Isometries on extremely non-complex Banach spaces

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Introduction: notation, objectives and motivation

Basic notation and main objectives

Notation

X real or complex Banach space.

- S_X unit sphere, B_X closed unit ball.
- X^* dual space.
- $L(X)$ bounded linear operators.
- $W(X)$ weakly compact linear operators.
- $\mathrm{Iso}(X)$ surjective isometries group.

Objective

- Construct spaces X with small $\mathrm{Iso}(X)$ and big $\mathrm{Iso}(X^*)$.
- To cases:
  - $\mathrm{Iso}(X)$ does not have uniformly continuous one-parameter semigroups but $\mathrm{Iso}(X^*) \supset \mathrm{Iso}(\ell_2)$.
  - $\mathrm{Iso}(X) = \{\pm \mathrm{Id}\}$ but $\mathrm{Iso}(X^*) \supset \mathrm{Iso}(\ell_2)$.

Motivation

X Banach space.

Autonomous dynamic system

$$(\diamondsuit) \qquad \begin{cases} x'(t) = Ax(t) \\ x(0) = x_0 \end{cases} \qquad x_0 \in X,\ A \text{ linear closed densely defined.}$$

One-parameter semigroup of operators

$\Phi : \mathbb{R}_0^+ \longrightarrow L(X)$ such that $\Phi(t+s) = \Phi(t)\Phi(s)$ $\forall t, s \in \mathbb{R}_0^+$, $\Phi(0) = \mathrm{Id}$.

- *Uniformly continuous*: $\Phi : \mathbb{R}_0^+ \longrightarrow (L(X), \|\cdot\|)$ continuous.
- *Strongly continuous*: $\Phi : \mathbb{R}_0^+ \longrightarrow (L(X), \mathrm{SOT})$ continuous.

Relationship (Hille-Yoshida, 1950's)

- *Bounded case:*
  - If $A \in L(X) \implies \Phi(t) = \exp(tA)$ solution of $(\diamondsuit)$ uniforly continuous.
  - Φ uniformly continuous $\implies A = \Phi'(0) \in L(X)$ and Φ solution of $(\diamondsuit)$.
- *Unbounded case:*
  - Φ strongly continuous $\implies A = \Phi'(0)$ closed and Φ solution of $(\diamondsuit)$.
  - If $(\diamondsuit)$ has solution Φ strongly continuous $\implies A = \Phi'(0)$ and $\Phi(t) =$ "$\exp(tA)$".

Sketch of the talk

Bounded or uniformly continuous case

M. Martín
The group of isometries of a Banach space and duality.
J. Funct. Anal. (2008).

Hilbert spaces

Hilbert space Numerical range (Toeplitz, 1918)

- A $n \times n$ real or complex matrix

$$W(A) = \Big\{ (Ax \mid x) \ : \ x \in \mathbb{K}^n, \ (x \mid x) = 1 \Big\}.$$

- H real or complex Hilbert space, $T \in L(H)$,

$$W(T) = \Big\{ (Tx \mid x) \ : \ x \in H, \ \|x\| = 1 \Big\}.$$

Some properties

H Hilbert space, $T \in L(H)$:

- $W(T)$ is convex.
- In the complex case, $\overline{W(T)}$ contains the spectrum of T.
- If, moreover, T is normal, $\overline{W(T)} = \overline{\mathrm{co}}\, Sp(T)$.

Banach spaces

Banach space numerical range (Bauer 1962; Lumer, 1961)

X Banach space, $T \in L(X)$,

$$V(T) = \left\{ x^*(Tx) \ : \ x^* \in S_{X^*},\ x \in S_X,\ x^*(x) = 1 \right\}$$

Some properties

X Banach space, $T \in L(X)$:

- $V(T)$ is connected (not necessarily convex).
- In the complex case, $\overline{V(T)}$ contains the spectrum of T.
- Actually,

$$\overline{\mathrm{co}}\, Sp(T) = \bigcap \overline{\mathrm{co}}\, V(T),$$

  the intersection taken over all numerical ranges $V(T)$ corresponding to equivalent norms on X.

Numerical radius

X real or complex Banach space, $T \in L(X)$,

$$v(T) = \sup\Big\{|\lambda| \ : \ \lambda \in V(T)\Big\}.$$

- v is a seminorm with $v(T) \leqslant \|T\|$.
- $v(T) = v(T^*)$ for every $T \in L(X)$.

Numerical index (Lumer, 1968)

X real or complex Banach space,

$$\begin{aligned} n(X) &= \inf\Big\{v(T) \ : \ T \in L(X),\ \|T\| = 1\Big\} \\ &= \max\{k \geqslant 0 \ : \ k\|T\| \leqslant v(T)\ \forall T \in L(X)\Big\}. \end{aligned}$$

Remarks

- $n(X) = 1$ iff $v(T) = \|T\|$ for every $T \in L(X)$.
- If there is $T \neq 0$ with $v(T) = 0$, then $n(X) = 0$.
- If X is complex, then $n(X) \geqslant 1/\mathrm{e}$.

