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Diametral notions for elements of the unit ball of a Banach space

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Contents

Abstract

We introduce extensions of Δ-points and Daugavet points in which slices are replaced by relatively weakly open subsets (super Δ-points and super Daugavet points) or by convex combinations of slices (ccs Δ-points and ccs Daugavet points). These notions represent the extreme opposite to denting points, points of continuity, and strongly regular points. We first give a general overview of these new concepts and provide some isometric consequences on the spaces. As examples:

(1) If a Banach space contains a super Δ-point, then it does not admit an unconditional FDD (in particular, unconditional basis) with suppression constant smaller than 2.
(2) If a real Banach space contains a ccs Δ-point, then it does not admit a one-unconditional basis.
(3) If a Banach space contains a ccs Daugavet point, then every convex combination of slices of its unit ball has diameter 2.

We next characterize the notions in some classes of Banach spaces, showing, for instance, that all the notions coincide in L_1-predual spaces and that all the notions but ccs Daugavet points coincide in L_1-spaces. We next comment on some examples which have previously appeared in the literature, and we provide some new intriguing examples: examples of super Δ-points which are as close as desired to strongly exposed points (hence failing to be Daugavet points in an extreme way); an example of a super Δ-point which is strongly regular (hence failing to be a ccs Δ-point in the strongest way); a super Daugavet point which fails to be a ccs Δ-point. The extensions of the diametral notions to points in the open unit ball and consequences on the spaces are also studied. Lastly, we investigate the Kuratowski measure of relatively weakly open subsets and of convex combinations of slices in the presence of super Δ-points or ccs Δ-points, as well as for spaces enjoying diameter-two properties. We conclude the paper with some open problems.

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1. Introduction

It is fair to say that one of the most studied properties of Banach spaces is the *Radon–Nikodým property* (*RNP*). Indeed, the large amount of its geometric, analytic, and measure-theoretic characterizations has made it an invaluable tool in several fields of Banach space theory such as representation of bounded linear operators, representation of dual spaces or representation of certain tensor product spaces (see [18, 20]).

A famous geometric characterization of the Radon–Nikodým property is related to the size of slices. Recall that a *slice* of a bounded non-empty subset C of a Banach space X is simply the (non-empty) intersection of C with a half-space. A Banach space X has RNP if and only if every non-empty closed and bounded subset of X admits slices of arbitrarily small diameter (see e.g. [18]).

A closely related and equally important geometric property of Banach spaces is the point of continuity property. Recall that a Banach space X has the *point of continuity property* (*PCP*) if every non-empty closed and bounded subset of X admits non-empty relatively weakly open subsets of arbitrarily small diameter. Let us emphasize here as an example the striking equivalence between the Radon–Nikodým property and the weak* version of the point of continuity property for dual spaces, and the related characterization of Asplund spaces as preduals of RNP spaces (see e.g. [19]). In his proof of the determination of the Radon–Nikodým property by subspaces with a finite-dimensional decomposition (FDD) in [17], Bourgain also introduced an important weakening of the point of continuity property, which he called property "$(*)$", and which is nowadays referred to as the convex point of continuity property. Recall that a Banach space X has the *convex point of continuity property* (*CPCP*) if every non-empty closed, convex and bounded subset of X admits non-empty relatively weakly open subsets of arbitrarily small diameter.

In fact, Bourgain implicitly used in his work the notion of strong regularity, which, as he showed, is implied by CPCP. Recall that a Banach space X is *strongly regular* (*SR*) if every non-empty closed, convex and bounded subset of X contains *convex combinations of slices* of arbitrarily small diameter. Observe that the convexity of the subset is required in this definition in order to guarantee that it contains all the convex combinations of its slices. It later turned out that strong regularity had important applications to the famous (still open) question of the equivalence between the Radon–Nikodým property and the Krein–Milman property. Recall that a Banach space X has the *Krein–Milman property* (*KMP*) if every non-empty closed, convex and bounded subset C of X admits an extreme point. RNP implies KMP (see e.g. [18, Theorem 3.3.6]), and it follows from [48] that every strongly regular space with KMP has RNP. Also recall that it was proved in [29] RNP and KMP are equivalent in dual spaces.

From the definitions it follows that RNP⇒PCP⇒CPCP and it is also known that CPCP⇒SR. None of the above implications can be reversed (see e.g. [49] and references therein). In order to show that strong regularity is implied by CPCP, Bourgain made an important geometric observation: in every non-empty bounded and convex subset of a Banach space X, every non-empty relatively weakly open subset contains a convex combination of slices. We will discuss this "Bourgain lemma" and its applications to the subject of the present paper in more detail in Chapter 2.

Another classical refinement of the above characterization of the Radon–Nikodým property is related to the notion of denting points. Recall that a point x_0 of a bounded subset C of X is a *denting point* of C if there are slices of C containing x_0 of arbitrarily small diameter. A Banach space X has RNP if and only if every closed, convex and bounded subset contains a denting point. Actually, every non-empty closed, convex and bounded subset C of a Banach space X with RNP is equal to the closure of the convex hull of the set of its denting points (see e.g. [18, Corollary 3.5.7]).

For PCP and CPCP, a similar role is played by points of weak-to-norm continuity. Given a bounded subset C of X, we say that a point $x_0 \in C$ is a *point of weak-to-norm continuity* (*point of continuity* for short) if the identity mapping $i\colon (C, w) \to (C, \tau)$ is continuous at the point x_0 or, equivalently, if x_0 belongs to relatively weakly open subsets of C of arbitrarily small diameter. Note that a classical result by Lin–Lin–Troyanski [39] establishes that a point $x_0 \in C$ is a denting point if, and only if, x_0 is simultaneously a point of continuity and an extreme point of C. In a space with PCP every non-empty closed and bounded subset contains a point of continuity. Moreover, the set of all points of continuity of a given closed, convex and bounded subset C of a Banach space X with CPCP is weakly dense in C (see e.g. [22, Theorem 1.13]).

In relation to strong regularity, a point x_0 of a bounded, convex subset C of X is a *point of strong regularity* if there are convex combinations of slices of C containing x_0 of arbitrarily small diameter. Then the set of all points of strong regularity of a given closed, convex and bounded subset C of a strongly regular Banach space X is norm dense in C (see [25, Theorem 3.6]). Observe that points of strong regularity may be in the interior of a set, while denting points (and points of continuity in the infinite-dimensional case) belong always to the border of the set (just observe that relatively weakly open subsets of a bounded convex closed set always intersect the border in the infinite-dimensional case).

In [3] the extreme opposite notion to denting point of the unit ball was introduced in the following sense. An element x in the unit sphere of a Banach space X is a Δ-*point* if we can find, in every slice of B_X containing x, points which are at distance from x as close as we wish to the maximal possible distance in the ball (distance 2). A similar yet stronger notion appeared simultaneously in relation to another quite famous property of Banach spaces, the Daugavet property. Recall that a Banach space X has the *Daugavet property* (*DPr*) if the Daugavet equation

$$\|\mathrm{Id} + T\| = 1 + \|T\| \tag{DE}$$

holds for every rank-one operator $T\colon X \to X$, where Id denotes the identity operator. In this case, all weakly compact operators also satisfy (DE). We refer the reader to the seminal paper [35] for the background. Recent results can be found in [42] and references therein. The Daugavet property admits a beautiful geometric characterization involving slices related to the notion of Daugavet points. An element x on the unit sphere of a Banach space X is a *Daugavet point* if in every slice of B_X (not necessarily containing the point x) there are points which are at distance from x as close as we wish to 2. With this definition in mind, [35, Lemma 2.1] states that X has DPr if and only if all elements in S_X are Daugavet points. Let us comment that the Daugavet property imposes severe restrictions on a Banach space. If X is a Banach space with DPr, then it fails RNP and it has no unconditional basis (actually, it cannot be embedded into a Banach space with unconditional basis).

On the other hand, Δ- and Daugavet points have proved to be far more flexible than the global properties that they define. For example:

- There exists a Banach space with RNP and a Daugavet point [51] (see Section 4.3.1).
- There exists a Banach space with a one-unconditional basis and a large subset of Daugavet points [6] (see Section 4.3.3).
- There exists a Banach space X such that all elements in S_X are Δ-points and such that every convex combination of slices of B_X intersecting S_X has diameter 2, but there are convex combinations of slices of B_X with arbitrarily small diameter [2] (see Section 4.3.2).

Nonetheless, it has recently been proved that Δ-points have some influence on the isometric structure of the space. For example, it is shown in [5] that *uniformly non-square* spaces do not contain Δ-points. Actually, it has been very recently proved in [37] that a Δ-point cannot be a *locally uniformly non-square point*. Also, combining the results from [5] and [52], *asymptotic uniformly smooth* spaces and their duals do not contain Δ-points. However, it is still an important open problem to understand whether Δ- or Daugavet points have any influence on the isomorphic structure of the space.

In this paper, our main aim is to study natural strengthenings of the notions of Daugavet points and Δ-points obtained by replacing slices by non-empty relatively weakly open subsets ("super points") or convex combination of slices ("ccs points") in order to provide new diametral notions which are extreme opposites to points of continuity and to strongly regular points, respectively. See Definitions 2.5 and 2.4 for details. Our main goal will be to understand the influence, for a given Banach space, of the existence of such points on its geometry, and to study the different diametral notions in several families of Banach spaces. A particular emphasis will be put on trying to distinguish between various formally different notions.

Let us end this chapter by giving a brief description about the organization of the paper and the main results obtained. Chapter 2 contains the necessary notation (which is standard, anyway), needed definitions, and some preliminary results. We include in Chapter 3 some characterizations of the new diametral point notions and some necessary conditions on the existence of such points. In particular, we study the existence of super Δ-points and ccs Δ-points in spaces with a one-unconditional basis. We first give an analogue for ccs Δ-points to a result from [6] which implicitly states that such spaces contain no super Δ-points. Second, we provide sharper and improved versions of this super Δ result in the context of unconditional FDDs with a small unconditional constant, and more generally in the context of spaces in which special families of operators are available. The chapter finishes with the study of the behaviour of super Δ-points and super Daugavet points with respect to absolute sums somehow analogous to the known one for Δ-points and Daugavet points. However, not all the results extend to ccs Δ-points and ccs Daugavet points, but we also give some partial results. Chapter 4 is devoted to examples and counterexamples. We first characterize the diametral notions in some families of classical Banach spaces: we show that all notions are equivalent in L_1-preduals and Müntz spaces (Section 4.1); all notions but ccs Daugavet points also coincide in L_1-spaces (Section 4.2). We next give in Section 4.3 some remarks on examples which have previously appeared in the literature, discussing the new diametral notions on them, and showing that they may help to distinguish between the diametral notions. The most complicated and tricky examples are produced in the last three sections of this chapter. We present super Δ-points which are as closed as desired to strongly exposed points (hence failing to be Daugavet points in an extreme way) in Section 4.4; super Δ-points which are strongly regular (hence failing to be ccs Δ-points in an extreme way) in Section 4.5; and super Daugavet points which belong to convex combinations of slices of diameter as small as desired (hence failing to be ccs Δ-points in an extreme way) in Section 4.6. We finish this section with a summary of relations between all the diametral notions. The idea in Chapter 5 is to generalize the diametral notions to elements of the open unit ball, and use these notions to characterize some geometric properties. In particular, we properly localize the result by Kadets that DSD2P is equivalent to the Daugavet property. Chapter 6 deals with Kuratowski index of non-compactness of slices, relatively weakly open subsets, and convex combinations of slices. We find that every relatively weakly open subset (respectively, every convex combination of slices) in a space with the diameter 2 property (respectively, with the strong diameter 2 property) has Kuratowski measure 2; these results extends the analogous result for slices and the the local diameter 2 property proved in [21, Proposition 3.1]. Also, we show that every relatively weakly open subset that contains a super Δ-point has Kuratowski measure 2, and a similar result is obtained with convex combinations of relatively weakly open subsets containing a ccw Δ-point; these results extend [52, Corollary 2.2]. Finally, Chapter 7 collects some interesting open questions and some remarks on them.

2. Notation and preliminary results

We will use standard notation as in the books [8], [23], and [24], for instance. Given a Banach space X, B_X (respectively, S_X) stands for the closed unit ball (respectively, the unit sphere) of X. We denote by X^* the topological dual of X and we write $J_X : X \to X^{**}$ for the canonical injection. We denote by $\operatorname{dent}(B_X)$ and $\operatorname{ext}(B_X)$ the sets of all denting points of B_X and of all extreme points of B_X, respectively. The set of preserved extreme points of B_X (i.e. those $x \in B_X$ such that $J_X(x) \in \operatorname{ext}(B_{X^{**}})$) is denoted by $\operatorname{pre-ext}(B_X)$. For Banach spaces X and Y, $\mathcal{L}(X,Y)$, $\mathcal{F}(X,Y)$, $\mathcal{K}(X,Y)$ denote, respectively, the sets of all (bounded linear) operators, finite-rank operators, and compact operators. The properties we are interested in only deal with the real structure of the Banach spaces involved, but we do not restrict the study to real spaces in order to consider real or complex examples. We will use the notation $\mathbb{K}$ to denote either $\mathbb{R}$ or $\mathbb{C}$, $\operatorname{Re}(z)$ to denote the real part of z (which is just the identity when dealing with a real space), and $\mathbb{T}$ to represent the set of scalars of modulus 1.

Given a non-empty subset C of X, we will denote by $\operatorname{co}(C)$ the convex hull of C and by $\operatorname{span}(C)$ the linear hull of C. Also we denote by $\overline{\operatorname{co}}(C)$ (respectively, $\overline{\operatorname{span}}(C)$) the norm closure of the convex hull (respectively, of the linear hull) of C. By a *slice* of C we will mean any subset of C of the form

$$S(x^*, \delta; C) := \{x \in C\colon\ \operatorname{Re} x^*(x) > M - \delta\}$$

where $x^* \in X^*$ is a continuous linear functional on X, $\delta > 0$ is a positive real number, and $M := \sup_{x \in C} \operatorname{Re} x^*(x)$. For slices of the unit ball we will simply write $S(x^*, \delta) := S(x^*, \delta; B_X)$. By a *relatively weakly open subset* of C we mean as usual any subset of C obtained as the (non-empty) intersection of C with an open set of X in the weak topology.

If C is assumed to be convex we will mean by a *convex combination of slices* of C (*ccs* of C for short) any subset of C of the form

$$\sum_{i=1}^{n} \lambda_i S_i,$$

where $\lambda_1, \ldots, \lambda_n \in (0, 1]$ are such that $\sum_{i=1}^{n} \lambda_i = 1$ and S_i is a slice of C for every $i \in \{1, \ldots, n\}$. Observe that convex combinations of slices are convex sets. We define in the same way *convex combinations of relatively weakly open subsets of* C (*ccw* of C for short).

The following lemma from [30] is a very useful tool when working with Δ-points.

Lemma 2.1 ([30, Lemma 2.1]). *Let X be a Banach space, and let $x^* \in S_{X^*}$ and $\alpha > 0$. For every $x \in S(x^*, \alpha)$ and every $0 < \beta < \alpha$ there exists $y^* \in S_{X^*}$ such that*

$$x \in S(y^*, \beta) \subseteq S(x^*, \alpha).$$

We also often rely on the following result, due to Bourgain and already mentioned in the introduction. We provide a proof below, following the one from [25, Lemma II.1], for the sake of completeness and for further discussion.

Lemma 2.2 (Bourgain). *Let X be a Banach space and let C be a bounded convex closed subset of X. Then every non-empty relatively weakly open subset W of C contains a convex combination of slices of C.*

Proof. Assume with no loss of generality that $W := \bigcap_{i=1}^m S(f_i, \alpha_i, C)$, write

$$\widetilde{C} = \overline{J_X(C)}^{w^*} \subset X^{**}, \quad \text{and} \quad W^{**} := \bigcap_{i=1}^{m} S\big(J_{X^*}(f_i), \alpha_i; \widetilde{C}\big),$$

which is a non-empty relatively weak* open subset of $\widetilde{C}$. By the Krein–Milman theorem (see e.g. [24, Theorem 3.37]), it follows that

$$\widetilde{C} = \overline{\operatorname{co}}^{w^*}(\operatorname{ext}\widetilde{C}), \quad \text{so} \quad \operatorname{co}\big(\operatorname{ext}\widetilde{C}\big) \cap W^{**} \neq \emptyset.$$

Pick a convex combination $\sum_{i=1}^n \lambda_i e_i^{**}$ of extreme points contained in W^{**}. By the continuity of the sum we can find, for every $1 \leqslant i \leqslant n$, a weak* open subset W_i^{**} with $e_i^{**} \in W_i^{**}$ and such that $\sum_{i=1}^n \lambda_i W_i^{**} \subset W^{**}$.

Since each e_i^{**} is an extreme point of $\widetilde{C}$, by Choquet's lemma (see [24, Lemma 3.40], for instance) there are weak* slices $S\big(J_{X^*}(g_i), \beta_i; \widetilde{C}\big)$ with $e_i^{**} \in S\big(J_{X^*}(g_i), \beta_i; \widetilde{C}\big) \subseteq W_i^{**}$ for every $i \in \{1, \dots, n\}$. Hence

$$\sum_{i=1}^{n} \lambda_i S\big(J_{X^*}(g_i), \beta_i; \widetilde{C}\big) \subseteq \sum_{i=1}^{n} \lambda_i W_i^{**} \subseteq W^{**}.$$

Now, if we take

$$U := \sum_{i=1}^{n} \lambda_i S(g_i, \beta_i, C)$$

it is not difficult to prove that $U \subseteq W$, as desired. ∎

Remark 2.3. Observe that, in general, it is unclear from the above proof whether or not, if we fix $x \in W$, we can guarantee that there exists a convex combination of slices U of C such that $x \in U \subseteq W$.

On the other hand, the result holds true if $x \in W \cap \operatorname{co}(\text{pre-ext}(C))$ in view of the above proof. Indeed, in that situation, if we write $x = \sum_{i=1}^n \lambda_i x_i \in W$ with $x_1, \dots, x_n \in \text{pre-ext}(C)$ and $\lambda_1, \dots, \lambda_n \in (0,1]$ with $\sum_{i=1}^n \lambda_i = 1$, by the weak continuity of the sum, we can find, for every $1 \leqslant i \leqslant n$, a non-empty relatively weakly open subset V_i with $x_i \in V_i$ for every i and such that $x = \sum_{i=1}^n \lambda_i x_i \in \sum_{i=1}^n \lambda_i V_i \subseteq W$ (in fact, let $Y := (X^n, \|\cdot\|_\infty)$ and consider the map $T\colon Y \to X$ given by $T(y_1, \dots, y_n) := \sum_{i=1}^n \lambda_i y_i$. Then T is clearly linear and continuous, hence weak-to-weak continuous. Furthermore, $T(B_Y) \subset B_X$, so the restriction of T to B_Y defines a continuous map from B_Y into B_X with respect to the corresponding relative weak topologies. In particular, there exists a neighborhood V of $\tilde{x} := (x_1, \dots, x_n)$ for the relative weak topology of B_Y such that $T(V) \subset W$. Without loss of generality, V is of the form

$$V := \{y \in B_Y \colon \ |\varphi_j(\tilde{x} - y)| < \varepsilon \text{ for every } j \in \{1, \dots, m\}\}$$

for some $m \in \mathbb{N}$, $\varphi_1, \dots \varphi_m \in Y^*$ and $\varepsilon > 0$. For every $j \in \{1, \dots, m\}$, write $\varphi_j := (y^*_{j,1}, \dots, y^*_{j,n})$ with $y^*_{j,1}, \dots, y^*_{j,n} \in X^*$. Then, for every $i \in \{1, \dots, n\}$, let

$$V_i := \{z \in B_X \colon \ \left|y^*_{j,i}(x_i - z)\right| < \varepsilon/n \text{ for every } j \in \{1, \dots, m\}\}.$$

On the one hand, V_i is a relatively weakly open neigborhood of x_i in B_X, and on the other hand, $\prod_{i=1}^n V_i \subset V$. Hence, $\sum_{i=1}^n \lambda_i V_i = T(\prod_{i=1}^n V_i) \subset W$, and we have constructed the desired sets). Now, observe that, since each x_i is a preserved extreme point of C, slices of C containing x_i are a neighbourhood basis for x_i in the weak topology. Hence, we can find, for $1 \leqslant i \leqslant n$, a slice S_i of C with $x_i \in S_i \subseteq V_i$, and so $x = \sum_{i=1}^n \lambda_i x_i \in \sum_{i=1}^n \lambda_i S_i \subseteq \sum_{i=1}^n \lambda_i V_i \subseteq W$, so $U := \sum_{i=1}^n \lambda_i S_i$ is the desired convex combination of slices.

