Rango numérico e igualdades de normas para operadores.

Prehistoria, historia y resultados recientes

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Estructura del curso

1. Un poco de prehistoria...

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1. Un poco de prehistoria...
2. Historia
3. C^*-álgebras
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1. Un poco de prehistoria...
2. Historia
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4. Igualdades de normas para operadores
5. El grupo de isometrías de un Espacio de Banach y la dualidad

Notación básica

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X espacio de Banach real o complejo

- $\mathbb{K}$ cuerpo base (puede ser $\mathbb{R}$ o $\mathbb{C}$),
- $\mathbb{T}$ conjunto de escalares de módulo uno:
- S_X esfera unidad, B_X bola unidad,
- X^* espacio dual topológico,
- $L(X)$ algebra de los operadores lineales y continuos en X,
- $T^* \in L(X^*)$ operador adjunto de $T \in L(X)$,
- $\operatorname{Iso}(X)$ isometrías sobreyectivas.

Capítulo 1
Un poco de prehistoria

F. F. Bonsall and J. Duncan
Numerical Ranges (volúmenes I y II).
London Math. Soc. Lecture Note Series, 1971 – 1973.

I. K. Daugavet
On a property of completely continuous operators in the space C.
Uspekhi Mat. Nauk (1963)

J. Duncan, C. M. McGregor, J. D. Pryce, and A. J. White
The numerical index of a normed space.
J. London Math. Soc. (1970)

D. Werner,
An elementary approach to the Daugavet equation, in: *Interaction between Functional Analysis, Harmonic Analysis and Probability* (N. Kalton, E. Saab and S. Montgomery-Smith editors).
Lecture Notes in Pure and Appl. Math. **175** (1996)

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$$\|\mathrm{Id} + T\| = 1 + \|T\| \qquad \text{(DE)}$$

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   Todo operador compacto en $L_1[0,1]$ verifica (DE).
4. **Abramovich, Holub, y otros, años 80:**
   $X = C(K)$, K compacto perfecto o $X = L_1(\mu)$, μ medida sin átomos
   $\Longrightarrow$ todo operador débilmente compacto $T \in L(X)$ verifica (DE).

Ver otro fichero

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Un motivo...

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En el origen...

Si tomamos $\mathcal{A} = K(X)$ o $\mathcal{A} = W(X)$, estamos probando que la mejor aproximación de Id en $\mathcal{A}$ es cero y que con cualquier otra aproximación nos alejamos de la peor manera posible.

Rango numérico: espacios de Hilbert

Rango numérico en espacios de Hilbert (Toeplitz, 1918)

- A $n \times n$ matriz real o compleja

$$W(A) = \Big\{(Ax \mid x) \ : \ x \in \mathbb{K}^n, \ (x \mid x) = 1\Big\}.$$

- H espacio de Hilbert real o complejo, $T \in L(H)$,

$$W(T) = \Big\{(Tx \mid x) \ : \ x \in H, \ \|x\| = 1\Big\}.$$

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Algunas propiedades

H espacio de Hilbert, $T \in L(H)$:

- $W(T)$ es convexo.
- En caso complejo, $\overline{W(T)}$ contiene al espectro de T.
- Si además T es normal, entonces $\overline{W(T)} = \overline{\mathrm{co}}\,\mathrm{Sp}(T)$.

Rango numérico: espacios de Banach

Rango numérico en espacios de Banach (Bauer 1962; Lumer, 1961)

X espacio de Banach, $T \in L(X)$,

$$V(T) = \Big\{x^*(Tx) \ : \ x^* \in S_{X^*},\ x \in S_X,\ x^*(x) = 1\Big\}$$

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X espacio de Banach, $T \in L(X)$:

- $V(T)$ es conexo (no necesariamente convexo).
- En caso complejo, $\overline{W(T)}$ contiene al espectro de T.
- De hecho,

$$\overline{\mathrm{co}}\, Sp(T) = \bigcap \overline{\mathrm{co}}\, V(T),$$

  donde tomamos la intersección sobre los rangos numéricos $V(T)$ correspondientes a todas las posibles normas equivalentes en X.

Rango numérico: álgebras normadas

Rango numérico de álgebra (Stampfli–Williams, 1968; Bonsall, 1969)

A álgebra normada con unidad e, $a \in A$:

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Relación

X espacio de Banach, $T \in L(X)$, entonces

$$\overline{\mathrm{co}}\big(V(T)\big) = V_{L(X)}(T).$$

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Motivos en espacios de Banach y álgebras normadas

- Permite definir sin dificultad operador (o elemento) hermitiano, disipativo...
- Su mayor éxito es la demostración de parte del Teorema de Vidav-Palmer: toda C^*-álgebra es una B^*-álgebra y se caracteriza por descomponer como hermitianos + i hermitianos.

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4. **Werner, 1997:**
   Todo operador en un álgebra de funciones satisface (aDE). Un álgebra de funciones es una subalgebra de $C(K)$ que contiene a las constante y separa los puntos de K.

Relación entre rango numérico y ecuación de Daugavet

Proposición (Duncan-McGregor-Pryce-White, 1970)

X espacio de Banach, $T \in L(X)$. Entonces

$$\sup \operatorname{Re} V(T) = \|T\| \quad \Longleftrightarrow \quad T \text{ satisface (DE)}$$
$$\sup |V(T)| = \|T\| \quad \Longleftrightarrow \quad T \text{ satisface (aDE)}$$

Capítulo 2

Un poco de historia.

