

- Then $|x^*(Tv)| = \left| x^*(x_0) - x^*\left(T\left(\frac{y_0}{\|Ty_0\|} - v\right)\right)\right| \sim \|T\|$.
- By a convexity argument, $\exists\, i$ such that $|x^*(Tx_i)| \sim \|T\|$ and $\operatorname{Re} x^*(x_i) \sim 1$.
- Then $\max_{\omega \in \mathbb{T}} \|\mathrm{Id} + \omega\, T\| \sim 1 + \|T\| \implies v(T) \sim \|T\|$. ✓

Reformulations of lushness and applications

Proposition

X Banach space. TFAE:

- X is lush,
- Every separable $E \subset X$ is contained in a separable lush Y with $E \subset Y \subset X$.

Separable lush spaces (real case)

X real separable. TFAE:

- X is lush.
- There is $G \subseteq S_{X^*}$ norming such that

$$B_X = \overline{\operatorname{aconv}}\left(\{x \in B_X \ : \ x^*(x) = 1\}\right) \qquad (x^* \in G).$$

Therefore, $|x^{**}(x^*)| = 1 \ \forall x^{**} \in \operatorname{ext}\left(B_{X^{**}}\right) \ \forall x^* \in G$.

Consequences (real case)

- X lush, $\dim(X) > 1 \implies X$ not smooth nor strictly convex.
- $X \subseteq C[0,1]$ strictly convex or smooth $\implies C[0,1]/X$ contains $C[0,1]$.

An important consequence

Proved in the previous slide...

X lush separable, $\dim(X)=\infty \implies$ there is $G\in S_{X^*}$ infinite such that

$$|x^{**}(x^*)|=1 \qquad \big(x^{**}\in \operatorname{ext}\left(B_{X^{**}}\right),\ x^*\in G\big).$$

Proposition (López–M.–Payá, 1999)

X real, $A\subset S_X$ infinite such that

$$|x^*(a)|=1 \qquad \big(x^*\in \operatorname{ext}\left(B_{X^*}\right),\ a\in A\big).$$

Then, $X\supseteq c_0$ or $X\supseteq \ell_1$.

Main consequence

X real lush, $\dim(X)=\infty \implies X^*\supseteq \ell_1$.

Question

What happens if just $n(X)=1$ **?** The same, we will prove later.

Lushness is not equivalent to numerical index one

Example (Kadets–M.–Merí–Shepelska, 2009)

There is a separable Banach space $\mathcal{X}$ such that

- $\mathcal{X}^*$ is lush but $\mathcal{X}$ is not lush.
- Since $n(\mathcal{X}^*) = 1$, also $n(\mathcal{X}) = 1$.
- The set

$$\{x^* \in S_{\mathcal{X}^*} \ : \ |x^{**}(x^*)| = 1 \text{ for every } x^{**} \in \operatorname{ext}(B_{\mathcal{X}^{**}})\}$$

is empty.

Remark

We cannot expect to show that $X^* \supseteq \ell_1$ using the ideas for lush spaces in the general case when $n(X) = 1$.

5 Banach spaces with numerical index one

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A. Avilés, V. Kadets, M. Martín, J. Merí, and V. Shepelska
Slicely Countably Determined Banach spaces
Trans. Amer. Math. Soc. (to appear)

SCD sets and spaces: Definitions and examples

SCD sets

A convex subset of B-space X. A is Slicely Countably Determined (SCD) if there is a sequence $\{S_n : n \in \mathbb{N}\}$ of slices of A satisfying one of the following equivalent conditions:

- every slice of A contains one of the S_n's,
- $A \subseteq \overline{\text{conv}}(B)$ if $B \subseteq A$ satisfies $B \cap S_n \neq \emptyset\ \forall n$,
- given $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \in S_n\ \forall n \in \mathbb{N}$, $A \subseteq \overline{\text{conv}}\big(\{x_n : n \in \mathbb{N}\}\big)$.

SCD spaces

X is Slicely Countably Determined (SCD) if so are its convex bounded subsets.

Examples

1. (Easy): X separable RNP $\Longrightarrow$ X is SCD,
2. (Easy): X separable Asplund $\Longrightarrow$ X is SCD,
3. $C[0,1]$ and $L_1[0,1]$ are not SCD,
4. (Main): $X \not\supseteq \ell_1 \Longrightarrow X$ is SCD.

SCD is a link between ADP and lushness

Theorem

X (separable) SCD,

$$n(X) = 1(\text{actually ADP}) \quad \Longrightarrow \quad X \text{ lush.}$$

Main consequence

X (arbitrary) such that $X \not\supseteq \ell_1$,

$$n(X) = 1(\text{actually ADP}) \quad \Longrightarrow \quad X \text{ lush.}$$

Corollary

X real $+ \dim(X) = \infty + n(X) = 1 \implies X^* \supseteq \ell_1.$

Proof.

- If $X \supseteq \ell_1 \implies X^*$ contains ℓ_∞ as a quotient, so X^* contains ℓ_1 as a quotient, and the lifting property gives $X^* \supseteq \ell_1$ ✓
- If $X \not\supseteq \ell_1 \implies X$ is SCD $+ n(X) = 1$, so X is lush.
- Lush $+ \dim(X) = \infty \implies X^* \supseteq \ell_1$ ✓