Throughout the text, we will often be discussing various "diameter-two properties". We use the notation introduced in [7]. A Banach space X has the *local or slice diameter-two property* (*LD2P*) if every slice of B_X has diameter 2; X has the *diameter-two property* (*D2P*) if every non-empty relatively weakly open subset of B_X has diameter 2; finally, X has the *strong diameter-two property* (*SD2P*) whenever every ccs of B_X has diameter 2 (and then every ccw has diameter 2 due to Lemma 2.2). For definitions and examples concerning those properties, we refer to [2, 14, 15, 45]. In particular, let us comment that the three properties are different, a result which was not easy to show [14]. Our paper is closely related to the diametral versions of those properties which have been implicitly studied for a long time in the literature, but whose formal definitions and names were fixed in [13]. A Banach space X has the *diametral local diameter-two property* (*DLD2P*) if for every slice S of B_X and every $x \in S \cap S_X$, $\sup_{y \in S} \|x - y\| = 2$. If slices are replaced by non-empty relatively weakly open subsets of B_X, we obtain the *diametral diameter-two property* (*DD2P*). It is immediate that these properties are not satisfied by any finite-dimensional space. Clearly, DLD2P implies LD2P, DD2P implies D2P (and none of these implications reverses, e.g. $X = c_0$), and DD2P implies DLD2P. It is unknown whether DLD2P and DD2P are equivalent. In fact, it is even unknown whether DLD2P implies D2P. For the analogous definition using ccs, we have to discuss a little bit more. Even for an infinite-dimensional space X, it is not true that every ccs of B_X intersects S_X. Actually, this happens if and only if X has a property stronger than SD2P (see [41, Theorem 3.4]). Thus, the definition of the *diametral strong diameter-two property* (*DSD2P*) given in [13] deals with all points in B_X as follows: for every ccs C and every $x \in C$, $\sup_{y \in C} \|x - y\| = \|x\| + 1$. This definition allows us to show that DSD2P implies SD2P. But, actually, it has recently been shown by V. Kadets [34] that DSD2P is equivalent to the Daugavet property. We will discuss this in detail in Chapter 5. On the other hand, we will use the following property which is weaker than DSD2P: a Banach space X has the *restricted DSD2P* if for every ccs C and every $x \in C \cap S_X$, $\sup_{y \in C} \|x - y\| = 2$. This property is strictly weaker than DSD2P; see Section 4.3.2.

Let us now introduce all notions of diametral points that we will consider in the text. Let us start with the more closely related ones to the definitions above.

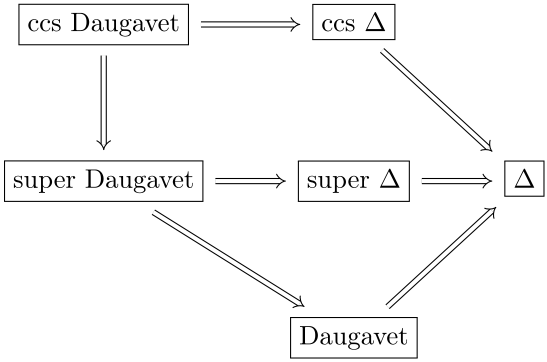

Figure 1. Relations between the diametral notions

Definition 2.4. Let X be a Banach space and let $x \in S_X$. We say that

(1) ([3]) x is a *Δ-point* if $\sup_{y\in S} \|x-y\| = 2$ for every slice S of B_X containing x,
(2) x is a *super Δ-point* if $\sup_{y\in V} \|x-y\| = 2$ for every non-empty relatively weakly open subset V of B_X containing x,
(3) x is a *ccs Δ-point* if $\sup_{y\in C} \|x-y\| = 2$ for every slice ccs C of B_X containing x.

Δ-points were introduced in [3] as a natural localization of DLD2P (i.e. X has DLD2P if and only if every element of S_X is a Δ-point). The other two definitions are new. Clearly, super Δ-points are the natural localization of DD2P, that is, X has DD2P if and only if every element of S_X is a super Δ-point. Moreover, ccs Δ-points are the localization of restricted DSD2P, as X has restricted DSD2P if and only if every element of S_X is a ccs Δ-point.

In relation to the Daugavet property, we have the following notions for points.

Definition 2.5. Let X be a Banach space and let $x \in S_X$. We say that

(1) ([3]) x is a *Daugavet point* if $\sup_{y\in S} \|x-y\| = 2$ for every slice S of B_X,
(2) x is a *super Daugavet point* if $\sup_{y\in V} \|x-y\| = 2$ for every non-empty relatively weakly open subset V of B_X,
(3) x is a *ccs Daugavet point* if $\sup_{y\in C} \|x-y\| = 2$ for every ccs C of B_X.

Let us recall that Daugavet points were introduced in [3] as a natural localization of the *Daugavet property* in the sense that a Banach space X has the Daugavet property if and only if every point in S_X is a Daugavet point [35, Lemma 2.1]. From the geometric characterization given in [50, Lemma 3] and the implicit result contained in its proof, it follows that super Daugavet points as well as ccs Daugavet points are also natural localizations of the Daugavet property.

Since every slice of B_X is relatively weakly open, and since by Bourgain's lemma (see Lemma 2.2) every non-empty relatively weakly open subset of B_X contains a ccs of B_X, we clearly have the diagram of Figure 1.

We will show throughout the text that none of the above implications reverses: there are Daugavet points which are not super Δ-points, super Daugavet points which are not ccs Δ-points, and so on. We refer to Section 4.7 for a description of all the relations and counterexamples. However, let us point out right away that we do not know whether there exist ccs Δ-points which are not super Δ. In view of Remark 2.3 such examples may exist since Bourgain's lemma is not localizable. Also, it follows again from Bourgain's lemma that a ccs Daugavet point $x \in S_X$ also satisfies $\sup_{y\in D} \|x - y\| = 2$ for every convex combination of relatively weakly open subsets (ccw for short) D of B_X. Again this is not clear for ccs Δ-points and we could thus naturally distinguish between ccs Δ-points and a formally stronger notion of "ccw Δ-points" where arbitrary ccw of the unit ball would be involved instead of only ccs. Since we do not have concrete examples at hand that allow distinguishing between these two notions, we will focus on convex combinations of slices and specifically point out any available ccw behavior throughout the text.

Let us also comment that it is clear that if every ccs of the unit ball of a given Banach space is weakly open (respectively, has non-empty relative weak interior), then every super Δ-point (respectively, every super Daugavet point) in this space is a ccs Δ-point (respectively, a ccs Daugavet point). Several properties of this kind were introduced and studied in [1, 4, 41]. We refer to those papers for some background and for examples.

Remark 2.6. There are natural weak* versions in dual spaces of all the notions of diametral points introduced in the present chapter where slices and relatively weakly open subsets are respectively replaced with weak* slices (i.e. slices defined by elements of the predual) and relatively weak* open subsets. With obvious terminology, it then follows from [35, Lemma 2.1] and from [50, Lemma 3] that a Banach space X has the Daugavet property if and only if every element in S_{X^*} is a weak* Daugavet point if and only if every element in S_{X^*} is a weak* ccs Daugavet point. It also follows from [2, Theorem 3.6] that X has DLD2P if and only if every point in S_{X^*} is a weak* Δ-point. However, the relationship between DD2P in X and weak* super Δ-points in S_{X^*} is currently unknown.

Observe that a direct consequence of those results is that weak* diametral points and their weak counterparts might differ in a very strong way since, for instance, the unit ball of the space $C[0, 1]^*$ admits denting points. Yet clearly all the results from the following chapters concerning the different notions of diametral points admit obvious analogues for their weak* counterparts. We leave the details to the reader to avoid unnecessary repetitions, but let us still point out that it follows from Goldstine's theorem and from the lower weak* semicontinuity of the norm in dual spaces that there is a natural correspondence between diametral properties of points in S_X and weak* properties of their image in the bidual under the canonical embedding J_X. Namely:

(1) $x \in S_X$ is a Daugavet point (respectively, a ccs Daugavet point) if and only if $J_X(x)$ is a weak* Daugavet point (respectively, a weak* ccs Daugavet point).
(2) $x \in S_X$ is a super Daugavet point if and only if $J_X(x)$ is a weak* super Daugavet point if and only if for every $y \in B_X$ there exists a net (y_s^{**}) in $B_{X^{**}}$ which converges to $J_X(y)$ in the weak* topology and such that $\|\pi_X(x) - y_s^{**}\| \to 2$ (see Chapter 3).
(3) $x \in S_X$ is a Δ-point (respectively, a super Δ-point) if and only if $J_X(x)$ is a Δ-point (respectively, a super Δ-point).
(4) $x \in S_X$ is a ccs Δ-point if and only if $J_X(x)$ is a weak* ccs Δ-point.

Let us point out that (3) essentially follows from the obvious fact that Δ-points and super Δ-points naturally pass to superspaces, that is, if Y is a subspace of X and if $x \in S_Y$ if a Δ-point (respectively, a super Δ-point) in Y, then x is a Δ-point (respectively, a super Δ-point) in X. This property is unclear for ccs Δ-points, so assertion (4) is not analogous to assertion (3).

3. Characterizations of diametral notions and implications on the geometry of the ambient space

In view of the definitions of diametral points, it is natural to expect that the presence of any kind of Daugavet element or Δ-element in a given Banach space will affect, by the severe restrictions it inflicts on the nature of the point considered, its global isometric geometry or even its topological structure. However, previous studies in this context have shown that the situation is much more complicated than one could expect at first sight. For example, a Banach space X with RNP and admitting a Daugavet point, and a Banach space with a one-unconditional basis and admitting a weakly dense subset of Daugavet points, were respectively constructed in [51] and in [6]. In this chapter, we provide useful characterizations of the new diametral notions, and investigate the immediate effect of the presence of such points on the geometry of the space considered.

We start with an intuitive but not completely trivial observation.

Observation 3.1. By definition, it is clear that super Δ-points do not exist in finite-dimensional spaces because the weak and norm topology coincide in this context. Also, it was proved in [5, Theorem 4.4] that finite-dimensional spaces also fail to contain Δ-points (hence ccs Δ-points). In fact, they fail to contain them in a stronger way [5, Corollary 6.10]. Consequently, the study of diametral notions only makes sense in infinite dimension, and from now on we will assume unless otherwise stated that all the Banach spaces we consider are infinite-dimensional.

Let us next prove a bunch of characterizations for super Daugavet points and super Δ-points.

Let X be a Banach space. For every $x \in S_X$ and for every $\varepsilon > 0$, define

$$\Delta_\varepsilon(x) := \{y \in B_X : \|x - y\| > 2 - \varepsilon\}.$$

We recall the following characterization of Daugavet points and Δ-points from [3].

Lemma 3.2 ([3, Lemmas 2.1 and 2.2]). *Let X be a Banach space.*

(1) *An element $x \in S_X$ is a Daugavet point if and only if $B_X = \overline{\text{co}}\,\Delta_\varepsilon(x)$ for every $\varepsilon > 0$.*
(2) *An element $x \in S_X$ is a Δ-point if and only if $x \in \overline{\text{co}}\,\Delta_\varepsilon(x)$ for every $\varepsilon > 0$.*

We have similar characterizations for super points.

Lemma 3.3. *Let X be a Banach space.*

(1) *An element $x \in S_X$ is a super Daugavet point if and only if $B_X = \overline{\Delta_\varepsilon(x)}^w$ for every $\varepsilon > 0$.*
(2) *An element $x \in S_X$ is a super Δ-point if and only if $x \in \overline{\Delta_\varepsilon(x)}^w$ for every $\varepsilon > 0$.*

Proof. Observe that for given $x \in S_X$, $y \in B_X$, and $\varepsilon > 0$, the point y belongs to the weak closure of the set $\Delta_\varepsilon(x)$ if and only if $\Delta_\varepsilon(x)$ has non-empty intersection with any neighborhood of y in the relative weak topology of B_X. Thus y belongs to $\overline{\Delta_\varepsilon(x)}^w$ for every $\varepsilon > 0$ if and only if $\sup_{z \in V} \|x - z\| = 2$ for every relatively weakly open subset V of B_X containing y. The conclusion easily follows. ∎

For any given $x \in B_X$, we denote by $\mathcal{V}(x)$ the set of all neighborhoods of x for the relative weak topology of B_X. We can provide characterizations of super points using nets, which is just a localization of [13, Proposition 2.5].

PROPOSITION 3.4. *Let X be an infinite-dimensional Banach space.*

(1) *An element $x \in S_X$ is a super Daugavet point if and only if for every $y \in B_X$ there exists a net (y_s) in B_X which converges weakly to y and such that $\|x - y_s\| \to 2$.*
(2) *An element $x \in S_X$ is a super Δ-point if and only if there exists a net (x_s) in B_X which converges weakly to x and such that $\|x - x_s\| \to 2$.*

In both cases we can moreover force the nets to be in S_X.

Proof. Let us fix $x \in S_X$. Given any $y \in B_X$, it is clear that if there exists a net (y_s) in B_X which converges weakly to y and such that $\|x - y_s\| \to 2$, then y belongs to the weak closure of $\Delta_\varepsilon(x)$ for every $\varepsilon > 0$. Conversely, let us pick $y \in B_X$ satisfying this property. We turn $S := \mathcal{V}(y) \times (0, \infty)$ into a directed set by $(V, \varepsilon) \leqslant (V', \varepsilon')$ if and only if $V' \subset V$ and $\varepsilon' \leqslant \varepsilon$. By the assumptions, $V \cap \Delta_\varepsilon(x)$ is a non-empty subset of B_X for every couple $s := (V, \varepsilon)$ in S. Picking any y_s in this set will then provide the desired net.

Finally, observe that for $x \in B_X$ and $\varepsilon > 0$, the set $B_X \backslash \Delta_\varepsilon(x) = \{y \in B_X : \|x - y\| \leqslant 2 - \varepsilon\}$ is weakly closed by the lower semicontinuity of the norm, so that $\Delta_\varepsilon(x)$ is a relatively weakly open subset of B_X. Thus $V \cap \Delta_\varepsilon(x)$ is a non-empty relatively weakly open subset of B_X for every couple $s := (V, \varepsilon)$ in S. Since X is infinite-dimensional, this set has to intersect S_X, and we can actually pick y_s in $V \cap \Delta_\varepsilon(x) \cap S_X$. ∎

REMARK 3.5. In [36] an example of a Banach space satisfying simultaneously the Daugavet property and the Schur property was provided. The example shows that there is no hope to get a version of the above result involving sequences.

Observe that the following result, similar to [32, Lemmas 2.1 and 2.2], is included in the preceding proof.

PROPOSITION 3.6. *Let X be a Banach space and let $x \in S_X$.*

(1) *If x is a super Daugavet point, then for every $\varepsilon > 0$ and every non-empty relatively weakly open subset V of B_X we can find a non-empty relatively weakly open subset U of B_X which is contained in V and such that $\|x - y\| > 2 - \varepsilon$ for every $y \in U$.*
(2) *If x is a super Δ-point, then for every $\varepsilon > 0$ and every non-empty relatively weakly open subset V of B_X containing x we can find a non-empty relatively weakly open subset U of B_X which is contained in V and such that $\|x - y\| > 2 - \varepsilon$ for every $y \in U$.*

Proof. Fix any $x \in S_X$ and any $y \in B_X$ which belongs to the weak closure of $\Delta_\varepsilon(x)$ for every $\varepsilon > 0$. Then, for every $V \in \mathcal{V}(y)$ and every $\varepsilon > 0$, the set $U := V \cap \Delta_\varepsilon(x)$ is a non-empty relatively weakly open subset of B_X. ∎

It is clear from the definition that denting points of B_X cannot be Δ-points. Also it was first observed in [32, Proposition 3.1] that every Daugavet point in a Banach space X has to be at distance 2 from every denting point of the unit ball of X. This elementary observation turned out to play an important role in the study of Daugavet points in Lipschitz-free spaces in [32] and [51]. We have similar observations for super points.

LEMMA 3.7. *Let X be a Banach space and let $x \in S_X$. If x is a super Δ-point, then x cannot be a point of continuity. If, moreover, x is a super Daugavet point, then x has to be at distance 2 from every point of continuity of B_X.*

Proof. If an element y of B_X is a point of continuity, then it is contained in relatively weakly open subsets of B_X of arbitrarily small diameter. Clearly no super Δ-point can have this property, and any super Daugavet point has to be at distance 2 from any such points. ∎

This lemma provides quite a few examples of Banach spaces which fail to contain super points. Following [26] let us recall that X has the *Kadets property* if the norm topology and the weak topology coincide on S_X, and that X has the *Kadets–Klee* property if weakly convergent sequences in S_X are norm convergent. Let us also recall that any LUR space has the Kadets–Klee property, and that any space with the Kadets–Klee property which fails to contain ℓ_1 has the Kadets property. By Proposition 3.4 we clearly have the following result.

PROPOSITION 3.8. *If X has the Kadets property, then X fails to contain super Δ-points.*

As a corollary we obtain the following. Recall that a Banach space is *asymptotic uniformly convex* (*AUC* for short) [31] if its modulus of asymptotic uniform convexity

$$\overline{\delta}_X(t) := \inf_{x \in S_X} \sup_{\dim X/Y < \infty} \inf_{y \in S_Y} \|x + ty\| - 1$$

is strictly positive for every $t > 0$.

COROLLARY 3.9. *Let X be AUC. Then X fails to contain super Δ-points.*

Proof. In an AUC space, every element of the unit sphere is a point of continuity of B_X (see [31, Proposition 2.6]). ∎

REMARK 3.10. It was proved in [5, Theorem 3.4] that any *reflexive* AUC space fails to contain Δ-points. Also, combining the observations from [5, end of Section 4] about weak* quasi-denting points in the unit ball of AUC* duals and [52, Corollary 2.4] about the maximality of the Kuratowski index of weak* slices containing weak* Δ-points, we find that every AUC* dual space fails to contain weak* Δ-points. However, it is currently unknown whether non-reflexive AUC spaces (and, in particular, whether the dual of the James tree spaces JT*) may contain Daugavet points or Δ-points.

It turns out that Daugavet points are characterized by their distances to denting points in RNP spaces (because the unit ball of an RNP space X can be written as the closed convex hull of the set of its denting points) as well as in Lipschitz-free spaces ([32, Theorem 3.2] for compact metric spaces and [51, Theorem 2.1] for a general statement). In the same way we can characterize super Daugavet points in terms of their distances to points of continuity of B_X in spaces with CPCP.