Aparece la propiedad de Daugavet

Y. Abramovich, and C. Aliprantis,
An invitation to operator theory.
Graduate Studies in Math. **50**, AMS, 2002.

Y. Abramovich, and C. Aliprantis,
Problems in operator theory.
Graduate Studies in Math. **51**, AMS, 2002.

V. Kadets, R. Shvidkoy, G. Sirotkin, and D. Werner,
Banach spaces with the Daugavet property.
Trans. Amer. Math. Soc. (2000)

D. Werner,
Recent progress on the Daugavet property.
Irish Math. Soc. Bulletin (2001)

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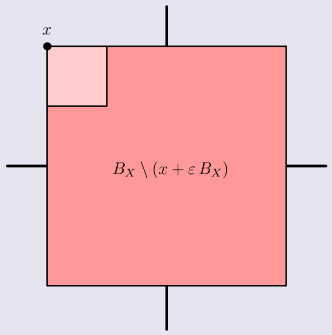

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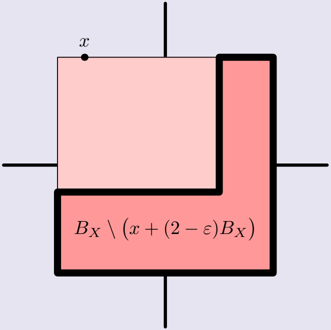
- $C[0,1]$ and $L_1[0,1]$ have an extremely opposite property: for every $x \in S_X$ and every $\varepsilon > 0$

$$\overline{\mathrm{co}}\Big(B_X \setminus \big(x + (2-\varepsilon)B_X\big)\Big) = B_X\,.$$



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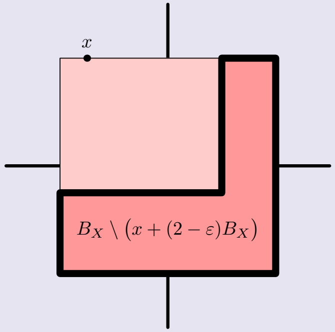
- This geometric property is equivalent to a property of operators on the space.



The Daugavet property

- A Banach space X is said to have the Daugavet property if every rank-one operator on X satisfies (DE).

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$$C[0,1] \text{ has it but } C[0,1]^* \text{ not.}$$

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6. The predual of a von Neumann algebra has the Daugavet property if and only if the algebra is non-atomic.

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Ver otro fichero

Geometric characterizations

Theorem [KSSW]

- X has the Daugavet property.

Every rank-one operator $T \in L(X)$ satisfies

$$\|\mathrm{Id} + T\| = 1 + \|T\|.$$

Geometric characterizations

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- X has the Daugavet property.
- For every $x \in S_X$, $x^* \in S_{X^*}$, and $\varepsilon > 0$, there exists $y \in S_X$ such that
$$\operatorname{Re} x^*(y) > 1 - \varepsilon \quad \text{and} \quad \|x - y\| \geqslant 2 - \varepsilon.$$
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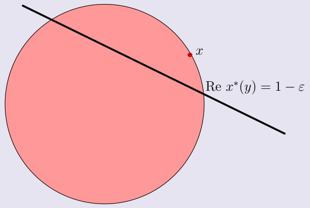

Geometric characterizations

Theorem [KSSW]

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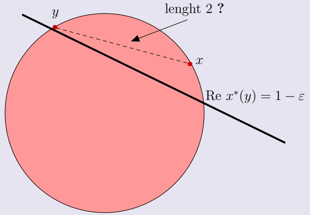

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- For every $x \in S_X$ and every $\varepsilon > 0$, we have
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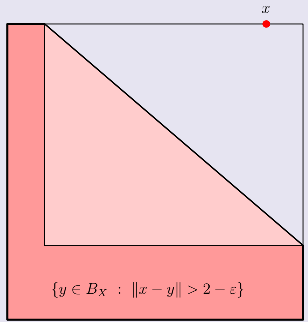

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Let X be a Banach space with the Daugavet property.

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3. Actually, X does not embed into an unconditional sum of Banach spaces without a copy of ℓ_1.

   (Shvidkoy, 2000)

Capítulo 3
C^*-álgebras

J. Becerra Guerrero and M. Martín,
The Daugavet Property of C^*-algebras, JB^*-triples, and of their isometric preduals.
Journal of Functional Analysis (2005)

M. Martín,
The alternative Daugavet property of C^*-algebras and JB^*-triples.
Mathematische Nachrichten (to appear)

M. Martín and T. Oikhberg,
An alternative Daugavet property.
Journal of Mathematical Analysis and Applications (2004)

T. Oikhberg,
The Daugavet property of C^*-algebras and non-commutative L_p-spaces.
Positivity (2002)

3 C^*-álgebras

- The known results
- A new sufficient condition
- Application: C^*-algebras and von Neumann preduals
  - von Neumann preduals
  - C^*-algebras
- The alternative Daugavet equation
  - Definitions and basic results
  - Geometric characterizations
  - C^*-algebras and preduals

4 Igualdades de normas para operadores

- Motivación
- Las ecuaciones
  - $\|\mathrm{Id} + T\| = f(\|T\|)$
  - $\|g(T)\| = f(\|T\|)$
  - $\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$
- Problemas abiertos
- Espacios $C(K)$ extremadamente no complejos
  - Motivación: estructura compleja
  - Los ejemplos

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Our objectives here

- To give "geometrical" proofs of the above results;
- At the same time, to give characterizations of the Daugavet property for C^*-algebras and von Neumann preduals in terms of the "geometry" of the underlining Banach space (without any algebra).

A new sufficient condition

Theorem

Let X be a Banach space such that

$$X^* = Y \oplus_1 Z$$

with Y and Z norming subspaces. Then, X has the Daugavet property.

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A closed subspace $W \subseteq X^*$ is norming if

$$\|x\| = \sup \{|w^*(x)| \ : \ w^* \in W, \ \|w^*\| = 1\}$$

or, equivalently, if B_W is w^*-dense in B_{X^*}.