Relationship with semigroups of operators

A motivating example

A real or complex $n \times n$ matrix. TFAE:

- A is skew-adjoint (i.e. $A^* = -A$).
- $\operatorname{Re}(Ax \mid x) = 0$ for every $x \in H$.
- $B = \exp(\rho A)$ is unitary for every $\rho \in \mathbb{R}$ (i.e. $B^*B = \operatorname{Id}$).

In term of Hilbert spaces

H (n-dimensional) Hilbert space, $T \in L(H)$. TFAE:

- $\operatorname{Re} W(T) = \{0\}$.
- $\exp(\rho T) \in \operatorname{Iso}(H)$ for every $\rho \in \mathbb{R}$.

For general Banach spaces

X Banach space, $T \in L(X)$. TFAE:

- $\operatorname{Re} V(T) = \{0\}$.
- $\exp(\rho T) \in \operatorname{Iso}(X)$ for every $\rho \in \mathbb{R}$.

Characterizing uniformly continuous semigroups of operators

Theorem

X real or complex Banach space, $T \in L(X)$. TFAE:

- $\operatorname{Re} V(T) = \{0\}$.
- $\|\exp(\rho T)\| \leqslant 1$ for every $\rho \in \mathbb{R}$.
- $\left\{\exp(\rho T) \ : \ \rho \in \mathbb{R}_0^+\right\} \subset \operatorname{Iso}(X)$.
- T belongs to the tangent space of $\operatorname{Iso}(X)$ at Id, i.e. exists a function $f : [-1,1] \longrightarrow \operatorname{Iso}(X)$ with $f(0) = \operatorname{Id}$ and $f'(0) = T$.
- $\displaystyle\lim_{\rho\to 0} \frac{\|\operatorname{Id} + \rho T\| - 1}{\rho} = 0$, i.e. the derivative or the norm of $L(X)$ at Id in the direction of T is null.

Consequences

- For every $T \in L(X)$
$$\left\|\exp(\rho T)\right\| \leqslant e^{v(T)\,\rho} \qquad \left(\rho \in \mathbb{R}\right)$$
and $v(T)$ is the smaller possibility.
- Then, $n(X) = 1$ is the worst possibility to find uniformly continuous one-parameter semigroups of isometries.

The main example

Spaces $C_E(K\|L)$

K compact, $L \subset K$ closed nowhere dense, $E \subset C(L)$.

$$C_E(K\|L) = \{f \in C(K) : f|_L \in E\}.$$

Theorem

$$C_E(K\|L)^* \equiv E^* \oplus_1 C_0(K\|L)^* \quad \& \quad n\big(C_E(K\|L)\big) = 1.$$

Consequence: the example

Take $K = [0,1]$, $L = \Delta$, $E = \ell_2 \subset C(\Delta)$.

- $\operatorname{Iso}\big(C_{\ell_2}([0,1]\|\Delta)\big)$ has no uniformly continuous one-parameter semigroups.
- $C_{\ell_2}([0,1]\|\Delta)^* \equiv \ell_2 \oplus_1 C_0([0,1]\|\Delta)^*$, so taken $S \in \operatorname{Iso}(\ell_2)$

$$\Longrightarrow \quad T = \begin{pmatrix} S & 0 \\ 0 & \mathrm{Id} \end{pmatrix} \in \operatorname{Iso}\big(C_{\ell_2}([0,1]\|\Delta)^*\big)$$

Then, $\operatorname{Iso}\big(C_{\ell_2}([0,1]\|\Delta)^*\big)$ contains infinitely many uniformly continuous one-parameter semigroups.

Isometries in finite-dimensional spaces

Theorem

X finite-dimensional real space. TFAE:

- $\mathrm{Iso}(X)$ is infinite.
- $n(X) = 0$.
- There is $T \in L(X)$, $T \neq 0$, with $v(T) = 0$.

Examples of spaces of this kind

1. Hilbert spaces.
2. $X_{\mathbb{R}}$, the real space subjacent to any complex space X.
3. An absolute sum of any real space and one of the above.
4. Moreover, if $X = X_0 \oplus X_1$ where X_1 is complex and

$$\left\| x_0 + e^{i\theta} x_1 \right\| = \|x_0 + x_1\| \qquad \big(x_0 \in X_0,\ x_1 \in X_1,\ \theta \in \mathbb{R}\big).$$

(Note that the other 3 cases are included here)

Question

Can every Banach space X with $n(X) = 0$ be decomposed as in 4 ?

Negative answer I

Infinite-dimensional case

There is an infinite-dimensional real Banach space X with $n(X) = 0$ but X is polyhedral. In particular, X does not contain $\mathbb{C}$ isometrically.