PROPOSITION 3.11. *If a Banach space X has CPCP, then a point $x \in S_X$ is a super Daugavet point if and only if it is at distance 2 from any point of continuity of B_X.*

Proof. If X has CPCP, then the set of all points of continuity of B_X is weakly dense in B_X (see for example [22, Proposition 3.9]), that is, every non-empty relatively weakly open subset of B_X contains a point of continuity. The conclusion follows easily. ∎

For ccs points, the situation is quite different. Indeed, although clearly ccs Δ-points may not be points of strong regularity, from [41, Theorem 3.1] we know that X has SD2P if and only if every convex combination of slices of B_X contains elements of norm arbitrarily close to 1. It readily follows that any space X which contains a ccs Daugavet point satisfies SD2P, so it is very far from being strongly regular. We will provide more details on this topic in Chapter 5, but for later reference let us state the following.

PROPOSITION 3.12. *Let X be a Banach space. If X contains a ccs Daugavet point, then it has SD2P (it fails to be strongly regular).*

Next, we show that extreme points have a nice behaviour with respect to diametral notions.

PROPOSITION 3.13. *Let X be a Banach space and let $x \in S_X$.*

(1) *If $x \in \text{pre-ext}(B_X)$ and it is a Δ-point, then x is a super Δ-point.*
(2) *If $x \in \text{ext}(B_X)$ and it is a super Δ-point, then x is a ccs Δ-point.*
(3) *In particular, if $x \in \text{pre-ext}(B_X)$ is a Δ-point, then x is a super Δ-point as well as a ccs Δ-point.*

Proof. It follows from Choquet's lemma (see for example [23, Lemma 3.69]) that slices form neighborhood bases in the relative weak topology of the unit ball of a Banach space for its preserved extreme points, so (1) immediately follows. For (2), if x is extreme and belongs to a ccs $C := \sum_{i=1}^{n} \lambda_i S_i$ of B_X then $x \in \bigcap_{i=1}^{n} S_i$, which is a relatively weakly open subset of B_X. ∎

REMARK 3.14. Observe that, in fact, any extreme super Δ-point is "ccw Δ-point" as we discussed in Chapter 2. Also, Choquet's lemma implies that every extreme weak* Δ-point in a dual space is a weak* ccw Δ-point.

3.1. Spaces with a one-unconditional basis and beyond. In [6], it was proved that no real Banach space with a subsymmetric basis contains a Δ-point. On the other hand, an example of a Banach space with a one-unconditional basis that contains a Δ-point was provided, and a more involved example of a Banach space with a one-unconditional basis that contains many Daugavet points was constructed. We will discuss this second example in detail in Section 4.3.3.

In the process, it was also implicitly shown that real Banach spaces with a one-unconditional basis cannot contain super Δ-points. In the present section, we prove that the same holds for ccs Δ-points. Also, we provide sharper and more general versions of [6, Proposition 2.12]. In the first part of this chapter, we follow [6] and restrict ourselves to real Banach spaces.

Let X be a real Banach space with a Schauder basis $(e_i)_{i\geqslant 1}$. We denote by $(e_i^*)_{i\geqslant 1}$ the corresponding sequence of biorthogonal functionals. Recall that $(e_i)_{i\geqslant 1}$ is said to be *unconditional* if the series $\sum_{i\geqslant 1} e_i^*(x)e_i$ converges unconditionally for every $x \in X$. Also, recall that an unconditional basis $(e_i)_{i\geqslant 1}$ is said to be *one-unconditional* if

$$\Big\| \sum_{i\geqslant 1} \theta_i e_i^*(x)e_i \Big\| = \Big\| \sum_{i\geqslant 1} e_i^*(x)e_i \Big\|$$

for every $(\theta_i)_{i\geqslant 1} \in \{-1,1\}^{\mathbb{N}}$ and for every $x \in X$. Moreover, if

$$\Big\| \sum_{i\geqslant 1} \theta_i e_i^*(x)e_{n_i} \Big\| = \Big\| \sum_{i\geqslant 1} e_i^*(x)e_i \Big\|$$

for every $(\theta_i)_{i\geqslant 1} \in \{-1,1\}^{\mathbb{N}}$, for every $x \in X$, and for every strictly increasing sequence $(n_i)_{i\geqslant 1}$ in $\mathbb{N}$, then the basis is called *subsymmetric*.

Observe that for spaces with a one-unconditional basis, it is enough, in order to study the various Daugavet and Δ-notions, to work in the *positive sphere*

$$S_X^+ := \{x \in S_X : e_i^*(x) \geqslant 0 \ \forall i\}$$

of the space X. Also, the following result is well known.

Lemma 3.15. *Let X be a real Banach space with a one-unconditional basis $(e_i)_{i\geqslant 1}$, and let $(a_i)_{i\geqslant 1}$ and $(b_i)_{i\geqslant 1}$ be sequences of real numbers. If the series $\sum_{i\geqslant 1} b_i e_i$ converges, and if $|a_i| \leqslant |b_i|$ for every i, then $\sum_{i\geqslant 1} a_i e_i$ converges as well, and we have*

$$\Big\| \sum_{i\geqslant 1} a_i e_i \Big\| \leqslant \Big\| \sum_{i\geqslant 1} b_i e_i \Big\|.$$

Let us now recall some notation and preliminary results from [6]. Let X be a real Banach space with a normalized one-unconditional basis $(e_i)_{i\geqslant 1}$. For every subset A of $\mathbb{N}$, we denote by P_A the projection on $\overline{\operatorname{span}}\{e_i : i \in A\}$. Then for every $x \in X$, we define

$$M(x) := \{A \subset \mathbb{N} : \|P_A(x)\| = \|x\|, \text{ and } \|P_A(x) - e_j^*(x)e_j\| < \|x\| \ \forall j \in A\}.$$

The set $M(x)$ can be seen as the set of all *minimal norm-giving subsets* of the support of x. We denote respectively by $M^{\mathcal{F}}(x)$ and $M^{\infty}(x)$ the subsets of all finite and infinite elements of $M(x)$. It follows from [6, Lemma 2.7] that the set $M(x)$ is never empty, and from [6, Proposition 2.15] that no element $x \in S_X$ satisfying $M^{\infty}(x) = \emptyset$ can be a Δ-point.

For every non-empty ordered subset $A := \{a_1 < a_2 < \cdots\}$ of $\mathbb{N}$, and for every $n \in \mathbb{N}$ smaller than or equal to $|A|$, we denote by $A(n) := \{a_1, \ldots, a_n\}$ the subset consisting of the n first elements of A. We will implicitly assume in the following that the elements of $M(x)$ are ordered subsets of $\mathbb{N}$. The next two results were proved in [6, Lemmas 2.8 and 2.11].

LEMMA 3.16. *Let X be a real Banach space with a normalized one-unconditional basis $(e_i)_{i\geqslant 1}$ and let $x \in S_X$. For every $n \in \mathbb{N}$, the sets*

$$\{A \in M(x)\colon\ |A| \leqslant n\} \quad \text{and} \quad \{A(n)\colon A \in M(x),\ |A| > n\}$$

are both finite.

LEMMA 3.17. *Let X be a real Banach space with a normalized one-unconditional basis and let $x \in S_X$. For every subset E of $\mathbb{N}$ such that $E \cap A \neq \emptyset$ for every $A \in M(x)$, we have $\|x - P_E(x)\| < 1$.*

With those tools at hand, we can now prove an analogue to [6, Proposition 2.13] for convex combinations of slices.

PROPOSITION 3.18. *Let X be a Banach space with a normalized one-unconditional basis and $x \in S_X^+$. Then there exists $\delta > 0$ and a ccs C of B_X containing x such that $\sup_{y\in C} \|x - y\| \leqslant 2 - \delta$.*

Proof. Let $x \in S_X^+$, and define $E = \bigcup_{A\in M(x)} A(1)$. From Lemmas 3.16 and 3.17, we know that E is a finite subset of $\mathbb{N}$ and $\|x - P_E(x)\| < 1$. In particular, there exists $\gamma > 0$ such that $\|x - P_E(x)\| \leqslant 1 - \gamma$. For every $i \in E$, we define

$$S_i := S\left(e_i^*, 1 - \frac{e_i^*(x)}{2}\right).$$

Then we consider the ccs

$$C := \frac{1}{|E|}\sum_{i\in E} S_i.$$

Since $x \in S_X^+$, it is clear that $x \in \bigcap_{i\in E} S_i$ and in particular $x \in C$.

So let us pick $y := \frac{1}{|E|}\sum_{i\in E} y^i$ in C with $y^i \in S_i$. Then $e_i^*(y^i) > e_i^*(x)/2$ for every i. In particular, $e_i^*(y^i) \geqslant 0$, and $\left|e_i^*(y^i) - e_i^*(x)\right| \leqslant e_i^*(y^i)$. Indeed, for any given non-negative real numbers α and β with $\beta \geqslant \alpha/2$, we have

$$|\beta - \alpha| = \beta - \alpha \leqslant \beta$$

if $\beta \geqslant \alpha$, and

$$|\beta - \alpha| = \alpha - \beta \leqslant \alpha - \frac{\alpha}{2} = \frac{\alpha}{2} \leqslant \beta$$

if $\beta \leqslant \alpha$. So in either case, $|\beta - \alpha| \leqslant \beta$ as desired.

It then follows from Lemma 3.15 that $\|y^i - e_i^*(x)e_i\| \leqslant \|y^i\| \leqslant 1$, and finally

$$\begin{aligned}\|x - y\| &\leqslant \left\|x - \frac{x}{|E|}\right\| + \left\|\frac{x}{|E|} - \frac{P_E(x)}{|E|}\right\| + \left\|\frac{P_E(x)}{|E|} - y\right\| \\ &\leqslant 1 - \frac{1}{|E|} + \frac{1-\gamma}{|E|} + \frac{1}{|E|}\sum_{i\in E}\|e_i^*(x)e_i - y^i\| \leqslant 2 - \frac{\gamma}{|E|}.\end{aligned}$$

The conclusion follows with $\delta := \gamma/|E|$. In particular, since x belongs to the relatively weakly open set $\bigcap_{i\in E} S_i \subset C$, we also find that x is not super Δ, recovering the result from [6]. ∎

So combining [6, Proposition 2.13] and Proposition 3.18, we immediately see that spaces with a normalized one-unconditional basis fail to contain super Δ-points and ccs Δ-points. So let us state the following result here for future reference.

Theorem 3.19. *Let X be a real Banach space with a normalized one-unconditional basis. Then X does not contain super Δ-points, and X does not contain ccs Δ-points.*

In the rest of the section, we aim at providing sharper and improved versions of [6, Proposition 2.13]. In particular, we will go back to working with either real or complex Banach spaces. The main result of this study is the following proposition.

Proposition 3.20. *Let X be a Banach space, and assume that there exists a subset $\mathcal{A} \subseteq \mathcal{F}(X, X)$ such that $\sup\{\|\mathrm{Id} - T\| : T \in \mathcal{A}\} < 2$ and for every $\varepsilon > 0$ and every $x \in X$, there exists $T \in \mathcal{A}$ such that $\|x - Tx\| < \varepsilon$. Then X contains no super Δ-point.*

Let us provide a lemma which is a localization of the above result from which its proof is immediate.

Lemma 3.21. *Let X be a Banach space, and let $x \in S_X$. If there exists a finite-rank operator T on X such that $\|x - Tx\| + \|\mathrm{Id} - T\| < 2$, then x is not a super Δ-point.*

Proof. Consider $\varepsilon > 0$ such that $K := \|x - Tx\| + \|\mathrm{Id} - T\| + \varepsilon < 2$. Since T has finite rank, we can find $N \geqslant 1$, $w_1, \ldots, w_N \in S_X$ and $f_1, \ldots, f_N \in X^*$ such that $T(z) = \sum_{n=1}^{N} f_n(z) w_n$ for every $z \in X$. Let

$$W := \left\{ y \in B_X \colon\ |f_n(x - y)| < \frac{\varepsilon}{2^{n+1}}\ \forall n \in \{1, \ldots, N\} \right\}.$$

It is a neighborhood of x in the relative weak topology of B_X, and for every $y \in W$, we have

$$\begin{aligned} \|x - y\| &\leqslant \|x - Tx\| + \|Tx - Ty\| + \|y - Ty\| \\ &\leqslant \|x - Tx\| + \|\mathrm{Id} - T\| + \sum_{n=1}^{N} |f_n(x - y)|\, \|w_n\| \\ &\leqslant \|x - Tx\| + \|\mathrm{Id} - T\| + \varepsilon \sum_{n=1}^{N} \frac{1}{2^{n+1}} \leqslant K < 2. \ \blacksquare \end{aligned}$$

Remark 3.22. It is unclear whether an analogue to Lemma 3.21 can be given for ccs Δ-points. So we do not know whether Proposition 3.20 extends to this notion.

As particular cases of Proposition 3.20, we have the following ones. Recall that a sequence $(E_n)_{n\geqslant 1}$ of finite-dimensional subspaces of a given Banach space X is called a *finite-dimensional decomposition* (*FDD*) for X if every element $x \in X$ can be represented in a unique way as a series $x := \sum_{n\geqslant 1} x_n$ with $x_n \in E_n$ for every $n \geqslant 1$. An FDD is said to be *unconditional* if the above series converges unconditionally for every $x \in X$. In this case, it is well known that the family $(P_A)_{A\subset\mathbb{N}}$, where P_A is the projection given by $P_A(x) := \sum_{n\in A} x_n$, is uniformly bounded, and the constant $K_S := \sup_{A\subset\mathbb{N}} \|P_A\|$ is called the *suppression-unconditional constant* of the FDD. We refer to [40, Section 1.g] for the details and to [8, Section 3.1] for the particular case of unconditional bases.

Corollary 3.23. *A Banach space X fails to have super Δ-points provided one of the following conditions is satisfied:*

(1) *There exists a family $\mathcal{A} \subseteq \mathcal{F}(X,X)$ such that* $\sup\{\|\mathrm{Id} - T\| : T \in \mathcal{A}\} < 2$ *and the identity mapping belongs to the strong operator topology (SOT) closure of $\mathcal{F}$.*
(2) *There exists a family $\{P_\lambda\}_{\lambda\in\Lambda}$ of finite-rank* projections *on X such that* $\overline{\bigcup_{\lambda\in\Lambda} P_\lambda(X)} = X$ *and* $\sup_{\lambda\in\Lambda}\|\mathrm{Id} - P_\lambda\| < 2$.
(3) *The space X admits an FDD with suppression-unconditional constant less than 2. In particular, X admits an unconditional basis with suppression-unconditional constant less than 2.*

Observe that the value 2 in the above results is sharp in several ways.

Remark 3.24. (1) The space $C[0,1]$ admits a monotone Schauder basis, so there exists a sequence $\{P_n\}_{n\geqslant 1}$ of norm-one finite-rank projections on this space which converges to Id in SOT topology. As $C[0,1]$ has the Daugavet property, all elements in S_X are super Daugavet points. Observe that $\|\mathrm{Id} - P_n\| = 2$ for every $n \geqslant 1$ by DPr.
(2) Let X be an arbitrary Banach space. For every $x \in S_X$ choose $f_x \in S_{X^*}$ such that $f_x(x) = 1$, and define $P_x(z) = f_x(z)x$ for every $z \in X$. Then $\{P_x : x \in S_X\}$ is a family of norm-one rank-one projections on X, $X = \bigcup_{x\in S_X} P_x(X)$, and $\|\mathrm{Id} - P_x\| \leqslant 2$ for every $x \in S_X$.
(3) The space c admits ccs Daugavet points (hence super Daugavet points) (see Theorem 4.2), but it is easy to check that its usual basis is 3-unconditional and 2-suppression-unconditional.
(4) It is shown in [30] that a Banach space has DLD2P if and only if $\|\mathrm{Id}-P\| \geqslant 2$ for every rank-one projection P. It follows that the suppression constant of an unconditional basis on a Banach space with DLD2P has to be greater than or equal to 2. Let us mention here that there is no local version of this result, as there are Banach spaces with one-unconditional basis and containing many Daugavet points [6] (see Section 4.3.3).

3.2. Absolute sums. In this section we look at the transfer of the diametral points through absolute sums of Banach spaces. Let us first recall the following definition.

Definition 3.25. A norm N on $\mathbb{R}^2$ is *absolute* if $N(a,b) = N(|a|,|b|)$ for every $(a,b) \in \mathbb{R}^2$, and *normalized* if $N(0,1) = N(1,0) = 1$.

If X and Y are Banach spaces, and if N is an absolute normalized norm on $\mathbb{R}^2$, we denote by $X \oplus_N Y$ the product space $X \times Y$ endowed with the norm $\|(x,y)\| = N(\|x\|,\|y\|)$. It is easy to check that $X \oplus_N Y$ is a Banach space, and that its dual can be expressed as $(X \oplus_N Y)^* \equiv X^* \oplus_{N^*} Y^*$ where N^* is the absolute norm given by the formula $N^*(c,d) = \max_{N(a,b)=1} |ac| + |bd|$. Classical examples of absolute normalized norms on $\mathbb{R}^2$ are the ℓ_p norms for $p \in [1,\infty]$, and more generally any ℓ_φ Orlicz norms. Information on absolute norms can be found in [16, §21] and [43] and references therein, for instance. Let us recall that for every absolute normalized sum N, given non-negative a,b,c,d in $\mathbb{R}$ with $a \leqslant b$ and $c \leqslant d$ we have $N(a,b) \leqslant N(c,d)$. In particular, $\|\cdot\|_\infty \leqslant N \leqslant \|\cdot\|_1$.

Similar to DD2P (see [13, Theorem 2.11]) and to Δ-points [28], super Δ-points transfer very well through absolute sums.

PROPOSITION 3.26. *Let X and Y be Banach spaces, and let N be an absolute normalized norm.*

(1) *If $x \in S_X$ and $y \in S_Y$ are super Δ-points, then (ax, by) is a super Δ-point in $X \oplus_N Y$ for every $(a,b) \in \mathbb{R}^2$ with $N(a,b) = 1$.*
(2) *If $x \in S_X$ is a super Δ-point, then $(x,0)$ is a super Δ-point in $X \oplus_N Y$. If $y \in S_Y$ is a super Δ-point, then $(0,y)$ is a super Δ-point in $X \oplus_N Y$.*

Proof. (1) We can find two nets $(x_s)_{s\in S}$ and $(y_t)_{t\in T}$ respectively in S_X and S_Y such that $x_s \xrightarrow{w} x$, $y_t \xrightarrow{w} y$, and $\|x - x_s\|, \|y - y_t\| \to 2$. Now, if we take $(a,b) \in \mathbb{R}^2$ with $N(a,b) = 1$ we clearly have $(ax_s, by_t) \xrightarrow[(s,t)\in S\times T]{w} (ax, by)$ and $\|(ax,by) - (ax_s, by_t)\| = N(a\|x - x_s\|, b\|y - y_t\|) \to 2N(a,b) = 2$, so (ax, by) is a super Δ-point in $X \oplus_N Y$.

(2) Just repeat the previous proof with $a = 1$ and $b = 0$ or with $a = 0$ and $b = 1$, and so only one of the points has to be a super Δ-point. ∎

For super Daugavet points the situation is more complicated and we need to distinguish between different kinds of absolute norms. The following definitions can be found, for instance, in [28].

DEFINITION 3.27. Let N be an absolute normalized norm on $\mathbb{R}^2$.