Proof of the theorem

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and that $\operatorname{Re} y_\mu^*(x_0) > 1 - \varepsilon$ (since $y_\mu^* \in U$).

Some immediate consequences

Corollary

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Proof

Just recall Goldstine and Krein-Milman Theorems.

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It was already known that...

- If $Y \subset L_1$ is reflexive, then L_1/Y has the Daugavet property.
  (Kadets–Shvidkoy–Sirotkin–Werner, 2000)
- If $Y \subset L_1$ is L-embedded, then L_1/Y does not have the RNP.
  (Harmand–Werner–Werner, 1993)

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Actually, much more can be proved:

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An atomic projection is an element $p \in X$ such that

$$p^2 = p^* = p \qquad \text{and} \qquad p\,X\,p = \mathbb{C}p.$$

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Theorem

If X is non-atomic, then N is norming. Therefore, X has the Daugavet property.
Example: $C[0,1]$

sketch of the proof of the theorem

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- But Y is an Asplund space, so $Y = 0$.

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- But $N^* \equiv \mathcal{N}$, so N is norming for $\mathcal{N}$ and now, also for X.

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- The norm of X is not Fréchet-smooth at any point.
- X is non-atomic.

The alternative Daugavet equation

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X Banach space, $T \in L(X)$

$$\max_{|\omega|=1} \|\mathrm{Id} + \omega\, T\| = 1 + \|T\| \qquad \text{(aDE)}$$

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- There exists $\omega \in \mathbb{T}$ such that $\omega\, T$ satisfies (DE).
- The numerical radius of T, $v(T)$, coincides with $\|T\|$, where

$$v(T) := \sup\{|x^*(Tx)| \ : \ x^* \in S_{X^*},\ x \in S_X,\ x^*(x) = 1\}.$$

Two possible properties

Let X be a Banach space.

- X is said to have the alternative Daugavet property (ADP) iff every rank-one operator on X satisfies (aDE).

(M.–Oikhberg, 2004; briefly appearance: Abramovich, 1991)

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The numerical index of a Banach space X is the greater constant k such that

$$v(T) \geqslant k\|T\|$$

for every operator $T \in L(X)$.

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Observation

No analogous property is possible for the Daugavet equation:

$$\|\mathrm{Id} + (-\mathrm{Id})\| = 0 \neq 1 + \|-\mathrm{Id}\|.$$

Numerical index 1

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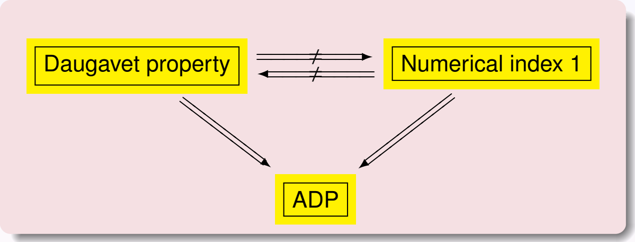
- In case $\dim(X) = \infty$, if X has numerical index 1 and the RNP, then $X \supseteq \ell_1$.

  (López–M.–Payá, 1999)

The alternative Daugavet property



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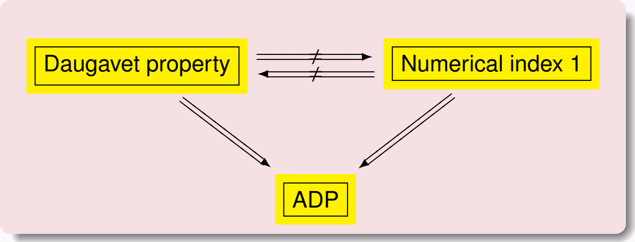

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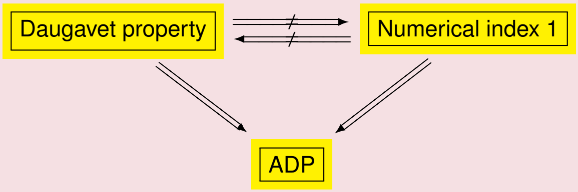

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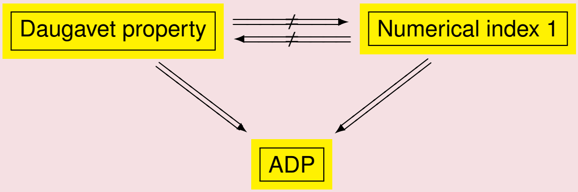

- $c_0 \oplus_\infty C([0,1],\ell_2)$ has the ADP, but neither the Daugavet property, nor numerical index 1.
- For RNP or Asplund spaces, the ADP implies numerical index 1.
- Every Banach space with the ADP can be renormed still having the ADP but failing the Daugavet property.

Geometric characterizations

Theorem

- X has the ADP.

Every rank-one operator $T \in L(X)$ (equivalently, every weakly compact operator) satisfies

$$\max_{|\omega|=1} \|\mathrm{Id} + \omega\, T\| = 1 + \|T\|.$$

Geometric characterizations

Theorem

- X has the ADP.
- For every $x \in S_X$, $x^* \in S_{X^*}$, and $\varepsilon > 0$, there exists $y \in S_X$ such that
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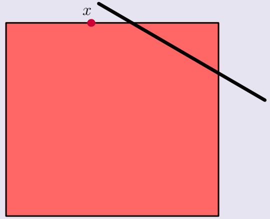

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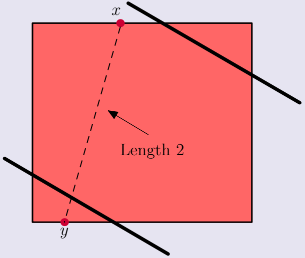