An easy example is

$$X = \left[\bigoplus_{n \geqslant 2} X_n \right]_{c_0}$$

X_n is the two-dimensional space whose unit ball is the regular polygon of $2n$ vertices.

Note

Such an example is not possible in the finite-dimensional case.

(Quasi affirmative) negative answer II

Finite-dimensional case

X finite-dimensional real space. TFAE:

- $n(X) = 0$.
- $X = X_0 \oplus X_1 \oplus \cdots \oplus X_n$ such that
  - X_0 is a (possible null) real space,
  - $X_1, \ldots, X_n$ are non-null complex spaces,

  there are $\rho_1, \ldots, \rho_n$ **rational** numbers, such that

$$\left\| x_0 + \mathrm{e}^{i\,\rho_1\,\theta}\, x_1 + \cdots + \mathrm{e}^{i\,\rho_n\,\theta}\, x_n \right\| = \left\| x_0 + x_1 + \cdots + x_n \right\|$$

  for every $x_i \in X_i$ and every $\theta \in \mathbb{R}$.

Remark

- The theorem is due to Rosenthal, but with real ρ's.
- The fact that the ρ's may be chosen as rational numbers is due to M.–Merí–Rodríguez-Palacios.

Consequences

Corollary

X real space with $n(X)=0$.

- If $\dim(X)=2$, then $X\equiv\mathbb{C}$.
- If $\dim(X)=3$, then $X\equiv\mathbb{R}\oplus\mathbb{C}$ (absolute sum).

Natural question

Are all finite-dimensional X's with $n(X)=0$ of the form $X=X_0\oplus X_1$ **?**

Answer

No.

Example

$X=(\mathbb{R}^4,\|\cdot\|)$, $\|(a,b,c,d)\|=\dfrac{1}{4}\displaystyle\int_0^{2\pi}\left|\mathrm{Re}\left(\mathrm{e}^{2it}(a+ib)+\mathrm{e}^{it}(c+id)\right)\right|\,dt.$

Then $n(X)=0$ but the unique possible decomposition is $X=\mathbb{C}\oplus\mathbb{C}$ with

$$\left\|\mathrm{e}^{it}x_1+\mathrm{e}^{2it}x_2\right\|=\|x_1+x_2\|.$$

The Lie-algebra of a Banach space

Lie-algebra

X real Banach space, $\mathcal{Z}(X) = \{T \in L(X) \ : \ v(T) = 0\}$.

- When X is finite-dimensional, $\mathrm{Iso}(X)$ is a Lie-group and $\mathcal{Z}(X)$ is the tangent space (i.e. its Lie-algebra).

Remark

If $\dim(X) = n$, then

$$0 \leqslant \dim(\mathcal{Z}(X)) \leqslant \frac{n(n-1)}{2}.$$

An open problem

Given $n \geqslant 3$, which are the possible $\dim\big(\mathcal{Z}(X)\big)$ over all n-dimensional X's?

Observation (Javier Merí, PhD)

When $\dim(X) = 3$, $\dim(\mathcal{Z}(X))$ cannot be 2.

Numerical index of Banach spaces

Numerical index (Lumer, 1968)

X real or complex Banach space,

$$n(X) = \max\{k \geqslant 0 \ : \ k\|T\| \leqslant v(T) \ \forall T \in L(X)\}.$$

Some examples

1. $C(K)$, $L_1(\mu)$ have numerical index 1.
2. H Hilbert space, $\dim(H) > 1$, then
$$n(H) = 0 \ \text{ real case} \qquad n(H) = \frac{1}{2} \ \text{ complex case.}$$
3. $n(L_p[0,1]) = n(\ell_p)$ but both are unknown.
4. If X_n is the two-dimensional space such that B_{X_n} is a $2n$-polygon, then
$$n(X_n) = \tan\left(\frac{\pi}{2n}\right) \ \text{ if } n \text{ is even} \qquad n(X_n) = \sin\left(\frac{\pi}{2n}\right) \ \text{ if } n \text{ is odd.}$$
5. If X is a C^*-algebra or the predual of a von Neumann algebra, then $n(X) = 1$ if the algebra is commutative and $n(X) = 1/2$ otherwise.

Numerical index and duality

Proposition

X Banach space.

- $v(T^*) = v(T)$ for every $T \in L(X)$.
- Therefore, $n(X^*) \leqslant n(X)$.

Question (1970)

Is it always $n(X) = n(X^*)$ **?**

Some positive partial answers

- When X is reflexive (evident).
- When X is a C^*-algebra or a von Neumann predual (1970's – 2000's).
- When X is L-embedded in X^{**} (2000's).
- If X has RNP and $n(X) = 1$, then $n(X^*) = 1$ (2000's).