(1) N has *property* (α) if for all $a, b \in \mathbb{R}_+$ with $N(a,b) = 1$ we can find a neighborhood W of (a,b) in $\mathbb{R}^2$ with $\sup_{(c,d)\in W} c < 1$ or $\sup_{(c,d)\in W} d < 1$ and such that any couple $(c,d) \in \mathbb{R}^2_+$ satisfying $N(c,d) = 1$ and $N((a,b) + (c,d)) = 2$ belongs to W.
(2) N is *A-octahedral* if there are $a, b \in \mathbb{R}_+$ with $N(a,b) = 1$ such that $N((a,b) + (c,d)) = 2$ for
$$c = \max\{e \in \mathbb{R}_+ : N(e,1) = 1\} \quad \text{and} \quad d = \max\{f \in \mathbb{R}_+ : N(1,f) = 1\}.$$
(3) N is *positively octahedral* if there exist $a, b \in \mathbb{R}_+$ such that $N(a,b) = 1$ and
$$N((a,b) + (0,1)) = N((a,b) + (1,0)) = 2.$$

Positively octahedral norms were introduced in [27] in order to characterize the absolute norms for which the corresponding absolute sum is *octahedral*. It is clear that property (α) and A-octhaedrality exclude each other and that every positively octahedral absolute normalized norm is A-octahedral (while there clearly exist absolute A-octahedral norms which are not positively octahedral). Moreover, it was proved in [28, Proposition 2.5] that every absolute normalized norm on $\mathbb{R}^2$ either has property (α) or is A-octahedral. For ℓ_p-norms, $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are both positively octahedral, and $\|\cdot\|_p$ has property (α) for every $p \in (1,\infty)$.

Observe that if an absolute normalized norm N on $\mathbb{R}^2$ is positively octahedral, and if (a,b) is as in the above definition, then the intersection of the unit sphere of N with the positive quadrant of $\mathbb{R}^2$ is equal to the union of the segments $[(1,0),(a,b)]$ and $[(0,1),(a,b)]$ (see [45, Section 3.3.1] for pictures). In particular, it follows that $N((a,b) + (c,d)) = 2$ for any non-negative c, d with $N(c,d) = 1$. Similar to the results from [3, Section 4] concerning Daugavet points, we have the following.

PROPOSITION 3.28. *Let X and Y be Banach spaces, and let N be an absolute normalized norm.*

(1) ([3, Proposition 4.6]) *If N has property (α), then $X \oplus_N Y$ has no Daugavet point (hence in particular no super Daugavet points).*
(2) *If N is positively octahedral and if $x \in S_X$ and $y \in S_Y$ are super Daugavet points, then (ax, by) is a super Daugavet point in $X \oplus_N Y$ for every $(a,b) \in \mathbb{R}^2_+$ as in the above definition.*

Proof. (2) Assume that N is positively octahedral, take $(a,b) \in \mathbb{R}^2_+$ as in the definition, and let $x \in S_X$ and $y \in S_Y$ be super Daugavet points. For any given $(u,v) \in X \oplus_N Y$ of norm $\|(u,v)\| = 1$ we can find two nets $(u_s)_{s\in S}$ and $(v_t)_{t\in T}$ respectively in S_X and S_Y such that $\|u\|u_s \xrightarrow{w} u$, $\|v\|v_t \xrightarrow{w} v$, and $\|x-u_s\|, \|y-v_t\| \to 2$. Then $(\|u\|u_s, \|v\|v_t) \xrightarrow[(s,t)\in S\times T]{w} (u,v)$. Since

$$\begin{aligned}\|ax - \|u\|u_s\| &= \|(x-u_s) - [(1-a)x - (1-\|u\|)u_s]\| \\ &\geqslant \|x-u_s\| - (1-a+1-\|u\|), \\ &= a + \|u\| - (2 - \|x-u_s\|),\end{aligned}$$

and, in the same way,

$$\|by - \|v\|v_t\| \geqslant b + \|v\| - (2 - \|y-v_t\|),$$

we have

$$\begin{aligned}\|(ax - \|u\|u_s, by - \|v\|v_t)\| &= N(\|ax - \|u\|u_s\|, \|by - \|v\|v_t\|) \\ &\geqslant N(a + \|u\| - (2 - \|x-u_s\|), b + \|v\| - (2 - \|y-v_t\|)) \\ &\to N((a+\|u\|, b+\|v\|) = 2.\end{aligned}$$

This shows that (ax, by) is a super Daugavet point in $X \oplus_N Y$. ∎

REMARK 3.29. Note that if $(a,b) = (1,0)$ (respectively, $(a,b) = (0,1)$) in the previous statement (for example, when $N = \|\cdot\|_1$), then we only need to assume that x (respectively, y) is super Daugavet in order to deduce that $(x,0)$ (respectively, $(0,y)$) is super Daugavet in $X \oplus_N Y$. Also, if $N = \|\cdot\|_\infty$, then we only need to assume that x (respectively, y) is super Daugavet in order to find that $(x, \beta y)$ (respectively, $(\alpha x, y)$) is super Daugavet in $X \oplus_N Y$ for every $\beta \in [0,1]$ (respectively, $\alpha \in [0,1]$).

In [28, Theorem 2.2] it is proved that regular Daugavet points also transfer through A-octahedral sums. We do not know if a similar result can be obtained for super Daugavet points. Indeed, observe that if N is an A-octahedral norm, and if c, d, and (a,b) are as in the above definition, then the intersection of the unit sphere of N with the positive quadrant of $\mathbb{R}^2$ is equal to the union of the segments $[(1,0),(1,d)]$, $[(1,d),(a,b)]$, $[(0,1),(c,1)]$ and $[(c,1),(a,b)]$. In particular, $N((a,b)+(e,f)) = 2$ for every couple (e,f) on the segments $[(1,d),(a,b)]$ and $[(c,1),(a,b)]$, but this is no longer true on the segments $[(1,0),(1,d)]$ and $[(0,1),(c,1)]$ and the argument in the above proof does not work anymore.

The situation for ccs Δ-points and ccs Daugavet point is not clear and the proofs of the above results do not seem to admit easy extensions. For instance, it follows from the next result that Remark 3.29 is not valid for ccs Daugavet points.

PROPOSITION 3.30. *Let X be an arbitrary Banach space, let Y be a Banach space containing a strongly exposed point $y_0 \in S_Y$, and let $E := X \oplus_1 Y$. Then there are convex combinations of slices of B_E around 0 of arbitrarily small diameter. In particular, E fails to contain ccs Daugavet points and also fails to have SD2P.*

Proof. Let $y_0^* \in S_{Y^*}$ strongly exposes y_0. Given $\varepsilon > 0$, there is $0 < \delta < \varepsilon$ such that $\|y - y_0\| < \varepsilon$ whenever $y \in B_Y$ satisfies $\operatorname{Re} y_0^*(y) > 1 - \delta$. Consider $f = (0, y_0^*) \in S_{E^*}$ and write

$$C := \tfrac{1}{2}(S(f, \delta; B_E) + S(-f, \delta, B_E))$$

Take $u := \frac{1}{2}(u_1 + u_2) \in C$ with $u_1 \in S(f, \delta; B_E)$ and $u_2 \in S(-f, \delta; B_E)$. So if write $u_1 := (x_1, y_1)$ and $u_2 := (x_2, y_2)$, we have

$$\operatorname{Re} y_0^*(y_1) = \operatorname{Re} f(x_1, y_1) > 1 - \delta \quad \text{and} \quad \operatorname{Re} y_0^*(y_2) = \operatorname{Re} f(x_2, y_2) < -1 + \delta.$$

On the one hand, it follows that $\|y_1 - y_0\| < \varepsilon$ and $\|y_2 + y_0\| < \varepsilon$. On the other hand, $\|y_1\|, \|y_2\| > 1 - \delta$, hence $\|x_1\| < \delta < \varepsilon$ and $\|x_2\| < \delta < \varepsilon$. Summarizing, we have

$$\|u\| = \tfrac{1}{2}\big(\|x_1 + x_2\| + \|y_1 + y_2\|\big) \leqslant \tfrac{1}{2}(2\varepsilon + 2\varepsilon) = 2\varepsilon. \ \blacksquare$$

REMARK 3.31. It is straightforward to adapt the previous proof to ℓ_p-sums for $1 < p < \infty$.

However, note that the situation is very different for ℓ_∞-sums.

THEOREM 3.32. *Let X and Y be Banach spaces, and let $E := X \oplus_\infty Y$. If $x \in S_X$ is a ccs Daugavet point, then $(x, y) \in S_E$ is a ccs Daugavet point for every $y \in B_Y$.*

Proof. Let $C := \sum_{i=1}^n \lambda_i S_i$ be a ccs of B_E. For every $i \in \{1, \ldots, n\}$, we can write $S_i := S(f_i, \delta_i)$ with $f_i := (x_i^*, y_i^*) \in S_{E^*}$ satisfying $1 = \|f_i\| = \|x_i^*\| + \|y_i^*\|$. Consider on the one hand

$$\tilde{S}_i := \{s \in B_X \colon \operatorname{Re} x_i^*(s) > \|x_i^*\| - \delta_i/2\},$$

and pick on the other hand any $t_i \in B_Y$ such that $\operatorname{Re} y_i^*(t_i) > \|y_i^*\| - \frac{\delta_i}{2}$. Since $\tilde{C} := \sum_{i=1}^n \lambda_i \tilde{S}_i$ is a ccs of B_X, we can find for every $\varepsilon > 0$ an element $s := \sum_{i=1}^n \lambda_i s_i$ in $\tilde{C}$ such that $\|x - s\| > 2 - \varepsilon$. Then, if we let $t := \sum_{i=1}^n \lambda_i t_i$, we get $(s_i, t_i) \in B_E$ and

$$\operatorname{Re} f_i(s_i, t_i) = \operatorname{Re} x_i^*(s_i) + y_i^*(t_i) > \|x_i^*\| + \|y_i^*\| - \delta_i = 1 - \delta_i$$

for every i, so that $(s_i, t_i) \in S_i$, and $(s, t) = \sum_{i=1}^n \lambda_i (s_i, t_i) \in C$. Finally,

$$\|(x, y) - (s, t)\| \geqslant \|x - s\| > 2 - \varepsilon,$$

so (x, y) is a ccs Daugavet point as stated. $\blacksquare$

4. Examples and counterexamples of diametral elements

In this chapter we give a number of examples and counterexamples of diametral elements on the unit spheres of Banach spaces. We first characterize the notion in some spaces which have natural relations with the Daugavet property, such as L_1-predual spaces, Müntz spaces, and L_1-spaces. Next, we comment on some examples which have previously appeared in the literature, including some improvements in some cases (as for Lipschitz free spaces). Finally, we include some complicated examples which will be needed to see that no implication in Figure 1 on page 13 reverses and also to negate some other possible implications between the relevant notions. A summary of all the relations between properties will be included in Section 4.7.

4.1. Characterization in $C(K)$-spaces, L_1-preduals, and Müntz spaces. It was shown in [3, Theorems 3.4 and 3.7] that the notions of Δ-point and Daugavet point coincide for L_1-preduals. The authors first characterize the Δ-points in $C(K)$-spaces and then get the result for L_1-preduals by using the principle of local reflexivity. Later on, a characterization of Δ-points (equivalently, Daugavet points) of L_1-preduals was provided in [42, Theorem 3.2], which implicitly proved that actually Δ-points and super Daugavet points coincide in this setting. Let us state this result here for further reference. Observe that the authors of [3] work with real Banach spaces, but it is immediate that the proof of [3, Theorems 3.4] works in the complex case as well; the paper [42] works in both the real and the complex cases.

Proposition 4.1 ([3, Theorems 3.4 and 3.7], [42, Theorem 3.2]). *Let X be an L_1-predual and let $x \in S_X$. The following assertions are equivalent:*

(1) *x is a Daugavet point.*
(2) *x is a Δ-point.*
(3) *For every $\delta > 0$, the weak* slice $S(J_X(x), \delta; B_{X^*})$ contains infinitely many linearly independent extreme points of B_{X^*}.*
(4) *For every $y \in B_X$, there exists a sequence (x_n^{**}) in $B_{X^{**}}$ with $\|x - x_n^{**}\| \to 2$ and*

$$\Big\| \sum_{n\geqslant 1} a_n (y - x_n^{**}) \Big\| \leqslant 2\|a\|_\infty$$

for every $a := (a_n) \in c_{00}$.
(5) *For every element $y \in B_X$, there exists a sequence (x_n^{**}) in $B_{X^{**}}$ which converges weak* to y and satisfies $\|x - x_n^{**}\| \to 2$.*

If $X = C(K)$ for a Hausdorff topological space K, the above is also equivalent to:

(6) *x attains its norm at an accumulation point of K.*

We will show that, in fact, Δ-points also coincide with the ccs versions for L_1-preduals. Our approach will be analogous to the one used in [3] for Δ-points and Daugavet points. We first prove the result for $C(K)$-spaces and then deduce it for all L_1-preduals using the fact that the bidual of an L_1-predual is a $C(K)$-space. For $C(K)$-spaces, we first prove a sufficient condition for ccs Daugavet points which, for the same price, can be proved for vector-valued spaces. Recall that given a compact Hausdorff topological space K and a Banach space X, $C(K, X)$ denotes the Banach space of continuous functions from K to X endowed with the supremum norm.

Theorem 4.2. *Let K be a compact Hausdorff topological space, X a Banach space, and let t_0 be an accumulation point of K. If a function $f \in S_{C(K,X)}$ satisfies $\|f(t_0)\| = 1$, then f is a ccs Daugavet point.*

Proof. Pick $x^* \in S_{X^*}$ such that $\operatorname{Re} x^*(f(t_0)) = \|f(t_0)\| = 1$. Let $C := \sum_{i=1}^{L} \lambda_i S_i$ be a convex combination of slices of $B_{C(K)}$. For every $i \in \{1, \dots, L\}$, pick a function $g_i \in S_i$. Since K is compact and t_0 is an accumulation point of K we have the following.

Claim. *There exists a sequence $(U_n)_{n\geqslant 0}$ of open neighborhoods of t_0 such that*

(1) *$U_0 = K$,*
(2) *$\overline{U_{n+1}}$ is a proper subset of U_n for every $n \geqslant 0$,*
(3) *$\operatorname{Re}(x^* \circ f)_{|\overline{U_n}} \geqslant 1 - 1/n$ and $\|g_{i|\overline{U_n}} - g_i(t_0)\| \leqslant 1/n$ for every $i \in \{1, \dots, L\}$ and every $n \geqslant 1$.*

Indeed, we construct the sequence inductively. Let $U_0 := K$ and assume $U_0, \dots, U_n$ have been constructed for some $n \geqslant 0$. Since K is normal, we can find an open subset U of U_n such that $t_0 \in U \subset \overline{U} \subset U_n$. Also since t_0 is an accumulation point of K and since K is Hausdorff, we can find an open subset V of U such that $\overline{V}$ is a proper subset of $\overline{U}$ (pick any point in U distinct from t_0 and separate the two points with open sets). By continuity of f and of the finitely many g_i's, we can then find an open subset W of V such that

$$\operatorname{Re}(x^* \circ f)_{|W} > 1 - \frac{1}{n+1} \quad \text{and} \quad \|g_{i|W} - g_i(t_0)\| < \frac{1}{n+1}$$

for every $i \in \{1, \dots, L\}$. The set $U_{n+1} := W$ does the job.

Now, pick $(U_n)_{n\geqslant 0}$ as in the claim and define $F_n := \overline{U_n} \backslash U_{n+1}$ for every $n \geqslant 0$. By construction, the F_n's are closed non-empty subsets of K and cover $K \setminus \bigcap_{n\geqslant 0} U_n$, and each F_n can only intersect its neighbors F_{n-1} and F_{n+1}. By Urysohn's lemma, for every $n \geqslant 1$ we can find a function $p_n \in C(K)$ satisfying

(1) $0 \leqslant p_n \leqslant 1$,
(2) $p_{n|F_{n+1}} = 1$,
(3) $p_{n|F_0 \cup \dots \cup F_{n-1} \cup \overline{U_{n+3}}} = 0$.

The sequence (p_n) is normalized and converges pointwise to 0, so it converges weakly to 0. Moreover, observe that

$$\|g_i - (1+g_i(t_0))p_n\|_\infty \leqslant 1 + \frac{1}{n}$$

for every $i \in \{1, \dots, L\}$ since $\|g_{i|\overline{U_n}} - g_i(t_0)\| \leqslant \frac{1}{n}$ and $p_{n|(K\setminus\overline{U_n})} = 0$ by construction. So all the functions

$$g_{i,n} := \frac{n}{n+1}(g_i - (1+g_i(t_0))p_n)$$

belong to $B_{C(K)}$ and the sequences $(g_{i,n})_{n\in\mathbb{N}}$ converge weakly to g_i for all $i \in \{1, \dots, L\}$. Since the finitely many S_i's are all weakly open, we may thus find some $N \geqslant 1$ such that $g_{i,n} \in S_i$ for every i and every $n \geqslant N$. In particular, the function

$$g_n := \sum_{i=1}^{L} \lambda_i g_{i,n}$$

belongs to C for every $n \geqslant N$. To conclude, fix $t \in F_{n+1} \subset \overline{U_{n+1}} \subset \overline{U_n}$ and observe that

$$\operatorname{Re} x^* f(t) \geqslant 1 - \frac{1}{n}$$

and that

$$\begin{aligned}\operatorname{Re} x^* g_{i,n}(t) &= \frac{n}{n+1} \operatorname{Re} x^*(g_i(t) - (1+g_i(t_0)) \\ &\leqslant \frac{n}{n+1} \operatorname{Re} x^* \Big(g_i(t_0) + \frac{1}{n} - (1+g_i(t_0)) \Big) = -1 + \frac{2}{n+1}\end{aligned}$$

for every $i \in \{1, \dots L\}$. Hence,

$$\|f - g_n\|_\infty \geqslant \operatorname{Re} x^* \Big(f(t) - \sum_{i=1}^{L} \lambda_i \operatorname{Re}(g_{i,n}(t)) \Big) \geqslant 2 - \frac{1}{n} - \frac{2}{n+1}. \qquad \blacksquare$$

Combining the previous result with Proposition 4.1, we get the promised characterization of diametral points in $C(K)$-spaces.

Corollary 4.3. *Let K be a Hausdorff topological compact space. Then the six concepts of diametral points are equivalent in $C(K)$.*

For vector-valued spaces, the situation is not that easy, but we can still provide some results. Observe that, clearly, if t_0 is an isolated point of a compact Hausdorff topological space K and X is a Banach space, then $C(K, X) = C(K \setminus \{t_0\}, X) \oplus_\infty X$.

Remark 4.4. Let K be a Hausdorff topological compact space, let X be a Banach space. and let $f \in C(K, X)$ be a function with $\|f\| = 1$.

(1) If $f \in C(K, X)$ with $\|f\| = 1$ attains its norm at an accumulation point of K, then f is a ccs Daugavet point (by Theorem 4.2), and hence f satisfies the six diametral notions.
(2) If $f \in C(K, X)$ with $\|f\| = 1$ attains its norm at an isolated point t_0 and $f(t_0)$ is a Daugavet (respectively, super Daugavet, ccs Daugavet) point, then f is a Daugavet (respectively, super Daugavet, ccs Daugavet) point (by [3, Section 4], Remark 3.29, and Theorem 3.32, respectively).

(3) Suppose that K contains an isolated point t_0, let $x_0 \in S_X$, and let $f \in C(K, X)$ be given by $f(t_0) = x_0$ and $f(t) = 0$ for every $t \in K \setminus \{t_0\}$. Then:

  (3.1) If x_0 is a Δ- (respectively, super Δ-) point of X, then f is a Δ- (respectively, super Δ-) point of $C(K, X)$ (by [3, Section 4] and Proposition 3.26, respectively).

  (3.2) If x_0 is a Daugavet (respectively, super Daugavet, ccs Daugavet) point of X, then f is a Daugavet (respectively, super Daugavet, ccs Daugavet) point of $C(K, X)$ (by [45, Proposition 3.3.11], Remark 3.29 Theorem 3.32, respectively).

  (3.3) If f is a Δ- (respectively, Daugavet) point of $C(K, X)$, then x_0 is a Δ- (respectively, Daugavet) point of X (by [45, Theorem 3.4.4], [45, Theorem 3.3.13], respectively).