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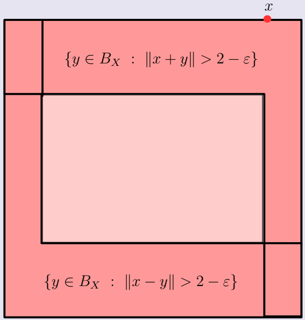
$$|y^*(x)| > 1 - \varepsilon \quad \text{and} \quad \|x^* - y^*\| \geqslant 2 - \varepsilon.$$
- For every $x \in S_X$ and every $\varepsilon > 0$, we have
$$B_X = \overline{\text{co}}\big(\mathbb{T}\{y \in B_X \ : \ \|x - y\| \geqslant 2 - \varepsilon\}\big).$$



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The alternative Daugavet property of V_* is equivalent to:

- the atomic projections of V are central, or
- $|v(v_*)| = 1$ for $v \in \operatorname{ext}(B_V)$ and $v_* \in \operatorname{ext}(B_{V_*})$, or
- $V = C \oplus_\infty N$, where C is commutative and N has no atomic projections.

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- $|x^{**}(x^*)| = 1$, for $x^{**} \in \operatorname{ext}(B_{X^{**}})$, and $x^* \in B_{X^*}$ w^*-strongly exposed, or
- $\exists$ a commutative ideal Y such that X/Y has the Daugavet property.

Capítulo 4

Igualdades de normas para operadores

V. Kadets, M. Martín y J. Merí,
Norm equalities for operators on Banach spaces.
Indiana U. Math. J. (2007)

P. Koszmider, M. Martín y J. Merí,
Extremely non-complex $C(K)$ spaces.
En preparación

4 Igualdades de normas para operadores
- Motivación
- Las ecuaciones
  - $\|\mathrm{Id} + T\| = f(\|T\|)$
  - $\|g(T)\| = f(\|T\|)$
  - $\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$
- Problemas abiertos
- Espacios $C(K)$ extremadamente no complejos
  - Motivación: estructura compleja
  - Los ejemplos

5 El grupo de isometrías de un Espacio de Banach y la dualidad
- The tool: numerical range of operators
- The example
- Some related results
  - Finite-dimensional spaces
  - Numerical index and duality

El problema que nos planteamos

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El problema

Estudiar la posibilidad de encontrar igualdades de normas para operadores que puedan ser válidas para todos los operadores de rango uno en un espacio de Banach.

Estudiamos tres casos:

1. $\|\mathrm{Id} + T\| = f(\|T\|)$ para f arbitraria,
2. $\|g(T)\| = f(\|T\|)$ para g entera y f arbitraria,
3. $\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$ para g entera y f *continua*.

Antecedentes: desigualdades tipo Daugavet

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Algunos ejemplos

- **Benyamini–Lin, 1985:**
  Para cada $1 < p < \infty$, $p \neq 2$, existe una función $\psi_p : (0,\infty) \longrightarrow (0,\infty)$ tal que
  $$\|\mathrm{Id} + T\| \geqslant 1 + \psi_p(\|T\|)$$
  para todo operador compacto T en $L_p[0,1]$.
- Si $p = 2$, existe un operador compacto T en $L_2[0,1]$ con $\|T\| = 1$ y tal que
  $$\|\mathrm{Id} + T\| = 1.$$
- **Boyko–Kadets, 2004:**
  Si llamamos ψ_p a la mejor función posible arriba, entonces
  $$\lim_{p\to 1^+} \psi_p(t) = t \qquad (t > 0).$$
- **Oikhberg, 2005:**
  En cualquier espacio $K(\ell_2) \subseteq X \subseteq L(\ell_2)$ se tiene que
  $$\|\mathrm{Id} + T\| \geqslant 1 + \frac{1}{8\sqrt{2}}\|T\|$$
  para todo operador compacto T en X.

Igualdades de la forma $\|g(T)\| = f(\|T\|)$

Nota

Si X tiene la propiedad de Daugavet, entonces $\|\mathrm{Id} + T\|$ depende sólo de $\|T\|$.

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Proposición

X real o complejo, $f : \mathbb{R}_0^+ \longrightarrow \mathbb{R}$ arbitraria, $a, b \in \mathbb{K}$. Si la igualdad

$$\|a\,\mathrm{Id} + b\,T\| = f(\|T\|)$$

se verifica para todo operador de rango uno T en X, entonces

$$f(t) = |a| + |b|\,t \qquad \left(t \in \mathbb{R}_0^+\right).$$

Si $a \neq 0$, $b \neq 0$, entonces X tiene la propiedad de Daugavet.

Demostración

Tenemos...

$\|a\,\mathrm{Id} + b\,T\| = f(\|T\|)\ \forall T \in L(X)$ de rango uno

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Queremos probar...

$f(t) = |a| + |b|\,t \qquad \left(t \in \mathbb{R}_0^+\right).$

- Trivial si $a \cdot b = 0$. Suponemos $a \neq 0$ y $b \neq 0$ y escribimos $\omega_0 = \frac{\overline{b}}{|b|}\frac{a}{|a|} \in \mathbb{T}$.

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$$T_t = t\,x_0^* \otimes x_0 \in L(X) \qquad \big(t \in \mathbb{R}_0^+\big).$$

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- Se sigue que

$$|a| + |b|\,t \geqslant f(t) = \|a\mathrm{Id} + b\,T_t\| \geqslant \big\|[a\mathrm{Id} + b\,T_t](x_0)\big\|$$

$$= \|a\,x_0 + b\,\omega_0 t\,x_0\| = |a + b\,\omega_0 t|\,\|x_0\| = \left|a + b\,\frac{\overline{b}}{|b|}\frac{a}{|a|}\,t\right| = |a| + |b|\,t.$$

Demostración

Tenemos...