Numerical index and duality. II

Answer

The answer is **NO**:

Example (Boyko-Kadets-M.-Werner, 2007)

$X = \{(x,y,z) \in c \oplus_\infty c \oplus_\infty c \ : \ \lim x + \lim y + \lim z = 0\}$.

With the previous construction it is easy to give examples:

Another example

- It is known: if X or X^* is a C^*-algebra, then $n(X) = n(X^*)$.
- Consider $Y = C_{K(\ell_2)}([0,1] \| \Delta)$. Then

$$n(Y) = 1 \qquad \text{and} \qquad Y^* \equiv K(\ell_2)^* \oplus_1 C_0([0,1] \| \Delta)^*.$$

So, $Y^{**} \equiv L(\ell_2) \oplus_\infty C_0([0,1] \| \Delta)^{**}$ is a C^*-algebra but $n(Y^*) \leqslant n\big(K(\ell_2)\big) = 1/2$.

Numerical index and duality. III

Remark

In all the examples there are another predual for which the numerical index coincides with the numerical index of its dual.

Open problems

We look for sufficient conditions assuring the equality between the numerical index of a Banach space and the one of its dual.

1. Asplundness is not such a property.
2. What's about RNP **?**
3. What's about if X^* has a unique predual **?** (it's true for L-embedded).
4. What's about if X does not contains a copy of c_0 **?**

Theorem

X separable Banach space containing (an isomorphic copy of) c_0, then there is an equivalent norm $|\cdot|$ on X such that

$$n\big((X,|\cdot|)^*\big) = 0, 1/\mathrm{e} \quad \text{and} \quad n\big((X,|\cdot|)\big) = 1.$$

Bibliography for the bounded case

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Problems with the unbounded or strongly continuous case

Numerical range of unbounded operators

Numerical range of unbounded operators (1960's)

X Banach space, $T : D(T) \longrightarrow X$ linear,

$$V(T) = \left\{ x^*(Tx) \ : \ x^* \in X^*, \ x \in D(T), \ x^*(x) = \|x^*\| = \|x\| = 1 \right\}.$$

Teorema (Stone, 1932)

H Hilbert space, A densely defined operator. TFAE:

- A generates an strongly continuous one-parameter semigroup of unitary operators (onto isometries).
- $A^* = -A$.
- $\mathrm{Re}(Ax \mid x) = 0$ for every $x \in D(A)$.

Numerical range of unbounded operators. II

Difficulty

Which Banach spaces have unbounded operators with numerical range zero?

Examples

- In $C_0(\mathbb{R})$, $\Phi(t)(f)(s) = f(t+s)$ is an strongly continuous one-parameter semigroup of isometries (generated by the derivative).
- In $C_E([0,1]\|\Delta)$ there are also strongly continuous one-parameter semigroup of isometries.

Consequence

We have to completely change our approach to the problem.

Extremely non-complex Banach spaces: motivation and first examples

P. Koszmider, M. Martín, and J. Merí.
Extremely non-complex $C(K)$ spaces.
J. Math. Anal. Appl. (2009).

Complex structures

Definition

X has complex structure if there is $T \in L(X)$ such that $T^2 = -\mathrm{Id}$.

Some remarks

- This gives a structure of vector space over $\mathbb{C}$:

$$(\alpha + i\beta)x = \alpha x + \beta T(x) \qquad \big(\alpha + i\beta \in \mathbb{C},\ x \in X\big)$$

- Defining

$$|\!|\!| x |\!|\!| = \max\big\{\|e^{i\theta}x\| \ :\ \theta \in [0, 2\pi]\big\} \qquad (x \in X)$$

  one gets that $(X, |\!|\!| \cdot |\!|\!|)$ is a complex Banach space.
- If T is an isometry, then actually the given norm of X is complex.
- Conversely, if X is a complex Banach space, then

$$T(x) = i\,x \qquad \big(x \in X\big)$$

  satisfies $T^2 = -\mathrm{Id}$ and T is an isometry.

Complex structures II

Some examples

1. If $\dim(X) < \infty$, X has complex structure iff $\dim(X)$ is even.
2. If $X \simeq Z \oplus Z$ (in particular, $X \simeq X^2$), then X has complex structure.
3. There are infinite-dimensional Banach spaces without complex structure:
   - **Dieudonné, 1952:** the James' space $\mathcal{J}$ (since $\mathcal{J}^{**} \equiv \mathcal{J} \oplus \mathbb{R}$).
   - **Szarek, 1986:** uniformly convex examples.
   - **Gowers-Maurey, 1993:** their H.I. space.
   - **Ferenczi-Medina Galego, 2007:** there are odd and even infinite-dimensional spaces X.
     - X is even if admits a complex structure but its hyperplanes does not.
     - X is odd if its hyperplanes are even (and so X does not admit a complex structure).