(4) It is now easy to show that the six diametral notions do not coincide in $C(K, X)$-spaces. Indeed, let K be a compact Hausdorff topological space containing an isolated point t_0, let X be a Banach space containing a Δ-point x_0 which is not a Daugavet point (e.g. any x_0 in the unit sphere of $X = C[0, 1] \oplus_2 C[0, 1]$; see Propositions 3.26 and 3.28), and consider the function $f \in C(K, X)$ given by $f(t_0) = x_0$ and $f(t) = 0$ for every $t \in K \setminus \{t_0\}$. Then f is a Δ-point by (3.1) but it is not a Daugavet point by (3.3).

We are now ready to extend Corollary 4.3 to general L_1-predual spaces.

Corollary 4.5. *Let X be an L_1-predual and let $x \in S_X$ be a Δ-point. Then x is a ccs Daugavet point. Hence the six diametral notions are equivalent for L_1-preduals.*

Proof. If x is a Δ-point in X, then as mentioned in Remark 2.6(3), $J_X(x)$ is a Δ-point in X^{**}. Now, X^{**} is isometric to a $C(K)$-space so Theorem 4.2 implies that $J_X(x)$ is a ccs Daugavet point in X^{**}. Then, using now Remark 2.6(4) (or using a straightforward argument based on the principle of local reflexivity as in [3, Theorem 3.7]), we conclude that x is a ccs Daugavet point in X. ∎

Observe that the proof of Theorem 4.2 also works for Müntz spaces (by using [3, Lemma 3.10] to provide suitable replacements for the functions p_n). We recall that given an increasing sequence $\Lambda = (\lambda_n)_{n=0}^{\infty}$ of non-negative real numbers with $\lambda_0 = 0$ such that $\sum_{i=1}^{\infty} \frac{1}{\lambda_i} < \infty$, the real Banach space

$$M(\Lambda) := \overline{\operatorname{span}}\,\{p_n : n \geqslant 0\} \subseteq C[0, 1]$$

given as the closure in $C[0, 1]$ of the linear span of the power functions $p_n : t \mapsto t^{\lambda_n}$, $t \in [0, 1]$, is called the *Müntz space* associated with Λ. Excluding the constant functions from $M(\Lambda)$, we have the subspace $M_0(\Lambda) := \overline{\operatorname{span}}\,\{p_n : n \geqslant 1\}$ of $M(\Lambda)$.

So, adapting the proof of Theorem 4.2 to Müntz spaces (for real-valued functions attaining their norm at $1 \in [0, 1]$) and also using [3, Proposition 3.12], we get the following result analogous to Corollaries 4.3 and 4.5.

Corollary 4.6. *Let $X = M(\Lambda)$ or $X = M_0(\Lambda)$ for an increasing sequence Λ of non-negative real numbers with $\lambda_0 = 0$ such that $\sum_{i=1}^{\infty} \frac{1}{\lambda_i} < \infty$. Then every Δ-point of X is a ccs Daugavet point (and hence the six diametral notions are equivalent).*

4.2. Characterization in L_1-spaces. In [3, Theorem 3.1] the equivalence between the notions of Daugavet point and Δ-point was obtained for elements of σ-finite L_1-spaces in the real case. Actually, it is not complicated to extend the results to arbitrary measures and also to the complex case.

PROPOSITION 4.7 ([3, Theorem 3.1] for the σ-finite real case). *Let (Ω, Σ, μ) be a measure space, and let f be a norm-one element in $L_1(\mu)$. Then the following assertions are equivalent:*

(1) *f is a Daugavet point.*
(2) *f is a Δ-point.*
(3) *The support of f contains no atom.*

Observe that (1)$\Rightarrow$(2) is immediate. For (2)$\Rightarrow$(3), suppose that f is a Δ-point and let A be an atom of finite measure (the only ones that can be contained in the support of an integrable function). Then clearly $L_1(\mu) = L_1(\mu|_{\Omega\setminus A}) \oplus_1 \mathbb{K}$ (as integrable functions are constant on atoms), and we may write $f = (f_1, c)$ for suitable $f_1 \in L_1(\mu|_{\Omega\setminus A})$ and $c = f(A) \in \mathbb{K}$. If $c \neq 0$, then $\|f_1\| \neq 1$ and it follows from [45, Theorem 3.4.4] that $1 \in \mathbb{K}$ is a Δ-point, a contradiction. This shows that the support of f contains no atom.

To show that (3)$\Rightarrow$(1), we actually prove the following more general result. Recall that given a measure space (Ω, Σ, μ) and a Banach space X, $L_1(\mu, X)$ denotes the Banach space of all Böchner-integrable functions from Ω to X (see [20, p. 49] for the definition and background).

THEOREM 4.8. *Let (Ω, Σ, μ) be a measure space, let X be a Banach space, and let f be a norm-one element in $L_1(\mu, X)$. If the support of the function f contains no atom, then f is a super Daugavet point.*

Proof. Let $S := \operatorname{supp} f$, which contains no atom by assumption. Let us first prove that f is a super Daugavet point. Since S contains no atoms, $L_1(\mu_{|S}, X)$ has the Daugavet property (see e.g. [54, Example in p. 81]). In particular, f is a super Daugavet point in this space. Since $L_1(\mu, X) = L_1(\mu_{|S}, X) \oplus_1 L_1(\mu_{|\Sigma\setminus S}, X)$, we conclude that f is a super Daugavet point in $L_1(\mu, X)$ by the transfer results from Section 3.2 (see Remark 3.29). ∎

Our next goal is to discuss the relationship to the ccs diametral notions. For real $L_1(\mu)$-spaces, and using a result from [1], we may actually deduce that real-valued integrable functions with atomless support are ccs Δ-points.

PROPOSITION 4.9. *Let (Ω, Σ, μ) be a measure space and let f be a norm-one element in* the real space *$L_1(\mu)$. If* supp *f contains no atom, then f is a ccs Δ-point.*

Proof. Take $\varepsilon > 0$ and $D := \sum_{i=1}^n \lambda_i S_i$ a ccs of $B_{L_1(\mu)}$ containing f, with $\lambda_i \in (0,1)$ and $\sum_{i=1}^n \lambda_i = 1$. Write $f := \sum_{i=1}^n \lambda_i g_i$ with $g_i \in S_i$ for every i. Consider the measurable subset $\tilde{S} := \operatorname{supp} f \cup \bigcup_{i=1}^n \operatorname{supp} g_i$ of Ω and let $\tilde{\mu}$ be the σ-finite measure $\tilde{\mu} := \mu_{|\tilde{S}}$ on $(\tilde{S}, \Sigma_{|\tilde{S}})$. Then D induces a ccs $\tilde{D}$ of $B_{L_1(\tilde{\mu})}$ by restriction of the support which contains the function $\tilde{f}$ which is just f viewed as an element of $L_1(\tilde{\mu})$, and hence $\operatorname{supp} \tilde{f}$ contains no toms. Since $\tilde{f}$ belongs to the unit sphere of the real space $L_1(\tilde{\mu})$, from [1, Theorem 5.5] we know that $\tilde{f}$ is an interior point of $\tilde{D}$ for the relative weak topology of $B_{L_1(\tilde{\mu})}$.

As we have already shown that $\tilde{f}$ is a super Daugavet point in Theorem 4.8 (and hence a super Δ-point), we can find $\tilde{g} \in \tilde{D}$ such that $\|\tilde{f} - \tilde{g}\| > 2 - \varepsilon$. By just considering the extension g of $\tilde{g}$ to the whole Ω by 0, we get $g \in D$ and $\|f - g\| = \|\tilde{f} - \tilde{g}\| > 2 - \varepsilon$. ■

Let us comment that it is not clear whether ccs Δ-points transfer through absolute sums, but we have used specific geometric properties of L_1-spaces in the previous proof.

Remark 4.10. Since [1, Theorem 5.5] is also valid for convex combinations of relatively weakly open subsets of $B_{L_1(\mu)}$, in fact every Δ-point in a real $L_1(\mu)$-space is a ccw Δ-point.

Putting together Proposition 4.7, Theorem 4.8, and Proposition 4.9, we get the following corollary.

Corollary 4.11. *Let (Ω, Σ, μ) be a measure space and let f be a norm-one element in $L_1(\mu)$. Then the following notions are equivalent for f: Δ-point, Daugavet point, super Δ-point, and super Daugavet point. Moreover, in the real case, the four notions are also equivalent to ccs Δ-point.*

We now deal with ccs Daugavet points in $L_1(\mu)$-spaces. Observe that if Ω admits an atom A of finite measure, then $L_1(\mu) \equiv L_1(\mu_{|\Omega\setminus A}) \oplus_1 \mathbb{K}$. In particular, in this case $L_1(\mu)$ fails to have ccs Daugavet points by Proposition 3.30. We then have the following characterization of the presence of a ccs Daugavet point in an L_1-space.

Proposition 4.12. *Let (Ω, Σ, μ) be a measure space. Then the following assertions are equivalent:*

(1) *$L_1(\mu)$ has the Daugavet property.*
(2) *$L_1(\mu)$ contains a ccs Daugavet point.*
(3) *$L_1(\mu)$ has* SD2P.
(4) *μ admits no atom of finite measure.*

Proof. (1)⇔(4) is well known (see [54, Section 2, Example (b)]); (1)⇒(2) is also known; (2)⇒(3) is contained in Proposition 3.12. Finally, (3)⇒(4) follows from Proposition 3.30 and the comment before the statement of this proposition. ■

4.3. Remarks on some examples from the literature

4.3.1. Lipschitz-free spaces. In [51], Veeorg constructed a surprising example of a space satisfying the Radon–Nikodým property and containing a Daugavet point. We slightly improve this result by showing that this point is also a ccs Δ-point by proving a general fact about extreme Δ-molecules in Lipschitz-free spaces. For the necessary definitions we refer to the cited paper [51] and to [9, 10, 32]. In particular, we will denote by $\mathcal{F}(M)$ the *Lipschitz-free space* over M (that is, the natural predual of the space of Lipschtiz functions on M). For further background on Lipschitz-free spaces, we refer to the book [53].

We start by recalling the following characterization of molecules which are Δ-points on Lipschitz-free spaces from [32].

Proposition 4.13 ([32, Theorem 4.7]). *Let M be a pointed metric space and let $x \neq y \in M$. Then the molecule $m_{x,y}$ is a Δ-point if and only if every slice S of $B_{\mathcal{F}(M)}$ containing $m_{x,y}$ also contains for every $\varepsilon > 0$ a molecule $m_{u,v}$ with $u \neq v \in M$ satisfying $d(u,v) < \varepsilon$.*

When the molecule is an extreme point, we have the following improved result.

Theorem 4.14. *Let M be a pointed metric space, and let $x \neq y \in M$. If the molecule $m_{x,y}$ is an extreme point and a Δ-point, then $m_{x,y}$ is a ccs Δ-point.*

Observe that this result cannot be obtained from Proposition 3.13: molecules of Lipschitz-free spaces which are preserved extreme points are denting points, hence very far from being Δ-points.

To prove the theorem, we need a result which is just an equivalent reformulation of a result in [32].

Lemma 4.15 ([32, Theorem 2.6]). *Let M be a pointed metric space, and let $\mu \in S_{\mathcal{F}(M)}$. For every $\varepsilon > 0$, there exists $\delta > 0$ such that given $u \neq v \in M$ with $d(u,v) < \delta$ we have $\|\mu \pm m_{u,v}\| > 2 - \varepsilon$.*

Using this result and a homogeneity argument similar to that for [15, Lemma 2.3], we can provide the pending proof.

Proof of Theorem 4.14. Let $C := \sum_{i=1}^n \lambda_i S_i$ be a ccs of $B_{\mathcal{F}(M)}$ containing $m_{x,y}$ and let $\varepsilon > 0$. Since $m_{x,y}$ is extreme, we see that $m_{x,y} \in \bigcap_{i=1}^n S_i$, and by Proposition 4.13 every S_i contains molecules of $\mathcal{F}(M)$ supported at arbitrarily close points. Using Lemma 4.15, we construct inductively for every $\eta > 0$ a finite sequence $(m_{u_i,v_i})_{i=1}^n$ of molecules in $\mathcal{F}(M)$ such that

(1) $m_{u_i,v_i} \in S_i$ for every i,
(2) $\|m_{x,y} - \sum_{i=1}^k \lambda_i m_{u_i,v_i}\| > 1 + \sum_{i=1}^k \lambda_i - \frac{k\varepsilon}{n}$ for every $k \leqslant n$.

Indeed, since S_1 contains molecules of $\mathcal{F}(M)$ supported at arbitrarily close points, we can find by Lemma 4.15 $u_1 \neq v_1 \in M$ such that $m_{u_1,v_1} \in S_1$ and $\|m_{x,y} - m_{u_1,v_1}\| > 2 - \frac{\varepsilon}{n}$. It follows that $\|m_{x,y} - \lambda_1 m_{u_1,v_1}\| \geqslant \|m_{x,y} - m_{u_1,v_1}\| - (1 - \lambda_1) > 1 + \lambda_1 - \frac{\varepsilon}{n}$. Assume that $m_{u_1,v_1}, \ldots, m_{u_k,v_k}$ are constructed as desired for a given $k \in \{1, \ldots, n-1\}$. Since S_{k+1} contains molecules of $\mathcal{F}(M)$ supported at arbitrarily close points, we can find by Lemma 4.15 $u_{k+1} \neq v_{k+1} \in M$ such that $m_{u_{k+1},v_{k+1}} \in S_{k+1}$ and

$$\left\| \frac{m_{x,y} - \sum_{i=1}^k \lambda_i m_{u_i,v_i}}{\|m_{x,y} - \sum_{i=1}^k \lambda_i m_{u_i,v_i}\|} - m_{u_{k+1},v_{k+1}} \right\| > 2 - \frac{\varepsilon}{n\|m_{x,y} - \sum_{i=1}^k \lambda_i m_{u_i,v_i}\|}.$$

Then

$$\begin{aligned} \left\| \frac{m_{x,y} - \sum_{i=1}^{k+1} \lambda_i m_{u_i,v_i}}{\|m_{x,y} - \sum_{i=1}^k \lambda_i m_{u_i,v_i}\|} \right\| &\geqslant \left\| \frac{m_{x,y} - \sum_{i=1}^k \lambda_i m_{u_i,v_i}}{\|m_{x,y} - \sum_{i=1}^k \lambda_i m_{u_i,v_i}\|} - m_{u_{k+1},v_{k+1}} \right\| \\ &\quad - \left(1 - \frac{\lambda_{k+1}}{\|m_{x,y} - \sum_{i=1}^k \lambda_i m_{u_i,v_i}\|}\right) \\ &> 1 + \frac{\lambda_{k+1}}{\|m_{x,y} - \sum_{i=1}^k \lambda_i m_{u_i,v_i}\|} - \frac{\varepsilon}{n\|m_{x,y} - \sum_{i=1}^k \lambda_i m_{u_i,v_i}\|}. \end{aligned}$$

By the assumption,

$$\Big\|m_{x,y}-\sum_{i=1}^{k+1}\lambda_i m_{u_i,v_i}\Big\| > \|m_{x,y}-\sum_{i=1}^{k}\lambda_i m_{u_i,v_i}\|+\lambda_{k+1}-\frac{\varepsilon}{n} > 1+\sum_{i=1}^{k+1}\lambda_i-\frac{(k+1)\varepsilon}{n}.$$

As a consequence, $\mu:=\sum_{i=1}^{n}\lambda_i m_{u_i,v_i}$ belongs to C and satisfies $\|m_{x,y}-\mu\|>2-\varepsilon$. ∎

In particular, as announced, the molecule $m_{x,y}$ in the example from [51] is a ccs Δ-point. Note that it cannot be a ccs Daugavet point by Proposition 3.12 since the space has RNP, but we do not know whether it is a super Δ-point or even a super Daugavet point. Let us state the result for further reference.

Example 4.16. Let M be the metric space constructed in [51, Example 3.1] and let x,y be the points described there. Then $\mathcal{F}(M)$ has RNP, the molecule $m_{x,y}$ is an extreme point of the unit ball of $\mathcal{F}(M)$ which is a Daugavet point. Hence, by our Theorem 4.14, $m_{x,y}$ is a ccs-Δ-point.

Let us finally remark that the space $\mathcal{F}(M)$ of Example 4.16 has RNP, so in particular it is strongly regular. Since then strongly regular points of its unit ball are norm dense, both examples have ccs Δ-points. They cannot contain ccs Daugavet points by Proposition 3.12.

Let us also comment that the use of Theorem 4.14 above cannot be omitted, as the molecule $m_{0,q}$ is not a preserved extreme point, hence Proposition 3.13 is again not applicable.

4.3.2. An example of a Banach space with DD2P, restricted DSD2P, but containing ccs of arbitrarily small diameter. In [2, Theorem 2.12], Abrahamsen, Hájek, Nygaard, Talponen, and Troyanski constructed a space X which has DLD2P, which is midpoint locally uniformly rotund (in particular, pre-ext$(B_X)=S_X$), and such that B_X contains convex combinations of slices of arbitrarily small diameter. It then follows from Proposition 3.13 that every element of S_X is actually a super Δ-point and a ccs Δ-point (that is, X has DD2P and restricted DSD2P). But since X contains ccs of arbitrarily small diameter, it fails SD2P in an extreme way. The obvious explanation for the failure of SD2P and the fact that every element in the unit sphere is a ccs Δ-point is that none of the convex combinations of slices of diameter strictly smaller than 2 intersects the unit sphere. On the other hand, the space X is constructed as the ℓ_2-sum of spaces, and so X does not contain Daugavet points by [3, Proposition 4.6] (see Proposition 3.28).

Observe further that X has restricted DSD2P and DD2P, but fails DSD2P (which is equivalent to the Daugavet property by [34]).

4.3.3. An example in a space with one-unconditional basis. Abrahamsen, Lima, Martiny, and Troyanski constructed in [6, Section 4] a Banach space $X_{\mathfrak{M}}$ with one-unconditional basis which contains a subset $D_B\subseteq S_{X_{\mathfrak{M}}}$ satisfying

- every element in D_B is both a Daugavet point and a point of continuity,
- $B_{X_{\mathfrak{M}}}=\overline{\mathrm{co}}(D_B)$,
- D_B is weakly dense in the unit ball.

Observe that no element of D_B is a super Δ-point (it is exactly the opposite!). By Theorem 3.19, no element of D_B is a ccs Δ-point.

4.4. A super Δ-point which fails to be a Daugavet point in an extreme way. In order to put into a context the following result, let us recall that Daugavet points are at distance 2 from any denting point (see [32, Proposition 3.1]). With this in mind, the following result can be interpreted as the existence of super Δ-points which fail to be Daugavet points in an extreme way.