$\|a\,\mathrm{Id} + b\,T\| = f(\|T\|)\ \forall T \in L(X)$ de rango uno

$\overset{?}{\Longrightarrow}$

Queremos probar...

$f(t) = |a| + |b|\,t \qquad \big(t \in \mathbb{R}_0^+\big).$

- Trivial si $a \cdot b = 0$. Suponemos $a \neq 0$ y $b \neq 0$ y escribimos $\omega_0 = \frac{\overline{b}}{|b|}\frac{a}{|a|} \in \mathbb{T}$.
- Ahora fijamos $x_0 \in S_X$, $x_0^* \in S_{X^*}$ con $x_0^*(x_0) = \omega_0$ y consideramos

$$T_t = t\,x_0^* \otimes x_0 \in L(X) \qquad (t \in \mathbb{R}_0^+).$$

- Obsérvese que $\|T_t\| = t$ y, por tanto,

$$f(t) = \|a\mathrm{Id} + b\,T_t\| \qquad (t \in \mathbb{R}_0^+).$$

- Se sigue que

$$|a| + |b|\,t \geqslant f(t) = \|a\mathrm{Id} + b\,T_t\| \geqslant \big\|[a\mathrm{Id} + b\,T_t](x_0)\big\|$$

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- Finalmente, fijado un operador de rango uno $T \in L(X)$, llamamos $S = \frac{a}{b}\,T$ y tenemos

$$|a|\big(1 + \|T\|\big) = |a| + |b|\,\|S\| = \|a\mathrm{Id} + b\,S\| = |a|\,\|\mathrm{Id} + T\|.$$

Igualdades de la forma $\|g(T)\| = f(\|T\|)$

Teorema

X real o complejo con $\dim(X) \geqslant 2$.
Supongamos que la igualdad

$$\|g(T)\| = f(\|T\|)$$

se verifica para todo operador de rango uno T en X, donde

- $g : \mathbb{K} \longrightarrow \mathbb{K}$ es entera,
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Entonces, tres casos son posibles:

- $\|\mathrm{Id}\| = 1$,
- $\|T\| = \|T\|$,

(casos triviales)

- $\|a\,\mathrm{Id} + b\,T\| = |a| + |b|\,\|T\|$, con $a \neq 0$, $b \neq 0$

(Propiedad de Daugavet)

Demostración (caso complejo)

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g es afín.

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Luego para cada $\lambda \in \mathbb{C}$ tenemos

$$\left\| a_0 \mathrm{Id} + \widetilde{g}(\lambda) T_1 \right\| = \|g(\lambda T_1)\| = f(|\lambda|) = \|g(\lambda T_0)\| = \|a_0 \mathrm{Id} + a_1 \lambda T_0\|.$$

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Usamos la desigualdad triangular para obtener que

$$|\widetilde{g}(\lambda)| \leqslant 2|a_0| + |a_1||\lambda| \qquad (\lambda \in \mathbb{C}),$$

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y por tanto $\widetilde{g}$ es un polinomio de grado uno (desigualdades de Cauchy).

Caso real en otro fichero

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Idea de la demostración en otro fichero

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Si $g(0) \neq -1/2$:

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Algunos problemas abiertos

- Obtener caracterizaciones geométricas de los espacios de Banach X en los que la igualdad
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Una pregunta de Godefroy

Pregunta de Godefroy (comunicación privada)

¿ Existe algún espacio de Banach real X (distinto de $\mathbb{R}$) para el que la igualdad

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Respuesta

La respuesta es afirmativa. De hecho, podemos tomar como X algunos espacios de tipo $C(K)$ con "pocos operadores" (construidos por Koszmider en 2004) y otros espacios $C(K)$ con "muchos operadores".

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Estructura compleja

Un espacio de Banach real X admite una estructura compleja si existe $T \in L(X)$ tal que

$$T^2 = -\mathrm{Id}.$$

Dicho de otra forma, X admite una norma equivalente que lo convierte en espacio normado complejo:

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- Por supuesto, $X = \mathbb{R}$ cumple esto.
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- Nuestro objetivo es probar que hay ejemplos de dimensión infinita.

Los primeros ejemplos

Multiplicador débil

Sea K un espacio topológico compacto. $T \in L\big(C(K)\big)$ es un multiplicador débil if

$$T^* = g\,\mathrm{Id} + S$$

donde g es una función de Borel y S es débilmente compacto.

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Nuestro principal resultado

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Teorema (Koszmider, 2004; Plebanek, 2004)

Existen espacios compactos perfecto K tales que todos los operadores en $C(K)$ son multiplicadores débiles. De hecho, hay diversos tipos de ejemplos:

- K conexo y tal que todo operador en $L(C(K))$ tiene la forma $g\,\mathrm{Id} + S$ con $g \in C(K)$ y S débilmente compacto (*multiplicación débil*).
- K totalmente disconexo y perfecto.

En particular, existen espacios $C(K)$ no isomorfos y extremadamente no complejos.

Demostrando un caso sencillo...

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- Basta probar que $\|\mathrm{Id} + (g\,\mathrm{Id} + S)\| = 1 + \|g\,\mathrm{Id} + S\|$ para $g \geqslant 0$ y S débilmente compacto.

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- **Paso 1:** Supongamos que $\|g\| \leqslant 1$ and $\min g(K) > 0$.

Demostrando un caso sencillo...

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Todo $T \in L\big(C(K)\big)$ tiene la forma $g\mathrm{Id} + S$, con $g \in C(K)$ y S débilmente compacto

Necesitamos...