Definition

X is extremely non-complex if $\operatorname{dist}(T^2, -\mathrm{Id})$ is the maximum possible, i.e.

$$\|\mathrm{Id} + T^2\| = 1 + \|T^2\| \qquad \big(T \in L(X)\big)$$

The Daugavet equation

What Daugavet did in 1963

The norm equality

$$\|\mathrm{Id} + T\| = 1 + \|T\|$$

holds for every compact $T \in L\big(C[0,1]\big)$.

The Daugavet equation

X Banach space, $T \in L(X)$, $\|\mathrm{Id} + T\| = 1 + \|T\|$ (DE).

Classical examples

1. **Daugavet, 1963:**
   Every compact operator on $C[0,1]$ satisfies (DE).
2. **Lozanoskii, 1966:**
   Every compact operator on $L_1[0,1]$ satisfies (DE).
3. **Abramovich, Holub, and more, 80's:**
   $X = C(K)$, K perfect compact space
   or $X = L_1(\mu)$, μ atomless measure
   $\Longrightarrow$ every weakly compact $T \in L(X)$ satisfies (DE).

The Daugavet property

The Daugavet property (Kadets–Shvidkoy–Sirotkin–Werner, 1997)

A Banach space X is said to have the Daugavet property iff every rank-one operator on X satisfies (DE).

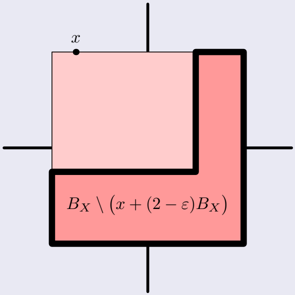
Some results

Let X be a Banach space with the Daugavet property. Then

- Every weakly compact operator on X satisfies (DE).
- X contains ℓ_1.
- X does not embed into a Banach space with unconditional basis.
- **Geometric characterization**: X has the Daugavet property iff for each $x \in S_X$

$$\overline{\mathrm{co}}\Big(B_X \setminus \big(x + (2-\varepsilon)B_X\big)\Big) = B_X\,.$$

(Kadets–Shvidkoy–Sirotkin–Werner, 1997 & 2000)



The Daugavet property II

More examples

The following spaces have the Daugavet property.

- **Wojtaszczyk, 1992:**
  The disk algebra and H^∞.
- **Werner, 1997:**
  "Nonatomic" function algebras.
- **Oikhberg, 2005:**
  Non-atomic C^*-algebras and preduals of non-atomic von Neumann algebras.
- **Becerra–M., 2005**:
  Non-atomic JB^*-triples and their preduals.
- **Becerra–M., 2006:**
  Preduals of $L_1(\mu)$ without Fréchet-smooth points.
- **Ivankhno, Kadets, Werner, 2007:**
  Lip(K) when $K \subseteq \mathbb{R}^n$ is compact and convex.

Daugavet–type inequalities

Some examples

- **Benyamini–Lin, 1985:**
  For every $1 < p < \infty$, $p \neq 2$, there exists $\psi_p : (0,\infty) \longrightarrow (0,\infty)$ such that
  $$\|\mathrm{Id} + T\| \geqslant 1 + \psi_p(\|T\|)$$
  for every compact operator T on $L_p[0,1]$.
  - If $p = 2$, then there is a non-null compact T on $L_2[0,1]$ such that
    $$\|\mathrm{Id} + T\| = 1.$$
- **Boyko–Kadets, 2004:**
  If ψ_p is the best possible function above, then
  $$\lim_{p\to 1^+} \psi_p(t) = t \qquad (t > 0).$$
- **Oikhberg, 2005:**
  If $K(\ell_2) \subseteq X \subseteq L(\ell_2)$, then
  $$\|\mathrm{Id} + T\| \geqslant 1 + \frac{1}{8\sqrt{2}}\|T\|$$
  for every compact T on X.

Norm equalities for operators

Motivating question

Are there other norm equalities which could define interesting properties of Banach spaces **?**

Concretely

We looked for non-trivial norm equalities of the forms

$$\|g(T)\| = f(\|T\|) \qquad \text{or} \qquad \|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$$

(g analytic, f arbitrary) satisfied by all rank-one operators on a Banach space.

Solution

We proved that there are few possibilities. . .

Norm equalities for operators: Occlusive results

Theorem

X real or complex with $\dim(X) \geqslant 2$.
Suppose that the norm equality

$$\|g(T)\| = f(\|T\|)$$

holds for every rank-one operator $T \in L(X)$, where

- $g : \mathbb{K} \longrightarrow \mathbb{K}$ is analytic,
- $f : \mathbb{R}_0^+ \longrightarrow \mathbb{R}$ is arbitrary.