THEOREM 4.17. *Let X be a Banach space with the Daugavet property. Then, for every $\varepsilon > 0$, there exists an equivalent norm $|\cdot|$ and two points $x, y \in B_{(X,|\cdot|)}$ such that*

(1) *y is a super Δ-point,*
(2) *x is strongly exposed,*
(3) *$|x - y| < \varepsilon$.*

Proof. Take a subspace $Y \subseteq X$ with $\dim(X/Y) = 1$. Observe that Y has the Daugavet property (see e.g. [50, Theorem 6(a)]). Take $x \in S_X$ with $0 < d(x, Y) < \varepsilon$ (this can be settled taking a non-zero element $v \in X/Y$ with quotient norm smaller than ε). Now, we can find an element $y \in S_Y$ such that $\|x - y\| < \varepsilon$. By the Hahn–Banach theorem, we can take $f \in S_{X^*}$ with $\operatorname{Re} f(x) > 0$ and $f = 0$ on Y. This means that x belongs to the slice $T := \{z \in B_X : \operatorname{Re} f(z) > \alpha\}$ for some $\alpha > 0$. Take $\delta > 0$ such that $\frac{\|x-y\|}{1-\delta} < \varepsilon$. By Lemma 2.1 we can find $x^* \in S_{X^*}$ such that $x \in S(x^*, \delta; B_X) \subseteq T$. By the above inclusion we conclude that $S(x^*, \delta; B_X) \cap B_Y = \emptyset$ or, in other words, that $\operatorname{Re} x^*(z) \leqslant 1 - \delta$ for every $z \in B_Y$. Set

$$B := \overline{\text{co}}(B_Y \cup (1 - \delta)B_X \cup \{\pm x\}).$$

Then B is the unit ball of an equivalent norm $|\cdot|$ which satisfies, in view of the inclusions $(1 - \delta)B_X \subseteq B \subseteq B_X$,

$$\|x\| \leqslant |x| \leqslant \frac{1}{1-\delta}\|x\|$$

for every $x \in X$. Let us prove that $|\cdot|$, x and y satisfy our requirements. First, observe that

$$|x - y| \leqslant \frac{\|x - y\|}{1 - \delta} < \varepsilon.$$

Next, we claim that y is a super Δ point. Indeed, since Y has the Daugavet property we can find a net $\{y_s\} \subseteq B_Y$ with $y_s \to y$ weakly and $\|y - y_s\| \to 2$. Notice that the weak convergence $y_s \to y$ is still guaranteed on X because $i\colon (Y, \|\cdot\|) \to (X, |\cdot|)$ is weak-to-weak continuous as $\|\cdot\|$ and $|\cdot|$ are equivalent. Moreover, $y_s \in B_Y \subseteq B$ for every s, so $|y_s| \leqslant 1$ for every s. Finally,

$$|y_s - y| \geqslant \|y_s - y\| \to 2,$$

and since $y \in B_Y \subseteq B$, we conclude $|y_s - y| \to 2$. Hence y is clearly a super Δ-point for the norm $|\cdot|$.

It remains to prove that x is strongly exposed. Indeed, we will prove that $\operatorname{Re} x^*$ strongly exposes B at x, for which it is enough to prove that $\operatorname{Re} x^*$ strongly exposes

$\mathrm{co}(B_Y \cup (1-\delta)B_X \cup \{\pm x\})$ at x. Take

$$z := \alpha u + \beta(1-\delta)v + (\gamma - \omega)x \in \mathrm{co}\big(B_Y \cup (1-\delta)B_X \cup \{\pm x\}\big)$$

with $\alpha+\beta+\gamma+\omega = 1$. Observe that $1-\delta < \operatorname{Re} x^*(x) \leqslant |x^*| \leqslant \|x^*\|$ due to the inclusion $B \subseteq B_X$. Taking into account that $\operatorname{Re} x^*(u) \leqslant 1-\delta$ since $u \in B_Y$ as $B_Y \cap S(x^*, \delta; B_X) = \emptyset$, we conclude

$$\operatorname{Re} x^*(z) \leqslant (1-\delta)(\alpha+\beta) + (\gamma-\omega) \operatorname{Re} x^*(x).$$

Since $\operatorname{Re} x^*(x) > 1-\delta$, we get $\sup\big\{\operatorname{Re} x^*(z) \colon z \in \mathrm{co}(B_Y \cup (1-\delta)B_X \cup \{\pm x\})\big\} = \operatorname{Re} x^*(x)$. If we take a sequence

$$z_n := \alpha_n u_n + \beta_n(1-\delta)v_n + (\gamma_n - \omega_n)x \in \mathrm{co}(B_Y \cup (1-\delta)B_X \cup \{\pm x\})$$

with $\alpha_n + \beta_n + \gamma_n + \omega_n = 1$ such that $\operatorname{Re} x^*(z_n) \to \operatorname{Re} x^*(x)$, it follows from the previous argument that $\alpha_n \to 0$, $\beta_n \to 0$, $\omega_n \to 0$ and $\gamma_n \to 1$, which means $z_n \to x$ in norm. ■

Remark 4.18. Using the previous theorem and Proposition 3.26 it is easy to construct (considering ℓ_2-sums, for instance) a Banach space X containing a sequence of super Δ-points (y_n) such that the distance from y_n to the set of strongly exposed points is going to zero.

4.5. A super Δ-point which is a strongly regular point. In the present section, as well as in the next one, we aim to distinguish the super and ccs notions of Δ- and Daugavet points. The following result shows that there are plenty of examples of spaces containing super Δ-points which are strongly regular points (hence far from being ccs Δ-points). We do the construction in real spaces for simplicity.

Theorem 4.19. *Every real Banach space with the Daugavet property can be equivalently renormed so that the new unit ball has a point which is simultaneously super-Δ and a point of strong regularity* (*hence, far away of being ccs Δ-point*).

We will use the following immediate result which follows from the fact that a convex combination of ccs is again a ccs.

Lemma 4.20. *Let X be a Banach space and let C be a closed, convex, bounded subset of X. Then the set of strongly regular points of C is a convex set.*

Proof of Theorem 4.19. Let X be a Banach space with the Daugavet property. Take a one-codimensional subspace Y of X. Since Y is complemented in X then $X = Y \oplus \mathbb{R}$, so we will see X in such way. Take $r > 0$, $y_0 \in S_Y$ and $f \in S_{X^*}$ such that $f(y_0) = 1$, and consider on $X = Y \oplus \mathbb{R}$ the equivalent norm $|\cdot|$ whose unit ball is

$$B := \overline{\mathrm{co}}\big(B_Y \times \{0\} \cup \{\pm(y_0, r)\} \cup \{\pm(y_0, -r)\}\big).$$

It follows that $|\cdot|$ agrees with the original norm $\|\cdot\|$ on the elements of the form $(y, 0)$.

We claim that $(y_0, 0)$ satisfies our requirements. First of all, let us prove that $(y_0, 0)$ is a super-Δ point. Since Y is one-codimensional, it has the Daugavet property (see e.g. [50, Theorem 6(a)]). Consequently, there exists a net $y_s \to y_0$ weakly in B_Y such that $\|y_0 - y_s\| \to 2$. Then $(y_s, 0) \to (y_0, 0)$ weakly in $(X, |\cdot|)$. Moreover, it is clear that

$(y_s, 0) \in B$ for every s. Finally,

$$|(y_s, 0) - (y_0, 0)| = |(y_s - y_0, 0)| = \|y_s - y_0\| \to 2.$$

Let us now prove that $(y_0, 0)$ is a point of strong regularity. To do so, it is enough, in view of Lemma 4.20, to show that $(y_0, \pm r)$ is a strongly exposed point (we will prove that for (y_0, r), the other case being completely analogous). Let us prove that $(f, 1)$ strongly exposes (y_0, r) in the set $B_Y \times \{0\} \cup \{\pm(y_0, r)\} \cup \{\pm(y_0, -r)\}$. On the one hand, we have

$$(f, 1)(y_0, r) = f(y_0) + r = 1 + r.$$

On the other hand, given $(y, 0) \in B_Y \times 0$ we have $(f, 1)(y, 0) = f(y) \leqslant 1 < 1 + r$. Moreover, $(f, 1)(y_0, -r) = 1 - r$ and $(f, 1)(-y_0, \pm r) = -1 \pm r < 1 + r$. Consequently,

$$\begin{aligned} \sup\{(f, 1)(a, b) \colon (a, b) \in B_Y \times \{0\} \cup \{\pm(y_0, r)\} \cup \{\pm(y_0, -r)\},\ (a, b) \neq (y_0, r)\} \\ \leqslant 1 < 1 + r = (f, 1)(y_0, r). \end{aligned}$$

This is enough to guarantee that $(f, 1)$ strongly exposes (y_0, r) in B, so we are done. ∎

4.6. A super Daugavet point which is not a ccs Δ-point. The previous example shows that we can distinguish the notion of super Δ-point and the one of ccs Δ-point. It seems natural then that we should be able to distinguish the notions of super Daugavet point and the one of ccs Δ-point. In order to do so, we need to consider an involved construction but, as a consequence, we will prove that there are super Daugavet points which are contained in convex combinations of slices of small diameter. The construction will be very similar to that of [14, Theorem 2.4], with a slight variation which makes the resulting norm with a stronger Daugavet flavour. As in the previous section, we will only work with real spaces here.

Let us recall a construction from Argyros, Odell, and Rosenthal [11]. Pick a nonincreasing null sequence $\{\varepsilon_n\}$ in $\mathbb{R}^+$. We construct an increasing sequence $\{K_n\}$ of closed, bounded and convex subsets in the real space c_0 and a sequence $\{g_n\}$ in c_0 as follows: First define $K_1 = \{e_1\}$, $g_1 = e_1$ and $K_2 = \mathrm{co}(e_1, e_1 + e_2)$. Choose $l_2 > 1$ and $g_2, \ldots, g_{l_2} \in K_2$ an ε_2-net in K_2. Assume that $n \geqslant 2$ and that m_n, l_n, K_n, and $\{g_1, \ldots, g_{l_n}\}$ have been constructed, with $K_n \subseteq B_{\mathrm{span}\{e_1, \ldots, e_{m_n}\}}$ and $g_i \in K_n$ for every $1 \leqslant i \leqslant l_n$. Define K_{n+1} as

$$K_{n+1} = \mathrm{co}(K_n \cup \{g_i + e_{m_n+i} \colon 1 \leqslant i \leqslant l_n\}).$$

Consider $m_{n+1} = m_n + l_n$ and choose $\{g_{l_n+1}, \ldots, g_{l_{n+1}}\} \in K_{n+1}$ so that $\{g_1, \ldots, g_{l_{n+1}}\}$ is an ε_{n+1}-net in K_{n+1}. Finally, we define $K_0 = \overline{\cup_n K_n}$. Then it follows that K_0 is a non-empty closed, bounded and convex subset of c_0 such that $x(n) \geqslant 0$ for every $n \in \mathbb{N}$ and $\|x\|_\infty \leqslant 1$ for every $x \in K_0$ and so $\mathrm{diam}(K_0) \leqslant 1$.

Now, for a fixed i, we see from the construction that $\{g_i + e_{m_n+i}\}_n$ is a sequence in K_0 (for n large enough) which is weakly convergent to g_i, and $\|(g_i - e_{m_n+i}) - g_i\| = \|e_{m_n+i}\| = 1$ holds for every n. Then $\mathrm{diam}(K_0) = 1$. We will freely use the set K_0 and the above construction throughout the section. Observe that, by the above construction,

$$K_0 = \overline{\{g_i \colon i \in \mathbb{N}\}}^w = \overline{\{g_i \colon i \in \mathbb{N}\}}.$$

Observe finally that, by the inductive construction, g_i has finite support for every $i \in \mathbb{N}$.

By [11, Theorem 1.2], K_0 contains convex combinations of slices of arbitrarily small diameter. However, all the points in K_0 are "super Daugavet points" in the following sense.

PROPOSITION 4.21. *For every $x_0 \in K_0$, every $\varepsilon > 0$, and every non-empty weakly open subset W of K_0, there exists $y \in W$ such that $\|x_0 - y\| > 1 - \varepsilon = \operatorname{diam}(K_0) - \varepsilon$.*

Proof. Take $\varepsilon > 0$ and a non-empty relatively weakly open subset of K_0. By a density argument, we can find $i \in \mathbb{N}$ such that $\|x_0 - g_i\| < \varepsilon$. Again by a density argument there exists $g_k \in W$ for certain $k \in \mathbb{N}$.

As explained above, by the definition of K_0, we have $g_k + e_{m_n+k} \in K_0$ for every $n \in \mathbb{N}$. Since $(g_k + e_{m_n+k})_{n\in\mathbb{N}} \to g_k$ weakly, we can find $n \in \mathbb{N}$ large enough so that $g_k + e_{m_n+k} \in W$ and $m_n + k \notin \operatorname{supp}(g_i) \cup \operatorname{supp}(g_k)$ (this is possible because the previous set is finite). So taking $y = g_k + e_{m_n+k}$, we get $y(m_m + k) = 1$ and so

$$\|g_i - y\| \geqslant y(m_n + k) - g_i(m_n + k) = 1 - 0 = 1.$$

As a consequence, $\|x_0 - y\| \geqslant \|g_i - y\| - \|g_i - x_0\| > 1 - \varepsilon$, and the proof is finished. ∎

It is time to construct the announced renorming of $C[0, 1]$. Take an infinite sequence of non-empty pairwise disjoint open subsets V_n of $[0, 1]$ such that $0 \notin \bigcup_{n\in\mathbb{N}} V_n$. By Urysohn lemma, we can find, for every $n \in \mathbb{N}$, a function $h_n \in S_{C[0,1]}$ with $0 \leqslant h_n \leqslant 1$ and such that $\operatorname{supp}(h_n) \subseteq V_n$. If we consider $Z := \overline{\operatorname{span}}\,\{h_n : n \in \mathbb{N}\}$, we find that Z is lattice isometrically isomorphic to c_0 (indeed, the mapping $e_n \mapsto h_n$ is an isometric Banach lattice isomorphism). Consequently, we can consider the set K_0 constructed in Z, finding that $K_0 \subseteq B_{C[0,1]}$ is a set of positive functions (because the latter linear isometry preserves the lattice structure) which contains convex combination of slices of arbitrarily small diameter but enjoying the property exhibited in Proposition 4.21. Moreover, by the construction of the functions h_n, $f(0) = 0$ for every $f \in Z$, so in particular $f(0) = 0$ for every $f \in K_0$.

Now, take $0 < \varepsilon < 1$ and write

$$B_\varepsilon := \overline{\operatorname{co}}\left(2\Big(K_0 - \frac{\mathbb{1}}{2}\Big) \cup 2\Big(-K_0 + \frac{\mathbb{1}}{2}\Big) \cup ((1-\varepsilon)B_{C[0,1]} + \varepsilon B_{\ker(\delta_0)})\right),$$

where $\mathbb{1}$ stands for the constant function 1 in $C[0, 1]$ and δ_0 is the evaluation functional at the point 0.

Consider $\|\cdot\|_\varepsilon$ the norm on (the real version of) $C[0, 1]$ whose unit ball is B_ε. As we have indicated, the renorming technique follows the scheme of the renorming given in [14, Theorem 2.4] with the difference that we use $B_{\ker(\delta_0)}$ instead of B_{c_0} in the last term because $\ker(\delta_0)$ has the Daugavet property (as it is one-codimensional in $C[0, 1]$ and we may use [35, Theorem 2.14]).

We have the following result.

THEOREM 4.22. *The space $(X, \|\cdot\|_\varepsilon)$ has the following properties:*

(1) *Every element of $2(K_0 - \frac{\mathbb{1}}{2})$ is a super Daugavet point.*

(2) *For every* $\eta > 0$ *there exists a convex combination of slices* D *of* B_ε *with* $\operatorname{diam}(D) < \eta$ *and such that* $D \cap 2(K_0 - \frac{\mathbb{1}}{2}) \neq \emptyset$.

In particular, there are super Daugavet points which are not ccs-Δ *points.*

Proof. (1). Take $a \in K_0$, and let us prove that $2a - \mathbb{1}$ is a super Daugavet point. In order to do so, pick a non-empty relatively weakly open subset W of B_ε. Write

$$A := 2(K_0 - \tfrac{\mathbb{1}}{2}) \quad \text{and} \quad B := (1-\varepsilon)B_{C[0,1]} + \varepsilon B_{\ker(\delta_0)}.$$

Since $B_\varepsilon = \overline{\text{co}}(A \cup -A \cup B)$ we see that W has non-empty intersection with the set $\text{co}(A \cup -A \cup B)$. Now, observe that $\frac{A-A}{2} = K_0 - K_0 \subseteq B_{\ker(\delta_0)} \subseteq B$ so that

$$\text{co}(A \cup -A \cup B) = \text{co}(A \cup B) \cup \text{co}(-A \cup B)$$

by [14, Lemma 2.4]. Consequently, either $W \cap \text{co}(A \cup B)$ or $W \cap \text{co}(-A \cup B)$ is non-empty. Let us distinguish several cases.

Assume first that $W \cap \text{co}(A \cup B)$ is non-empty, so pick $a' \in K_0$, $f \in B_{C[0,1]}$, $g \in B_{\ker(\delta_0)}$, and $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$ such that

$$\alpha(2a' - \mathbb{1}) + \beta((1-\varepsilon)f + \varepsilon g) \in W.$$

Take $\eta > 0$. By Proposition 4.21, there exists a net $a_s \to a'$ weakly with $a_s \in K_0$ for every s and $\|a - a_s\| \to 1$. Since $(2a_s - \mathbb{1}) \to 2a' - \mathbb{1}$ weakly, we can find s large enough so that

$$\alpha(2a_s - \mathbb{1}) + \beta((1-\varepsilon)f + \varepsilon g) \in W$$

and

$$\|(2a - \mathbb{1}) - (2a_s - \mathbb{1})\| = 2\|a - a_s\| > 2 - \eta.$$

Observe that $2a - \mathbb{1}$ and $2a_s - \mathbb{1}$ are functions in $B_{C[0,1]}$ since a, a_s are positive functions of norm at most 1. Since $\|(2a - \mathbb{1}) - (2a_s - \mathbb{1})\| > 2 - \eta$, there exists $t_0 \in [0,1]$ and $\theta \in \{-1, 1\}$ such that $\theta(2a - \mathbb{1})(t_0) > 1 - \eta$ and $\theta(2a_s - \mathbb{1})(t_0) < -1 + \eta$ (observe that $t_0 \neq 0$ since $a(t_0) = a_s(t_0) = 0$ by construction). Consequently, the set

$$U := \{t \in [0,1] \colon \theta(2a - \mathbb{1})(t) > 1 - \eta \quad \text{and} \quad \theta(2a_s - 1)(t) < -1 + \eta\}$$

is a non-empty open subset of $[0,1]$, and we can construct a sequence of non-empty pairwise disjoint open sets $W_n \subseteq U$. Observe that $0 \notin \bigcup_{n \in \mathbb{N}} W_n$ since $0 \notin U$. Take $p_n \in W_n$ for every $n \in \mathbb{N}$. We can construct, for every $n \in \mathbb{N}$, two functions f_n and g_n in the unit ball of $C[0,1]$ satisfying $f_n = f$ and $g_n = g$ in $[0,1] \setminus W_n$ and $f_n(p_n) = g_n(p_n) = -\theta$. The functions $f - f_n$ have pairwise disjoint supports, so $f - f_n \to 0$ weakly or, in other words, $f_n \to f$ weakly. A similar argument shows that $g_n \to g$ weakly. Notice also that, given $n \in \mathbb{N}$, since $0 \notin W_n$ then $g_n(0) = g(0) = 0$, so $(g_n) \subseteq \ker(\delta_0)$. Hence $\alpha(2a_s - \mathbb{1}) + \beta((1-\varepsilon)f_n + \varepsilon g_n)$ is a sequence in B_ε which converges in n weakly to $\alpha(2a_s - \mathbb{1}) + \beta((1-\varepsilon)f + \varepsilon g) \in W$. Consequently, we can find n large enough such that $\alpha(2a_s - \mathbb{1}) + \beta((1-\varepsilon)f_n + \varepsilon g_n) \in W$. Finally, the inclusion $B_\varepsilon \subseteq B_{C[0,1]}$ implies that

$\|z\| \leqslant \|z\|_\varepsilon$, so

$$\begin{aligned}
&\big\|(2a-\mathbb{1}) - \alpha(2a_s - \mathbb{1}) - \beta((1-\varepsilon)f_n + \varepsilon g_n)\big\|_\varepsilon \\
&\qquad \geqslant \big\|(2a-\mathbb{1}) - \alpha(2a_s - \mathbb{1}) - \beta((1-\varepsilon)f_n + \varepsilon g_n)\big\| \\
&\qquad \geqslant \theta((2a-\mathbb{1}) - \alpha(2a_s - \mathbb{1}) - \beta((1-\varepsilon)f_n)(p_n) \\
&\qquad = \theta(2a-\mathbb{1})(p_n) - \theta\alpha(2a_s-\mathbb{1})(p_n) \\
&\qquad\quad - \theta\beta((1-\varepsilon)f_n(p_n) + \theta\varepsilon g_n(p_n)) \\
&\qquad > 1-\eta-\alpha(-1+\eta)-\beta(-1) \\
&\qquad = 1+\alpha+\beta-(1+\alpha)\eta = 2-2\eta.
\end{aligned}$$

Since $\eta > 0$ was arbitrary, this finishes the case $W \cap \operatorname{co}(A \cup B) \neq \emptyset$.