$\|\mathrm{Id} + T^2\| = 1 + \|T^2\|$

- Si $T = g\mathrm{Id} + S$, entonces $T^2 = g^2\,\mathrm{Id} + S'$ con S' débilmente compacto.
- Basta probar que $\|\mathrm{Id} + (g\,\mathrm{Id} + S)\| = 1 + \|g\,\mathrm{Id} + S\|$ para $g \geqslant 0$ y S débilmente compacto.
- **Paso 1:** Supongamos que $\|g\| \leqslant 1$ and $\min g(K) > 0$.

Demostración

- Sabemos que $\max_{\pm}\|\mathrm{Id} \pm (g\,\mathrm{Id} + S)\| = 1 + \|g\,\mathrm{Id} + S\|$, luego basta probar que

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- $\|\mathrm{Id} - (g\,\mathrm{Id} + S)\| \leqslant \|(1-g)\mathrm{Id}\| + \|S\| = 1 - \min g(K) + \|S\|.$
- $\|g\,\mathrm{Id} + S\| = \|\mathrm{Id} + S + (g\,\mathrm{Id} - \mathrm{Id})\| \geqslant \|\mathrm{Id} + S\| - \|g\,\mathrm{Id} - \mathrm{Id}\|$
$$= 1 + \|S\| - \big(1 - \min g(K)\big) = \|S\| + \min g(K).$$

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Demostración

Basta recordar que el conjunto de los operadores que satisfacen (DE) es cerrado.

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- **Paso 3:** Por último, para cada $g \geqslant 0$ tenemos

$$\left\|\mathrm{Id} + \frac{1}{\|g\|}\,(g\,\mathrm{Id} + S)\right\| = 1 + \frac{1}{\|g\|}\|g\,\mathrm{Id} + S\|$$

y hemos acabado.

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y hemos acabado.

Nota

Si $\|u + v\| = \|u\| + \|v\| \quad\Longrightarrow\quad \|\alpha\, u + \beta\, v\| = \alpha\|u\| + \beta\|v\|$ para $\alpha, \beta \in \mathbb{R}_0^+$.

Más ejemplos

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Podría pensarse que ser extremadamente no complejo obliga a tener “pocos operadores”. Los siguientes ejemplos muestran que esto no es así.

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- Existe (en ZFC) un espacio topológico compacto K_1 verificando:
  - $C(K_1)$ es extremadamente no complejo y
  - $C(K_1)$ contiene una copia isomórfica complementada de $C[0, 1]$.

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- Existe (en ZFC) un espacio topológico compacto K_2 verificando:
  - $C(K_2)$ es extremadamente no complejo y
  - $C(K_2)$ contiene una copia isométrica (1-complementada) de ℓ_∞.

Capítulo 5

El grupo de isometrías de un Espacio de Banach y la dualidad

F. F. Bonsall and J. Duncan
Numerical Ranges (volúmenes I y II).
London Math. Soc. Lecture Note Series, 1971 – 1973.

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M. Martín, J. Merí, and A. Rodríguez-Palacios.
Finite-dimensional spaces with numerical index zero.
Indiana U. Math. J. (2004).

H. P. Rosenthal
The Lie algebra of a Banach space.
in: *Banach spaces* (Columbia, Mo., 1984), LNM, Springer, 1985.

4 Igualdades de normas para operadores
- Motivación
- Las ecuaciones
  - $\|\mathrm{Id} + T\| = f(\|T\|)$
  - $\|g(T)\| = f(\|T\|)$
  - $\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$
- Problemas abiertos
- Espacios $C(K)$ extremadamente no complejos
  - Motivación: estructura compleja
  - Los ejemplos

5 El grupo de isometrías de un Espacio de Banach y la dualidad
- The tool: numerical range of operators
- The example
- Some related results
  - Finite-dimensional spaces
  - Numerical index and duality

Objetivo principal de este capítulo

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Se construye un espacio de Banach real X tal que

- $\mathrm{Iso}(X)$ no contiene semigrupos uniparamétricos uniformemente continuos;
- pero $\mathrm{Iso}(X^*)$ contiene una cantidad infinita de tales semigrupos.

Hilbert spaces

Hilbert space Numerical range (Toeplitz, 1918)

- A $n \times n$ real or complex matrix

$$W(A) = \Big\{(Ax \mid x) \ : \ x \in \mathbb{K}^n, \ (x \mid x) = 1\Big\}.$$

- H real or complex Hilbert space, $T \in L(H)$,

$$W(T) = \Big\{(Tx \mid x) \ : \ x \in H, \ \|x\| = 1\Big\}.$$

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Some properties

H Hilbert space, $T \in L(H)$:

- $W(T)$ is convex.
- In the complex case, $\overline{W(T)}$ contains the spectrum of T.
- If, moreover, T is normal, $\overline{W(T)} = \overline{\mathrm{co}}\,\mathrm{Sp}(T)$.

Banach spaces

Banach space numerical range (Bauer 1962; Lumer, 1961)

X Banach space, $T \in L(X)$,

$$V(T) = \left\{ x^*(Tx) \ : \ x^* \in S_{X^*}, \ x \in S_X, \ x^*(x) = 1 \right\}$$

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Some properties

X Banach space, $T \in L(X)$:

- $V(T)$ is connected (not necessarily convex).
- In the complex case, $\overline{W(T)}$ contains the spectrum of T.
- Actually,

$$\overline{\mathrm{co}}\, Sp(T) = \bigcap \overline{\mathrm{co}}\, V(T),$$

  the intersection taken over all numerical ranges $V(T)$ corresponding to equivalent norms on X.

Numerical radius

X real or complex Banach space, $T \in L(X)$,

$$v(T) = \sup\left\{|\lambda| \ : \ \lambda \in V(T)\right\}.$$

- v is a seminorm with $v(T) \leqslant \|T\|$.
- $v(T) = v(T^*)$ for every $T \in L(X)$.