Then, there are $a, b \in \mathbb{K}$ such that

$$g(\zeta) = a + b\zeta \qquad (\zeta \in \mathbb{K}).$$

Corollary

Only three norm equalities of the form

$$\|g(T)\| = f(\|T\|)$$

are possible:

- $b = 0$: $\|a\,\mathrm{Id}\| = |a|$,
- $a = 0$: $\|bT\| = |b|\,\|T\|$,

  (trivial cases)
- $a \neq 0$, $b \neq 0$:
  $\|a\,\mathrm{Id} + bT\| = |a| + |b|\,\|T\|$,

  (Daugavet property)

Norm equalities for operators: Occlusive results II

Theorem

X complex with $\dim(X) \geqslant 2$. Suppose that the norm equality

$$\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$$

holds for every rank-one operator $T \in L(X)$, where

- $g : \mathbb{C} \longrightarrow \mathbb{C}$ is analytic, non constant and with $g(0) = 0$,
- $f : \mathbb{R}_0^+ \longrightarrow \mathbb{R}$ is *continuous*.

Then, X has the Daugavet property

Remarks

- We do not know if the result is true in the real case.
- It is true if g is onto.
- Even the simplest case, $g(t) = t^2$, is not solved. The only known thing is that, in this case, $f(t) = 1 + t$, leading to the equation

$$\|\mathrm{Id} + T^2\| = 1 + \|T^2\|$$

The question

Godefroy, private communication

Is there any real Banach space X (with $\dim(X) > 1$) such that

$$\|\mathrm{Id} + T^2\| = 1 + \|T^2\|$$

for every operator $T \in L(X)$ **?**

In other words, are there extremely non-complex Banach spaces other than $\mathbb{R}$ **?**

The first attempts

The first idea

We may try to check whether the known spaces without complex structure are actually extremely non-complex.

Some examples

1. If $\dim(X) < \infty$, X has complex structure iff $\dim(X)$ is even.
2. **Dieudonné, 1952:** the James' space $\mathcal{J}$ (since $\mathcal{J}^{**} \equiv \mathcal{J} \oplus \mathbb{R}$).
3. **Szarek, 1986:** uniformly convex examples.
4. **Gowers-Maurey, 1993:** their H.I. space.
5. **Ferenczi-Medina Galego, 2007:** there are odd and even infinite-dimensional spaces X.
    - X is even if admits a complex structure but its hyperplanes does not.
    - X is odd if its hyperplanes are even (and so X does not admit a complex structure).

(Un)fortunately...

This did not work and we moved to $C(K)$ spaces.

The first example: weak multiplications

Weak multiplication

Let K be a compact space. $T \in L\big(C(K)\big)$ is a weak multiplication if

$$T = g\,\mathrm{Id} + S$$

where $g \in C(K)$ and S is weakly compact.

Theorem

K perfect, $T = g\,\mathrm{Id} + S \in L\big(C(K)\big)$ weak multiplication
$\Longrightarrow \ \|\mathrm{Id} + T^2\| = 1 + \|T^2\|$

Proof of the theorem

We have $X = C(K)$, K perfect, $T = g\mathrm{Id} + S$

- $\max \|\mathrm{Id} \pm T\| = 1 + \|T\|$ (true for every K and every T)
- $\|\mathrm{Id} + S\| = 1 + \|S\|$ (if $S \in W(X)$, K perfect)

We need

$\|\mathrm{Id} + T^2\| = 1 + \|T^2\|$

- If $T = g\mathrm{Id} + S$, then $T^2 = g^2\,\mathrm{Id} + S'$ with S' weakly compact.
- We will prove that $\|\mathrm{Id} + g^2\,\mathrm{Id} + S\| = 1 + \|g^2\,\mathrm{Id} + S\|$ for $g \in C(K)$ and S weakly compact.
- **Step 1:** We assume $\|g^2\| \leqslant 1$ and $\min g^2(K) > 0$.
- **Step 2:** We can avoid the assumption that $\min g^2(K) > 0$.
- **Step 3:** Finally, for every g the above gives

$$\left\|\mathrm{Id} + \frac{1}{\|g^2\|}\left(g^2\,\mathrm{Id} + S\right)\right\| = 1 + \frac{1}{\|g^2\|}\|g^2\,\mathrm{Id} + S\|$$

which gives us the result.

The first example: weak multiplications. II

Weak multiplication

Let K be a compact space. $T \in L\big(C(K)\big)$ is a weak multiplication if

$$T = g\,\mathrm{Id} + S$$

where $g \in C(K)$ and S is weakly compact.

Theorem

K perfect, $T = g\,\mathrm{Id} + S \in L\big(C(K)\big)$ weak multiplication
$\implies \|\mathrm{Id} + T^2\| = 1 + \|T^2\|$

Example (Koszmider, 2004; Plebanek, 2004)

There are perfect compact spaces K such that all operators on $C(K)$ are weak multiplications.

Consequence

Therefore, there are extremely non-complex $C(K)$ spaces.