If $W \cap \operatorname{co}(-A \cup B) \neq \emptyset$, find $a' \in K_0$, $f \in B_{C[0,1]}$, $g \in B_{\ker(\delta_0)}$, and $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$ such that

$$\alpha(-2a' + \mathbb{1}) + \beta((1-\varepsilon)f + \varepsilon g) \in W.$$

This case is simpler because $\|(2a-\mathbb{1}) - (-2a'+\mathbb{1})\| \geqslant (2a-\mathbb{1}) - (-2a'+\mathbb{1})(0) = 2$. Now, an approximation argument for f_n and g_n similar to that of the above case (working on a non-empty open subset of $(0,1)$ in order to get $g_n(0) = 0$) finishes this case , and consequently the proof of (1).

(2). The first part of the proof is a repetition of the argument of [14, Theorem 2.4]. Fix $\gamma > 0$. From [11, Theorem 1.2] there exist slices $S_1, \ldots, S_n$ of K_0 such that

$$\operatorname{diam}\left(\frac{1}{n}\sum_{i=1}^n S_i\right) < \frac{1}{4}(1-\varepsilon)\gamma.$$

We can assume that $S_i = \{x \in K_0 \colon x_i^*(x) > 1 - \widetilde{\delta}\}$ where $0 < \widetilde{\delta} < 1$, $x_i^* \in C[0,1]^*$ and $\sup x_i^*(K_0) = 1$ holds for every $i = 1, \ldots, n$. It is clear that

$$\sup x_i^*\left(2\left(K_0 - \frac{\mathbb{1}}{2}\right)\right) = 2\left(1 - x_i^*\left(\frac{\mathbb{1}}{2}\right)\right),$$

for all $i = 1, \ldots, n$. We put $\rho, \delta > 0$ such that $\frac{1}{2}\rho\|x_i^*\| + \delta < \widetilde{\delta}$, $2\rho < \varepsilon$, $\rho\|x_i^*\| < 4\delta$, and $\frac{(7-2\varepsilon)\rho}{(1-\varepsilon)} < \gamma$, for all $i = 1, \ldots, n$. We consider the relatively weakly open set of B_ε given by

$$U_i := \left\{x \in B_\varepsilon \colon x_i^*(x) > 2\left(1 - \delta - x_i^*\left(\frac{\mathbb{1}}{2}\right)\right) + \frac{1}{2}\rho\|x_i^*\|,\ x(0) = \delta_0(x) < -1 + \rho^2\right\}$$

for every $i = 1, \ldots, n$. It is clear that $\|x_i^*\|_\varepsilon \leqslant \|x_i^*\|$ for every $i = 1, \ldots, n$ and $\|\delta_0\|_\varepsilon = \|\delta_0\| = 1$.

Since $\rho\|x_i^*\| < 4\delta$, we have $2\big(1 - x_i^*\big(\frac{\mathbb{1}}{2}\big)\big) > 2\big(1 - \delta - x_i^*\big(\frac{\mathbb{1}}{2}\big)\big) + \frac{1}{2}\rho\|x_i^*\|$. Now, we have $\sup x_i^*\big(2\big(K_0 - \frac{\mathbb{1}}{2}\big)\big) = 2\big(1 - x_i^*\big(\frac{\mathbb{1}}{2}\big)\big)$, so there exists $x \in K_0$ such that

$$x_i^*\left(2\left(x - \frac{\mathbb{1}}{2}\right)\right) > 2\left(1 - \delta - x_i^*\left(\frac{\mathbb{1}}{2}\right)\right) + \frac{1}{2}\rho\|x_i^*\| \quad \text{and} \quad \delta_0\left(2\left(x - \frac{\mathbb{1}}{2}\right)\right) = -1 < -1 + \rho^2.$$

This implies that $U_i \neq \emptyset$ for every $i = 1, \ldots, n$. In order to estimate the diameter of

$\frac{1}{n}\sum_{i=1}^n U_i$, it is enough to compute the diameter of

$$\frac{1}{n}\sum_{i=1}^{n} U_i \cap \mathrm{co}\left(2\left(K_0-\frac{\mathbb{1}}{2}\right)\cup -2\left(K_0-\frac{\mathbb{1}}{2}\right)\cup [(1-\varepsilon)B_X+\varepsilon B_{\ker(\delta_0)}]\right).$$

Since $2(K_0-\frac{\mathbb{1}}{2})$ and $(1-\varepsilon)B_{C[0,1]}+\varepsilon B_{\ker(\delta_0)}$ are convex subsets of B_ε, given $x\in B_\varepsilon$, we can assume that $x=\lambda_1 2(a-\frac{\mathbb{1}}{2})+\lambda_2 2(-b+\frac{\mathbb{1}}{2})+\lambda_3[(1-\varepsilon)x_0+\varepsilon y_0]$, where $\lambda_i\in[0,1]$ with $\sum_{i=1}^3\lambda_i=1$ and $a,b\in K_0$, $x_0\in B_{C[0,1]}$, and $y_0\in B_{\ker(\delta_0)}$.

So given $x,y\in\frac{1}{n}\sum_{i=1}^n U_i$, for $i=1,\dots,n$, there exist $a_i,a_i',b_i,b_i'\in K_0$, $\lambda_{(i,j)},\lambda'_{(i,j)}\in[0,1]$ with $j=1,2,3$ and, $x_i,x_i'\in B_{C[0,1]}$, and $y_i,y_i'\in B_{Ker(\delta_0)}$, such that

$$\begin{aligned} u_i &:= 2\lambda_{(i,1)}\left(a_i-\frac{\mathbb{1}}{2}\right)+2\lambda_{(i,2)}\left(-b_i+\frac{\mathbb{1}}{2}\right)+\lambda_{(i,3)}[(1-\varepsilon)x_i+\varepsilon y_i],\\ u_i' &:= 2\lambda'_{(i,1)}\left(a_i'-\frac{\mathbb{1}}{2}\right)+2\lambda'_{(i,2)}\left(-b_i'+\frac{\mathbb{1}}{2}\right)+\lambda'_{(i,3)}[(1-\varepsilon)x_i'+\varepsilon y_i'] \end{aligned}$$

belong to U_i for every $i\in\{1,\dots,n\}$, and such that

$$x=\frac{1}{n}\sum_{i=1}^n u_i \quad\text{and}\quad y=\frac{1}{n}\sum_{i=1}^n u_i'.$$

For $i\in\{1,\dots,n\}$ we have $u_i\in U_i$ so

$$\delta_0(u_i)=\delta_0\left(2\lambda_{(i,1)}\left(a_i-\frac{\mathbb{1}}{2}\right)+2\lambda_{(i,2)}\left(-b_i+\frac{\mathbb{1}}{2}\right)+\lambda_{(i,3)}[(1-\varepsilon)x_i+\varepsilon y_i]\right)<-1+\rho^2.$$

Observe that, by construction,

$$\delta_0\left(a_i-\frac{\mathbb{1}}{2}\right)=-\frac{1}{2},\delta_0\left(-b_i+\frac{\mathbb{1}}{2}\right)=\frac{1}{2}\quad\text{and}\quad\delta_0((1-\varepsilon)x_i+\varepsilon y_i)=\delta_0((1-\varepsilon)x_i)\geqslant-(1-\varepsilon).$$

This implies that

$$2\lambda_{(i,2)}+\lambda_{(i,3)}\varepsilon-1=-\lambda_{(i,1)}+\lambda_{(i,2)}-\lambda_{(i,3)}(1-\varepsilon)<-1+\rho^2.$$

Since $2\rho<\varepsilon$, we deduce that $\lambda_{(i,2)}+\lambda_{(i,3)}<\frac{1}{2}\rho$. As a consequence,

$$\lambda_{(i,1)}>1-\frac{1}{2}\rho, \tag{4.1}$$

and similarly

$$\lambda'_{(i,1)}>1-\frac{1}{2}\rho \tag{4.2}$$

for every $i = 1, \ldots, n$. Now, the previous inequalities imply that

$$\begin{aligned}
\|x-y\|_\varepsilon \leqslant & \frac{1}{n}\Big\|\sum_{i=1}^n 2\lambda_{(i,1)}\Big(a_i-\frac{\mathbb{1}}{2}\Big)-2\lambda'_{(i,1)}\Big(a'_i-\frac{\mathbb{1}}{2}\Big)\Big\|_\varepsilon \\
& +\frac{1}{n}\sum_{i=1}^n\Big\|2\lambda_{(i,2)}\Big(-b_i+\frac{\mathbb{1}}{2}\Big)\Big\|_\varepsilon+\frac{1}{n}\sum_{i=1}^n\Big\|2\lambda'_{(i,2)}\Big(-b'_i+\frac{\mathbb{1}}{2}\Big)\Big\|_\varepsilon \\
& +\frac{1}{n}\sum_{i=1}^n\|\lambda_{(i,3)}[(1-\varepsilon)x_i+\varepsilon y_i]\|_\varepsilon+\frac{1}{n}\sum_{i=1}^n\|\lambda'_{(i,3)}[(1-\varepsilon)x'_i+\varepsilon y'_i]\|_\varepsilon \\
\leqslant & \frac{1}{n}\Big\|\sum_{i=1}^n 2\lambda_{(i,1)}\Big(a_i-\frac{\mathbb{1}}{2}\Big)-2\lambda'_{(i,1)}\Big(a'_i-\frac{\mathbb{1}}{2}\Big)\Big\|_\varepsilon \\
& +\frac{1}{n}\sum_{i=1}^n(\lambda_{(i,2)}+\lambda_{(i,3)})+\frac{1}{n}\sum_{i=1}^n(\lambda'_{(i,2)}+\lambda'_{(i,3)})
\end{aligned}$$

and, by using (4.1),(4.2),

$$\begin{aligned}
& \leqslant \frac{1}{n}\Big\|\sum_{i=1}^n 2\lambda_{(i,1)}\Big(a_i-\frac{\mathbb{1}}{2}\Big)-2\lambda'_{(i,1)}\Big(a'_i-\frac{\mathbb{1}}{2}\Big)\Big\|_\varepsilon+\rho \\
& \leqslant \frac{2}{n}\Big\|\sum_{i=1}^n \lambda_{(i,1)}a_i-\lambda'_{(i,1)}a'_i\Big\|_\varepsilon+\frac{1}{n}\sum_{i=1}^n|\lambda_{(i,1)}-\lambda'_{(i,1)}|\,\|\mathbb{1}\|_\varepsilon+\rho \\
& \leqslant \frac{2}{n}\Big\|\sum_{i=1}^n \lambda_{(i,1)}a_i-\lambda'_{(i,1)}a'_i\Big\|_\varepsilon+\frac{(3-2\varepsilon)}{2(1-\varepsilon)}\rho.
\end{aligned}$$

Now,

$$\begin{aligned}
& \Big\|\sum_{i=1}^n \lambda_{(i,1)}a_i-\lambda'_{(i,1)}a'_i\Big\|_\varepsilon \\
& \qquad \leqslant \Big\|\sum_{i=1}^n(\lambda_{(i,1)}-1)a_i\Big\|_\varepsilon+\Big\|\sum_{i=1}^n a_i-a'_i\Big\|_\varepsilon+\Big\|\sum_{i=1}^n(\lambda'_{(i,1)}-1)a'_i\Big\|_\varepsilon \\
& \qquad \leqslant \frac{1}{1-\varepsilon}\Big\|\sum_{i=1}^n a_i-a'_i\Big\|+\sum_{i=1}^n\frac{1}{1-\varepsilon}|\lambda_{(i,1)}-1|\,\|a_i\|+\sum_{i=1}^n\frac{1}{1-\varepsilon}|\lambda'_{(i,1)}-1|\,\|a'_i\| \\
& \qquad \leqslant \frac{1}{1-\varepsilon}\Big\|\sum_{i=1}^n a_i-a'_i\Big\|+\frac{1}{1-\varepsilon}n\rho.
\end{aligned}$$

(In the previous estimate observe that $\|a_i\| \leqslant 1$ and $\|a'_i\| \leqslant 1$ since $a_i, a'_i \in K_0 \subseteq B_{\ker(\delta_0)} \subseteq B_\varepsilon$). Hence,

$$\|x-y\|_\varepsilon \leqslant \frac{2}{1-\varepsilon}\Big\|\frac{1}{n}\sum_{i=1}^n a_i-a'_i\Big\|+\frac{7-2\varepsilon}{2(1-\varepsilon)}\rho. \tag{4.3}$$

Now, in order to prove that the previous norm is small, we will prove that both elements $\frac{1}{n}\sum_{i=1}^n a_i, \frac{1}{n}\sum_{i=1}^n a'_i$ belong to the set $\frac{1}{n}\sum_{i=1}^n S_i$, which has small diameter. To this end, note that

$$x_i^*\left(2\lambda_{(i,1)}\left(a_i-\frac{\mathbb{1}}{2}\right)+2\lambda_{(i,2)}\left(-b_i+\frac{\mathbb{1}}{2}\right)+\lambda_{(i,3)}[(1-\varepsilon)x_i+\varepsilon y_i]\right) > 2\left(1-\delta-x_i^*\left(\frac{\mathbb{1}}{2}\right)\right)+\frac{\rho}{2}\|x_i^*\|,$$

for every $i\in\{1,\dots,n\}$. Then

$$\begin{aligned} x_i^*\left(2\lambda_{(i,1)}\left(a_i-\frac{\mathbb{1}}{2}\right)\right)+\frac{1}{2}\rho\|x_i^*\| &\geqslant x_i^*\left(2\lambda_{(i,1)}\left(a_i-\frac{\mathbb{1}}{2}\right)\right)+\lambda_{(i,2)}\|x_i^*\|+\lambda_{(i,3)}\|x_i^*\| \\ &\geqslant x_i^*\left(2\lambda_{(i,1)}\left(a_i-\frac{\mathbb{1}}{2}\right)\right)+\lambda_{(i,2)}\|x_i^*\|_\varepsilon+\lambda_{(i,3)}\|x_i^*\|_\varepsilon \\ &\geqslant x_i^*\left(2\lambda_{(i,1)}\left(a_i-\frac{\mathbb{1}}{2}\right)+2\lambda_{(i,2)}\left(-b_i+\frac{\mathbb{1}}{2}\right)\right. \\ &\qquad \left.+\lambda_{(i,3)}[(1-\varepsilon)x_i+\varepsilon y_i]\right). \end{aligned}$$

We have

$$x_i^*\left(2\lambda_{(i,1)}\left(a_i-\frac{\mathbb{1}}{2}\right)\right) > 2\left(1-\delta-x_i^*\left(\frac{\mathbb{1}}{2}\right)\right),$$

and hence

$$x_i^*(\lambda_{(i,1)}a_i) > 1-\delta-(1-\lambda_{(i,1)})x_i^*\left(\frac{\mathbb{1}}{2}\right) \geqslant 1-\delta-\frac{1}{2}\rho\|x_i^*\|.$$

We recall that $\delta+\frac{1}{2}\rho\|x_i^*\|<\widetilde{\delta}$, so $x_i^*(\lambda_{(i,1)}a_i)>1-\widetilde{\delta}$. It follows that $x_i^*(a_i)>1-\widetilde{\delta}$. Then $a_i\in K_0\cap S_i$ and, similarly, we get $a_i'\in K_0\cap S_i$ for every $i=1,\dots,n$. Therefore,

$$\frac{1}{n}\sum_{i=1}^n a_i,\ \ \frac{1}{n}\sum_{i=1}^n a_i' \in \frac{1}{n}\sum_{i=1}^n S_i.$$

Since the diameter of $\frac{1}{n}\sum_{i=1}^n S_i$ is less than $\frac{1}{4}(1-\varepsilon)\gamma$, we deduce that $\frac{1}{n}\|\sum_{i=1}^n a_i-a_i'\|<\frac{1}{4}(1-\varepsilon)\gamma$. Finally, we conclude from (4.3) and the above estimate that $\|x-y\|_\varepsilon\leqslant\gamma$. Hence, the set $C:=\frac{1}{n}\sum_{i=1}^n U_i$ has diameter at most γ for the norm $\|\cdot\|_\varepsilon$.

Now, Bourgain's lemma (see Lemma 2.2) ensures the existence of a convex combination of slices $\sum_{j=1}^{p_i}\alpha_{ij}T_{ij}\subseteq U_i$ for every $1\leqslant i\leqslant n$. Using this fact, we will find a convex combination of slices of B of diameter smaller than $\gamma+\frac{4\rho^2}{(1-\frac{\rho^2}{\varepsilon})\varepsilon}$ and such that every slice contains points of $2(K_0-\frac{\mathbb{1}}{2})$. Since ρ and γ can be taken as small as we wish, we will be done. In order to do so, fix $1\leqslant i\leqslant n$ and define

$$A_i:=\left\{j\in\{1,\dots,p_i\}\colon T_{ij}\cap\left(2K_0-\frac{\mathbb{1}}{2}\right)=\emptyset\right\},\quad B_i:=\{1,\dots,p_i\}\setminus A_i.$$

Given $x_{ij}\in T_{ij}$ we find that, for $j\in A_i$, $\delta_0(x_{ij})\geqslant-1+\varepsilon$ by the definition of the unit ball B_ε. Since $\sum_{j=1}^{p_i}\alpha_{ij}x_{ij}\in\sum_{j=1}^{p_i}\alpha_{ij}T_{ij}\subseteq U_i$ we derive $-1+\rho^2>\delta_0(\sum_{j=1}^{p_i}\alpha_{ij}x_{ij})$. Hence

$$-1+\rho^2>\sum_{j\in A_i}\alpha_{ij}\delta_0(x_{ij})+\sum_{i\in B_i}\alpha_{ij}\delta_0(x_{ij})\geqslant(-1+\varepsilon)\sum_{j\in A_i}\alpha_{ij}-\sum_{j\in B_i}\alpha_{ij}=-1+\varepsilon\sum_{j\in A_i}\alpha_{ij}.$$

From the above inequality we infer that $\sum_{j\in A_i}\alpha_{ij} < \rho^2/\varepsilon$ for every $1\leqslant i\leqslant n$. Now, we set $\Lambda_i := \sum_{j\in B_i}\lambda_{ij}$, which belongs to the interval $[1-\rho^2/\varepsilon, 1]$ for $1\leqslant i\leqslant n$ and set

$$D := \frac{1}{n}\sum_{i=1}^{n}\sum_{j\in B_i}\frac{\alpha_{ij}}{\Lambda_I}T_{ij}.$$

Observe that D is a convex combination of slices of B_ε since every T_{ij} is a slice of B_ε and since

$$\frac{1}{n}\sum_{i=1}^{n}\sum_{j\in B_I}\frac{\alpha_{ij}}{\Lambda_i}\alpha_{ij} = 1.$$

We claim that $D\subseteq C + \frac{2}{1-\frac{\rho^2}{\varepsilon}}\frac{\rho^2}{\varepsilon}B_\varepsilon$. This is enough to finish the proof because the above condition implies that

$$\operatorname{diam}(D)\leqslant \operatorname{diam}(C) + \frac{4}{1-\frac{\rho^2}{\varepsilon}}\frac{\rho^2}{\varepsilon}\leqslant \gamma + \frac{4}{1-\frac{\rho^2}{\varepsilon}}\frac{\rho^2}{\varepsilon}.$$

So let us prove the above inclusion. Take $z := \frac{1}{n}\sum_{i=1}^{n}\sum_{j\in B_i}\frac{\alpha_{ij}}{\Lambda_i}x_{ij}\in D$ for certain $x_{ij}\in T_{ij}$. Write $z' := \frac{1}{n}\sum_{i=1}^{n}\sum_{j\in B_i}\alpha_{ij}x_{ij}$. Then

$$|z-z'|\leqslant \frac{1}{n}\sum_{i=1}^{n}\sum_{j\in B_{ij}}\left|1-\frac{1}{\Lambda_i}\right|\alpha_{ij}|x_{ij}| < \frac{1}{1-\frac{\rho^2}{\varepsilon}}\frac{\rho^2}{\varepsilon}.$$

On the other hand, for $1\leqslant i\leqslant n$ and $j\in A_i$ take $x_{ij}\in T_{ij}$. Define

$$z'' := \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{p_i}\alpha_{ij}x_{ij}\in \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{p_i}\alpha_{ij}T_{ij}\subseteq \frac{1}{n}\sum_{i=1}^{n}U_i = C.$$

Moreover, we have

$$|z'-z''|\leqslant \frac{1}{n}\sum_{i=1}^{n}\sum_{j\in A_i}\alpha_{ij} < \frac{\rho^2}{\varepsilon}.$$

Consequently $z = z'' + (z-z'')\in C + \frac{2}{1-\frac{\rho^2}{\varepsilon}}\frac{\rho^2}{\varepsilon}B_\varepsilon$ since

$$|z-z''|\leqslant |z-z'| + |z'-z''| < \frac{1}{1-\frac{\rho^2}{\varepsilon}}\frac{\rho^2}{\varepsilon} + \frac{\rho^2}{\varepsilon} < \frac{2}{1-\frac{\rho^2}{\varepsilon}}\frac{\rho^2}{\varepsilon}. \blacksquare$$

4.7. A summary of relations between the properties. Figure 2 below is a scheme which complements Figure 1 with the counterexamples following from known results and from the results in this chapter.