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Numerical index (Lumer, 1968)

X real or complex Banach space,

$$\begin{aligned} n(X) &= \inf\Big\{v(T) \ : \ T \in L(X),\ \|T\| = 1\Big\} \\ &= \max\big\{k \geqslant 0 \ : \ k\|T\| \leqslant v(T)\ \forall T \in L(X)\Big\}. \end{aligned}$$

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Remarks

- $n(X) = 1$ iff $v(T) = \|T\|$ for every $T \in L(X)$.
- If there is $T \neq 0$ with $v(T) = 0$, then $n(X) = 0$.
- The converse is not true.

Relationship with semigroups of operators

A motivating example

A real or complex $n \times n$ matrix. TFAE:

- A is skew-adjoint (i.e. $A^* = -A$).
- $B = \exp(\rho A)$ is unitary for every $\rho \in \mathbb{R}$ (i.e. $B^*B = \mathrm{Id}$).

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In term of Hilbert spaces

H (n-dimensional) Hilbert space, $T \in L(H)$. TFAE:

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For general Banach spaces

X Banach space, $T \in L(X)$. TFAE:

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Characterizing uniformly continuous semigroups of operators

Theorem

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- $\operatorname{Re} V(T) = \{0\}$.
- $\|\exp(\rho T)\| \leqslant 1$ for every $\rho \in \mathbb{R}$.
- $\left\{\exp(\rho T) \ : \ \rho \in \mathbb{R}_0^+\right\} \subset \operatorname{Iso}(X)$.
- T belongs to the tangent space of $\operatorname{Iso}(X)$ at Id, i.e. exists a function $f : [-1, 1] \longrightarrow \operatorname{Iso}(X)$ with $f(0) = \mathrm{Id}$ and $f'(0) = T$.
- $\lim_{\rho \to 0} \dfrac{\|\mathrm{Id} + \rho T\| - 1}{\rho} = 0$, i.e. the derivative or the norm of $L(X)$ at Id in the direction of T is null.

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- $\displaystyle \lim_{\rho\to 0} \frac{\|\mathrm{Id} + \rho\, T\| - 1}{\rho} = 0$, i.e. the derivative or the norm of $L(X)$ at Id in the direction of T is null.

This follows from the "exponential formula"

$$\sup \operatorname{Re} V(T) = \lim_{\beta\downarrow 0} \frac{\|\mathrm{Id} + \beta\, T\| - 1}{\beta} = \sup_{\alpha>0} \frac{\log\|\exp(\alpha\, T)\|}{\alpha}.$$

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Uniformly continuous one-parameter semigroups

$T \in L(X)$ satisfying the above conditions.

- $\left\{\exp(\rho T) \ : \ \rho \in \mathbb{R}_0^+\right\}$ is a uniformly continuous one-parameter semigroups of surjective isometries.
- $T \in L(X)$ is the generator of the semigroup.

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Remark

If X is complex, there always exists uniformly continuous one-parameter semigroups of surjective isometries:

$$t \longmapsto e^{it}\,\mathrm{Id} \qquad \text{generator: } i\,\mathrm{Id}.$$

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Main consequence for us

If X is a real Banach space with $n(X) > 0$, then $\operatorname{Iso}(X)$ is “small”:

- it does not contain any uniformly continuous one-parameter semigroups,
- the tangent space of $\operatorname{Iso}(X)$ at Id is zero.

Characterizing uniformly continuous semigroups of operators

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- For every $T \in L(X)$, one has
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Demostración en otro fichero

If X is a real Banach space with $n(X) > 0$, then Iso(X) is "small":

- it does not contain any uniformly continuous one-parameter semigroups,
- the tangent space of Iso(X) at Id is zero.

The main example

The construction

E separable Banach space. We construct a Banach space $X(E)$ such that

$$n\big(X(E)\big) = 1 \qquad \text{and} \qquad X(E)^* \equiv E^* \oplus_1 L_1(\mu)$$

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The main consequence

Take $E = \ell_2$ (real). Then

- $n\big(X(\ell_2)\big) = 1$, so $\mathrm{Iso}\big(X(\ell_2)\big)$ is "small".
- Since $X(\ell_2)^* \equiv \ell_2 \oplus_1 L_1(\mu)$, given $S \in \mathrm{Iso}(\ell_2)$, the operator

$$T = \begin{pmatrix} S & 0 \\ 0 & \mathrm{Id} \end{pmatrix}$$

  is a surjective isometry of $X(\ell_2)^*$.
- Therefore, $\mathrm{Iso}\big(X(\ell_2)^*\big)$ contains infinitely many semigroups of isometries.

Sketch of the construction I

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Define (viewing $E \hookrightarrow C[0,1]$)

$$Y = \Big\{ f \in C([0,1] \times [0,1]) \ : \ f(\cdot, 0) = 0 \Big\}$$
$$X(E) = \Big\{ f \in C([0,1] \times [0,1]) \ : \ f(\cdot, 0) \in E \Big\}$$

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Proving that $X(E)^* \equiv E^* \oplus_1 L_1(\mu)$

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We need

$$X(E)^* \equiv E^* \oplus_1 L_1(\mu) \quad \& \quad n\big(X(E)\big) = 1$$

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  - $\widetilde{\Phi} : X(E)/Y \longrightarrow E$ is a surjective isometry since:
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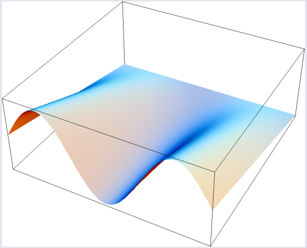

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If $f_0(\xi_0) \sim 1$, then we were done. This our goal.