More examples: weak multipliers

Weak multiplier

Let K be a compact space. $T \in L\big(C(K)\big)$ is a weak multiplier if

$$T^* = g\,\mathrm{Id} + S$$

where g is a Borel function and S is weakly compact.

Theorem

If K is perfect and all operators on $C(K)$ are weak multipliers, then $C(K)$ is extremely non-complex.

Example (Koszmider, 2004)

There are infinitely many different perfect compact spaces K such that all operators on $C(K)$ are weak multipliers.

Corollary

There are infinitely many non-isomorphic extremely non-complex Banach spaces.

Further examples

Proposition

There is a compact infinite totally disconnected and perfect space K such that all operators on $C(K)$ are weak multipliers.

Consequence

There is a family $(K_i)_{i\in I}$ of pairwise disjoint perfect and totally disconnected compact spaces such that

- every operator on $C(K_i)$ is a weak multiplier,
- for $i \neq j$, every $T \in L(C(K_i), C(K_j))$ is weakly compact.

Theorem

There are some compactifications $\widetilde{K}$ of the above family $(K_i)_{i\in I}$ such that the corresponding $C(\widetilde{K})$'s are extremely non-complex.

Further examples II

Main consequence

There are perfect compact spaces $K_1,\ K_2$ such that:

- $C(K_1)$ and $C(K_2)$ are extremely non-complex,
- $C(K_1)$ contains a complemented copy of $C(\Delta)$.
- $C(K_2)$ contains a 1-complemented isometric copy of ℓ_∞.

Observation

- $C(K_1)$ and $C(K_2)$ have operators which are not weak multipliers.
- They are not indecomposable spaces.

Related open questions

Question 1

Find topological characterization of the compact Hausdorff spaces K such that the spaces $C(K)$ are extremely non-complex.

Question 2

Find topological consequences on K when $C(K)$ is extremely non-complex.
For instance:
If $C(K)$ is extremely non-complex and $\psi : K \longrightarrow K$ is continuous, are there an open subset U of K such that $\psi|_U = \mathrm{id}$ and $\psi(K \setminus U)$ is finite **?**

- We will show latter than $\varphi : K \longrightarrow K$ homeomorphism $\implies \varphi = \mathrm{id}$.

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Extremely non-complex Banach spaces: surjective isometries

P. Koszmider, M. Martín, and J. Merí.
Isometries on extremely non-complex Banach spaces.
Preprint (2008).

Extremely non-complex Banach spaces

Definition

X is extremely non-complex if $\operatorname{dist}(T^2, -\mathrm{Id})$ is the maximum possible, i.e.

$$\|\mathrm{Id} + T^2\| = 1 + \|T^2\| \qquad \big(T \in L(X)\big)$$

Examples

There are several extremely non-complex $C(K)$ spaces:

- If $T = g\mathrm{Id} + S$ for every $T \in L(C(K))$ (K Koszmider).
- If $T^* = g\mathrm{Id} + S$ for every $T \in L(C(K))$ (K weak Koszmider).
- One $C(K)$ containing a complemented copy of $C(\Delta)$.
- One $C(K)$ containing an isometric (1-complemented) copy of ℓ_∞.

Isometries on extremely non-complex spaces. I

Theorem

X extremely non-complex.

- $T \in \mathrm{Iso}(X) \implies T^2 = \mathrm{Id}$.
- $T_1, T_2 \in \mathrm{Iso}(X) \implies T_1 T_2 = T_2 T_1$.
- $T_1, T_2 \in \mathrm{Iso}(X) \implies \|T_1 - T_2\| \in \{0, 2\}$.
- $\Phi : \mathbb{R}_0^+ \longrightarrow \mathrm{Iso}(X)$ one-parameter semigroup $\implies \Phi(\mathbb{R}_0^+) = \{\mathrm{Id}\}$.

Consequences

- $\mathrm{Iso}(X)$ is a Boolean group for the composition operation.
- $\mathrm{Iso}(X)$ identifies with the set $\mathsf{Unc}(X)$ of unconditional projections on X:

$$P \in \mathsf{Unc}(X) \iff P^2 = P,\ 2P - \mathrm{Id} \in \mathrm{Iso}(X)$$
$$\iff P = \frac{1}{2}(\mathrm{Id} - T),\ T \in \mathrm{Iso}(X), T^2 = \mathrm{Id}.$$

- $\mathrm{Iso}(X) \equiv \mathsf{Unc}(X)$ is a Boolean algebra
  $\iff P_1 P_2 \in \mathsf{Unc}(X)$ when $P_1, P_2 \in \mathsf{Unc}(X)$
  $\iff \left\| \frac{1}{2}(\mathrm{Id} + T_1 + T_2 - T_1 T_2) \right\| = 1$ for every $T_1, T_2 \in \mathrm{Iso}(X)$.