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- Consider the non-empty open set
$$V = \big\{ \xi \in]0,1]\times[0,1] \ : \ f_0(\xi) \sim f_0(\xi_0) \big\}$$
and find $\varphi : [0,1]\times[0,1] \longrightarrow [0,1]$ continuous with $\operatorname{supp}(\varphi) \subset V$ and $\varphi(\xi_0) = 1$.

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- Therefore, there is $i \in \{1,2\}$ such that $\big|[T(f_i)](\xi_0)\big| \sim \|T\|$, but now $|f_i(\xi_0)| = 1$.
- Equivalently,
$$\Big|\delta_{\xi_0}\big(T(f_i)\big)\Big| \sim \|T\| \qquad \text{and} \qquad |\delta_{\xi_0}(f_i)| = 1,$$
meaning that $v(T) \sim \|T\|$.

Isometries in finite-dimensional spaces

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Theorem

Let X be a finite-dimensional real space. TFAE:

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Question

Can every Banach space X with $n(X) = 0$ be decomposed as in 4 ?

Negative answer I

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Infinite-dimensional case

There is an infinite-dimensional real Banach space X with $n(X) = 0$ but X is polyhedral. In particular, X does not contain $\mathbb{C}$ isometrically.

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Note

Such an example is not possible in the finite-dimensional case.

(Quasi affirmative) negative answer II

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  - X_0 is a (possible null) real space,
  - $X_1, \ldots, X_n$ are non-null complex spaces,

  there are $\rho_1, \ldots, \rho_n$ rational numbers, such that

$$\left\| x_0 + e^{i\rho_1 \theta} x_1 + \cdots + e^{i\rho_n \theta} x_n \right\| = \left\| x_0 + x_1 + \cdots + x_n \right\|$$

  for every $x_i \in X_i$ and every $\theta \in \mathbb{R}$.

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Remark

- The theorem is due to Rosenthal, but with real ρ's.
- The fact that the ρ's may be chosen as rational numbers is due to M.–Merí–Rodríguez-Palacios.

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Corollary

X real space with $n(X) = 0$.

- If $\dim(X) = 2$, then $X \equiv \mathbb{C}$.
- If $\dim(X) = 3$, then $X \equiv \mathbb{R} \oplus \mathbb{C}$ (absolute sum).

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Example

$$X = (\mathbb{R}^4, \|\cdot\|), \ \|(a,b,c,d)\| = \frac{1}{4}\int_0^{2\pi} \left| \mathrm{Re}\left(e^{2it}(a+ib) + e^{it}(c+id) \right) \right| \, dt.$$

Then $n(X) = 0$ but the only decomposition is $X = \mathbb{C} \oplus \mathbb{C}$ with

$$\left\| e^{it} x_1 + e^{2it} x_2 \right\| = \|x_1 + x_2\|.$$

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If $\dim(X) = n$, then

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An open problem

Given $n \geqslant 3$, which are the possible $\dim\big(\mathcal{Z}(X)\big)$ over all n-dimensional X's?

The Lie-algebra of a Banach space

Lie-algebra

X real Banach space, $\mathcal{Z}(X) = \{T \in L(X) \ : \ v(T) = 0\}$.

- When X is finite-dimensional, $\mathrm{Iso}(X)$ is a Lie-group and $\mathcal{Z}(X)$ is the tangent space (i.e. its Lie-algebra).

Remark

If $\dim(X) = n$, then

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Observation

When $\dim(X) = 3$, $\dim(\mathcal{Z}(X))$ cannot be 2.

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Numerical index (Lumer, 1968)

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4. If X_n is the two-dimensional space such that B_{X_n} is a $2n$-polygon, then
$$n(X_n) = \tan\left(\frac{\pi}{2n}\right) \ \text{ if } n \text{ is even} \qquad n(X_n) = \sin\left(\frac{\pi}{2n}\right) \ \text{ if } n \text{ is odd.}$$

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5. If X is a C^*-algebra or the predual of a von Neumann algebra, then $n(X) = 1$ if the algebra is commutative and $n(X) = 1/2$ otherwise.

Numerical index and duality

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- $v(T^*) = v(T)$ for every $T \in L(X)$.

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Some positive partial answers

- When X is reflexive (evident).
- When X is a C^*-algebra or a von Neumann predual (1970's – 2000's).
- When X is L-embedded in X^{**} (unpublished).
- If X has RNP and $n(X) = 1$, then $n(X^*) = 1$ (2000's).
- Written in the review of an article that the answer is YES (1980's).

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Answer (Boyko-Kadets-M.-Werner, 2007)

The answer is **NO**.

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Example

Take $X(\ell_2)$. Then $n\big(X(\ell_2)\big) = 1$ and $X(\ell_2)^* \equiv \ell_2 \oplus_1 L_1(\mu)$.
Since there is $S \in L\big(X(\ell_2)^*\big)$, $S \neq 0$ with $v(S) = 0$, then $n\big(X(\ell_2)^*\big) = 0$.

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- It is known: if X or X^* is a C^*-algebra, then $n(X) = n(X^*)$.

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Question

Is it always $n(X) = n(X^*)$ **?**

Another example

- It is known: if X or X^* is a C^*-algebra, then $n(X) = n(X^*)$.
- Consider $Y = X\big(K(\ell_2)\big)$. Then

$$n(Y) = 1 \qquad \text{and} \qquad Y^* \equiv K(\ell_2)^* \oplus_1 L_1(\mu).$$

Then, $Y^{**} \equiv L(\ell_2) \oplus_\infty L_\infty(\mu)$ is a C^*-algebra but $n(Y^*) \leqslant n\big(K(\ell_2)\big) = 1/2$.

Rango numérico e igualdades de normas para operadores.

Prehistoria, historia y resultados recientes

Miguel Martín

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17 a 22 de diciembre – Universitat de València