Extremely non-complex $C_E(K\|L)$ spaces.

Theorem

K perfect weak Koszmider, L closed nowhere dense, $E \subset C(L)$
$\implies C_E(K\|L)$ is extremely non-complex.

Proposition

K perfect $\implies \exists\, L \subset K$ closed nowhere dense with $C[0,1] \subset C(L)$.

Example

Take K perfect weak Koszmider, $L \subset K$ closed nowhere dense with $E = \ell_2 \subset C[0,1] \subset C(L)$:

- $C_{\ell_2}(K\|L)$ has no non-trivial one-parameter semigroup of isometries.
- $C_{\ell_2}(K\|L)^* = \ell_2 \oplus_1 C_0(K\|L)^*$, so $\mathrm{Iso}\big(C_{\ell_2}(K\|L)^*\big) \supset \mathrm{Iso}(\ell_2)$.

Observation

$C_{\ell_2}(K\|L)$ is not isomorphic to a $C(K')$ space since $\ell_2 \overset{\text{comp}}{\hookrightarrow} C_{\ell_2}(K\|L)^*$.

But we are able to give a better result...

Isometries on extremely non-complex $C_E(K\|L)$ spaces

Theorem

$C_E(K\|L)$ extremely non-complex, $T \in \operatorname{Iso}(C_E(K\|L))$
$\Longrightarrow$ exists $\theta : K \setminus L \longrightarrow \{-1,1\}$ continuous such that

$$[T(f)](x) = \theta(x) f(x) \qquad \big(x \in K \setminus L,\ f \in C_E(K\|L)\big)$$

Consequence: connected case

If K and $K \setminus L$ are connected, then

$$\operatorname{Iso}\big(C_E(K\|L)\big) = \{-\mathrm{Id}, +\mathrm{Id}\}$$

The main example

Koszmider, 2004

$\exists$ $\mathcal{K}$ connected weak Koszmider space such that $\mathcal{K} \setminus F$ is connected if $|F| < \infty$.

Observation on the above construction

There is $\mathcal{L} \subset \mathcal{K}$ closed nowhere dense with

- $\mathcal{K} \setminus \mathcal{L}$ connected
- $C[0,1] \subseteq C(\mathcal{L})$

The best example

Consider $X = C_{\ell_2}(\mathcal{K} \| \mathcal{L})$. Then:

$$\mathrm{Iso}(X) = \{-\mathrm{Id}, +\mathrm{Id}\} \qquad \text{and} \qquad \mathrm{Iso}(X^*) \supset \mathrm{Iso}(\ell_2)$$

Proof.

- $\mathcal{K}$ weak Koszmider, $\mathcal{L}$ nowhere dense, $\ell_2 \subset C(\mathcal{L})$ $\implies X$ well-defined and extremely non-complex.
- $\mathcal{K} \setminus \mathcal{L}$ connected $\implies \mathrm{Iso}(X) = \{-\mathrm{Id}, +\mathrm{Id}\}$.
- $X^* = \ell_2 \oplus_1 C_0(\mathcal{K} \| \mathcal{L})^*$, so $\mathrm{Iso}(\ell_2) \subset \mathrm{Iso}(X^*)$.

Open questions on extremely non-complex Banach spaces

Questions

X extremely non complex

- Does X have the Daugavet property **?**
- Stronger: Does Y have the Daugavet property if

$$\|\mathrm{Id} + T^2\| = 1 + \|T^2\| \quad \text{for every rank-one } T \in L(Y) \text{ ?}$$

- Is it true that $n(X) = 1$ **?**
  - We actually know that $n(X) \geqslant C > 0$.
- Is $\mathrm{Iso}(X) \equiv \mathrm{Unc}(X)$ a Boolean algebra **?**
- If $Y \leqslant X$ is 1-codimensional, is Y extremely non complex **?**
- Is it possible that $X \simeq Z \oplus Z \oplus Z$ **?**

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Questions conducting to the results presented here

Rafael Payá, Granada 1996; Ángel Rodríguez, Granada 1999

Are $n(X)$ and $n(X^*)$ always equal?

Gilles Godefroy, París 2005

Is there any X different from $\mathbb{R}$ such that $\|\mathrm{Id}+T^2\| = 1+\|T^2\|$ for every $T \in L(X)$?

Rafael Payá, ICM Madrid 2006

Is there X with $n(X) > 0$ such that there is a non-null $S \in L(X^*)$ with $v(S) = 0$? Equivalently, is there X such that $\mathrm{Iso}(X)$ has no uniformly continuous one-parameter semigroups of isometries but $\mathrm{Iso}(X^*)$ have?

Armando Villena, Granada 2007

Is it possible that $\mathrm{Iso}(X) = \{\pm \mathrm{Id}\}$ but $\mathrm{Iso}(X^*) \supset \mathrm{Iso}(\ell_2)$?