Let us list the corresponding counterexamples.

(a) The example in Section 4.6.
(b) The example in Section 4.5 negates this implication in the strongest possible way.
(c) Any of the elements in D_B in Section 4.3.3. They also show directly that Δ-points are not necessarily ccs Δ-points.
(d) Every element of the unit sphere of the space X given in Section 4.3.2 is a ccs Δ-point but not Daugavet point.

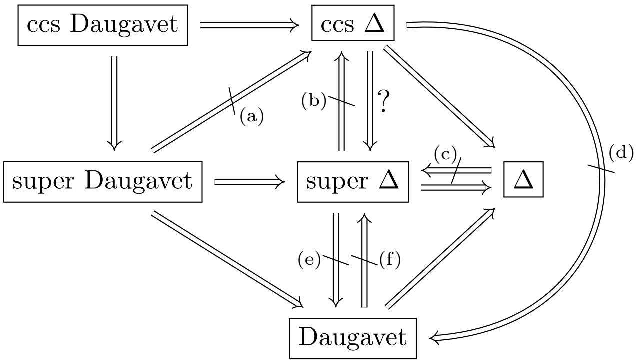

Figure 2. Scheme of all relations between the diametral notions

(e) In $X = C[0,1] \oplus_2 C[0,1]$, every element in the unit sphere is a super Δ-point (Proposition 3.26); but X contains no Daugavet point (Proposition 3.28). Also, every element of the unit sphere of the space X given in Section 4.3.2 is a super Δ-point but not a Daugavet point.

(f) Any of the elements in D_B in Section 4.3.3.

5. Diametral properties for elements of the open unit ball

As mentioned in Chapter 2, DSD2P is equivalent to the Daugavet property by [34], but the ccs Δ-points on the unit sphere of a Banach space do not characterize DSD2P, but restricted DSD2P, which is not equivalent to the Daugavet property (see Section 4.3.2). Actually, the elements in the open unit ball play a decisive role in the proof in [34] of the equivalence between DSD2P and the Daugavet property. Our objective here is to introduce and study the diametral notions for interior points, providing interesting applications, and to investigate the behavior of Daugavet elements and Δ-elements on rays in the unit ball of a given Banach space.

The definition of the Daugavet notions for elements in the open unit ball is the natural extension of the definitions for elements of norm one given in Definition 2.5.

Definition 5.1. Let X be a Banach space and let $x \in B_X$. We say that

(1) x is a *Daugavet point* if $\sup_{y\in S} \|x - y\| = \|x\| + 1$ for every slice S of B_X,
(2) x is a *super Daugavet point* if $\sup_{y\in V} \|x-y\| = \|x\|+1$ for every non-empty relatively weakly open subset V of B_X,
(3) x is a *ccs Daugavet point* if $\sup_{y\in C} \|x - y\| = \|x\| + 1$ for every ccs C of B_X.

It turns out that the existence of a non-zero Daugavet kind element actually forces the whole ray to which it belongs to be composed of similar elements.

Proposition 5.2. *Let X be a Banach space, and let $x \in S_X$. The following assertions are equivalent:*

(1) *x is a Daugavet (resp. super Daugavet, resp. ccs Daugavet) point.*
(2) *rx is a Daugavet (resp. super Daugavet, resp. ccs Daugavet) point for every $r \in [0, 1]$.*
(3) *rx is a Daugavet (resp. super Daugavet, resp. ccs Daugavet) point for some $r \in (0, 1)$.*

Let us recall the following elementary but very useful result from [34] due to Kadets.

Lemma 5.3 ([34, Lemma 2.2]). *Let X be a normed space. If $x, y \in X$ and $\varepsilon > 0$ satisfies that*

$$\|x + y\| > \|x\| + \|y\| - \varepsilon,$$

then for every $a, b > 0$,

$$\|ax + by\| > a\|x\| + b\|y\| - \max\{a, b\}\varepsilon.$$

Proof of Proposition 5.2. We will only do the proof for Daugavet points, the other cases being completely analogous. So let us first assume that x is a Daugavet point. Take

$r \in [0,1]$, $\varepsilon > 0$, and S a slice of B_X. Then there exists $y \in S$ such that $\|x-y\| > 2-\varepsilon$. In particular, $\|y\| > 1-\varepsilon$. As $\|y\| \leqslant 1$,

$$\|x-y\| > 2-\varepsilon \geqslant \|x\| + \|y\| - \varepsilon.$$

It follows from Lemma 5.3 that

$$\|rx-y\| > \|rx\| + \|y\| - \varepsilon > \|rx\| + 1 - 2\varepsilon.$$

Hence, rx is also a Daugavet point.

Now, let us assume that rx is a Daugavet point for some $r \in (0,1)$. Again take $\varepsilon > 0$ and S slice of B_X, and pick $y \in S$ such that $\|rx-y\| > \|rx\| + 1 - r\varepsilon$. In particular, $\|y\| > 1 - r\varepsilon$. As $\|y\| \leqslant 1$, we have

$$\|rx-y\| > \|rx\| + \|y\| - r\varepsilon.$$

Hence, by Lemma 5.3, we get

$$\|x-y\| > \|x\| + \|y\| - \varepsilon > 2-(1+r)\varepsilon$$

and so x is a Daugavet point. ■

As mentioned in the discussion preceding Proposition 3.12, the presence of a ccs Daugavet point in a given Banach space forces the space to satisfy SD2P. This can now be viewed as a consequence of the previous proposition and the following immediate reformulation of [41, Theorem 3.1].

Proposition 5.4 ([41, Theorem 3.1]). *Let X be a Banach space. Then X has SD2P if and only if* 0 *is a ccs Daugavet point.*

Note that c_0 has SD2P but has no non-zero Daugavet points (use Proposition 4.1, for instance).

Compare Proposition 5.4 with the following obvious remark.

Remark 5.5. A Banach space X is infinite-dimensional if and only if 0 is a super Daugavet point.

We also mention that 0 is always a Daugavet point (in finite or infinite dimension), as every slice of the unit ball has to intersect the unit sphere.

Let us also point out that [41, Theorem 3.1] admits the following scaled version.

Proposition 5.6. *Let X be a Banach space, and let $r \in (0,1]$. Then the following assertions are equivalent:*

(1) *Every ccs of B_X has diameter greater than or equal to* $2r$.
(2) $\sup\{\|x\| : x \in C\} \geqslant r$ *for every ccs C of B_X.*
(3) $\sup\{\|x\| : x \in D\} \geqslant r$ *for every symmetric ccs D of B_X (so containing* 0).

Proof. Suppose (1) holds. Then, for any given ccs C of B_X, and for any fixed $\varepsilon > 0$, there exist $x, y \in C$ such that $\|x-y\| > 2r - 2\varepsilon$. In particular, $\|x\| > r-\varepsilon$ or $\|y\| > r - \varepsilon$, giving (2).

(2)⇒(3) is immediate.

Suppose that (1) fails, that is, there exists a ccs C of B_X and $\varepsilon > 0$ such that $\operatorname{diam}(C) \leqslant 2r - 2\varepsilon$. We consider the ccs D of B_X given by $D := \frac{1}{2}(C-C)$. Then D is symmetric, and for every $u := \frac{x-y}{2}$ and $u' := \frac{x'-y'}{2}$ in D we have $\|u-u'\| = \|\frac{x+y'}{2} - \frac{x'+y}{2}\|$. Now, x, x', y, y' belong to C, and C is convex, so $\frac{x+y'}{2}$ and $\frac{x'+y}{2}$ also belong to C, so $\|u - u'\| \leqslant 2r - 2\varepsilon$. As D is symmetric, it implies that $D \subset (r - \varepsilon)B_X$, hence (3) fails. ∎

The following is a nice consequence of the proposition above outside the diametral notions.

Corollary 5.7. *Let X be a Banach space. Then B_X contains ccs of arbitrarily small diameter if and only if* 0 *is a strongly regular point of B_X.*

Now let us consider the Δ notions for points of the open unit ball which are just the adaptation of the notions given in Definition 2.4.

Definition 5.8. Let X be a Banach space and let $x \in B_X$. We say that

(1) x is a *Δ-point* if $\sup_{y\in S} \|x - y\| = \|x\| + 1$ for every slice S of B_X containing x,
(2) x is a *super Δ-point* if $\sup_{y\in V} \|x - y\| = \|x\| + 1$ for every non-empty relatively weakly open subset V of B_X containing x,
(3) x is a *ccs Δ-point* if $\sup_{y\in C} \|x - y\| = \|x\| + 1$ for every slice ccs C of B_X containing x.

With this definitions in hand, we may get an improvement of Proposition 5.4 from Proposition 5.6.

Corollary 5.9. *Let X be a Banach space. Then X has SD2P if and only if* 0 *is ccs Δ-point.*

Compare the previous corollary with the following obvious remark which is analogous to Remark 5.5.

Remark 5.10. A Banach space X is infinite-dimensional if and only if 0 is a super Δ-point.

Observe that the definition of ccs Δ-points for elements in B_X gives a localization of DSD2P, that is, X has DSD2P (and hence the Daugavet property [34]) if and only if all the elements of B_X are ccs Δ-points. Recall that DSD2P is not equivalent to restricted DSD2P (meaning that all points in S_X are ccs Δ-points); see Section 4.3.2.

The following result is a localization of Kadets' theorem [34] on the equivalence of DSD2P and DPr.

Theorem 5.11. *Let X be a Banach space and let $x \in S_X$. If rx is a ccs Δ-point for every $r \in (0, 1)$, then x is a ccs Daugavet point. Moreover, it is enough to assume that* $\inf\{r \in (0,1) \colon rx \text{ is a ccs } \Delta\text{-point}\} = 0$.

Proof. Fix a ccs C of B_X and $\varepsilon > 0$. Since $\tilde{C} := \frac{1}{2}(C - C)$ is also a ccs of B_X and since $0 \in \tilde{C}$ is a norm interior point of $\tilde{C}$ by [41, Proposition 2.1], we find that rx belongs to $\tilde{C}$ for every $r \in (0, \delta)$ for some $\delta > 0$. By hypothesis, there is $r > 0$ such that rx is a ccs Δ-point and $rx \in \tilde{C}$. So there exists $y \in \tilde{C}$ such that $\|rx - y\| > r + 1 - r\varepsilon$. Then if we write $y := y_1 - y_2$ with $y_1, y_2 \in C$, we have $\|rx - y_1\| > r + 1 - r\varepsilon$ or $\|rx - y_2\| > r + 1 - r\varepsilon$ by the triangle inequality; in particular, $\|y_1\| > 1 - r\varepsilon$ and $\|y_2\| > 1 - r\varepsilon$. In both cases, there is $y \in C$ such that $\|rx - y\| > r\|x\| + \|y\| - r\varepsilon$ and it follows from Lemma 5.3 that

$$\|x - y\| > \|x\| + \|y\| - \varepsilon > 2 - (1 + r)\varepsilon > 2 - 2\varepsilon. \blacksquare$$

At this point, it is natural to ask whether an equivalent formulation of Proposition 5.2 is valid for some of the various Δ-notions. For ccs Δ-points, the answer is negative as follows from Theorem 5.11 and, for instance, the example in Section 4.3.2. Another, maybe simpler example is the following one.

Example 5.12. Assume that a positive measure μ admits an atom of finite measure and also has a non-empty non-atomic part. Then the real space $L_1(\mu)$ contains no ccs Daugavet point by Proposition 4.12. However, it contains elements in the unit sphere which are ccs Δ-points and super Daugavet points by Theorem 4.8 and Proposition 4.9. In particular, as a consequence of Theorem 5.11 and of Proposition 5.2, there must exist f in the unit sphere which is a ccs Δ-point and $t \in (0, 1)$ such that tf is not a ccs Δ-point but it is a super Δ-point.

For Δ-points, we have the following result.

Proposition 5.13. *Let X be a Banach space, and let $x \in S_X$. If x is a Δ-point, then rx is a Δ-point for every $r \in (0, 1)$.*

Proof. Assume that x is a Δ-point and fix $r \in (0, 1)$. Take $\varepsilon > 0$ and a slice S of B_X containing rx. Now, either $x \in S$ or $-x \in S$. In the first case, we can find $y \in S$ such that $\|x - y\| \geqslant 2 - \varepsilon$, and using Lemma 5.3, we get $\|rx - y\| \geqslant \|rx\| + 1 - 2\varepsilon$. Else, $\|rx - (-x)\| = r + 1 = \|rx\| + 1$ and we are done. $\blacksquare$

For super Δ-points, it is currently quite obscure whether they behave like Δ-points up on rays.

6. Kuratowski measure and large diameters

Let M be a metric space. The *Kuratowski measure of non-compactness* $\alpha(A)$ of a non-empty bounded subset A of M is defined as the infimum of all real numbers $\varepsilon > 0$ such that A can be covered by a finite number of subsets of M of diameter $\leq \varepsilon$.

From the definition, we clearly have $\alpha(A) = 0$ if and only if A is *totally bounded* (or *precompact*). It follows that every complete subset A of M with α-measure 0 is compact, and in particular if M is a complete metric space, that $\alpha(A) = 0$ if and only if $\overline{A}$ is compact, where $\overline{A}$ stands for the closure of the set A. The α-measure can thus be seen as a way to measure how far a given (non-empty) bounded and closed subset of M is from being a compact space. It was introduced by K. Kuratowski [38] in order to provide a generalization of the famous intersection theorem of Cantor. A general theory of measures of non-compactness was later developed, and it turned out to provide important results in metric fixed point theory, and in particular to have applications in functional equations or optimal control. We refer e.g. to [12] for an introduction to the topic and for more precise applications.

Observe that $A \subseteq B$ implies $\alpha(A) \leqslant \alpha(B)$, and that $\alpha(\overline{A}) = \alpha(A)$. Also note that $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$ for any non-empty bounded subsets A, B of M. Furthermore, if $M = X$ is a normed space, then α is known to enjoy additional useful properties: it is symmetric, translation invariant, positively homogeneous, sub-additive, and satisfies $\alpha(\operatorname{co} A) = \alpha(A)$. The α-measure has proved to be a powerful tool for the study of the geometry of Banach spaces and we refer e.g. to [46, 47, 44] in connection with *property* (α), the *drop property*, and the isomorphic characterization of reflexive Banach spaces.

From the definition it is clear that the Kuratowski measure of A is smaller than or equal to its diameter. Obviously, equality does not always hold, but a fruitful relationship between the notion of Δ-points and the Kuratowski measure of slices was discovered in [5] and completed in [52]. In particular, the following result was obtained (see [52, Corollary 2.2]).

Theorem 6.1. *Let X be a Banach space and let $x \in S_X$. If x is a Δ-point, then $\alpha(S) = 2$ for every slice S of B_X containing x. Moreover, $\alpha(S(x, \delta; B_{X^*})) = 2$ in B_{X^*} for every $\delta > 0$.*

Observe that the converse does not hold in general, as the following example shows.

Example 6.2. Consider $X := L_1([0, 1]) \oplus_\infty \ell_1$. It follows that both X and X^* enjoy SD2P [15, Remark 2.6], so Theorem 6.3 below implies that given any slice $S = S(x^*, \delta; B_X)$ we have $\alpha(S) = 2$, and the same holds for the slices in the dual. However, there are points which are not Δ-points because X fails DLD2P [30, Theorem 3.2] since ℓ_1 does, so it remains to take any point $x \in S_X$ which is not a Δ-point to get the desired counterexample.

The connection between having big slices in diameter and having big slices in Kuratowski index goes beyond Theorem 6.1. The following result was first pointed out in [21].

THEOREM 6.3 ([21, Proposition 3.1]). *Let X be a Banach space and let $\beta \in (0, 2]$. The following assertions are equivalent:*

(1) *Every slice of B_X has diameter greater than or equal to β.*
(2) *Every slice of B_X has Kuratowski measure greater than or equal to β.*

In this chapter, we aim to prove analogues to this result for relatively weakly open subsets, as well as for convex combinations of slices or of weakly open sets, and to extend Veeorg's result to super Δ-points and ccw Δ-points.

6.1. Kuratowski measure and diameter-two properties. The analogue to Theorem 6.3 for non-empty weakly open subsets is the following.

THEOREM 6.4. *Let X be a Banach space and let $\beta \in (0, 2]$. The following assertions are equivalent:*

(1) *Every non-empty relatively weakly open subset of B_X has diameter at least β.*
(2) *Every non-empty relatively weakly open subset of B_X has Kuratowski measure at least β.*

Proof. (2)⇒(1) is immediate, so let us prove (1)⇒(2). To this end, fix $\beta \in (0, 2]$ and assume that every non-empty relatively weakly open subset of B_X has diameter at least β. Then pick $\varepsilon > 0$, and let us prove by induction on n that for every non-empty relatively weakly open subset W of B_X and for every finite collection $C_1, \dots, C_n$ of subsets of X with $\operatorname{diam}(C_i) \leqslant \beta - \varepsilon$ for every i, we have $W \not\subset \bigcup_{i=1}^{n} C_i$.

For $n = 1$, this is clear since by assumption $\operatorname{diam}(W) \geqslant \beta > \beta - \varepsilon$ for every non-empty relatively weakly open subset W of B_X.

So assume that the result is true for every non-empty relatively weakly open subset W of B_X and for every collection of n sets, and let us prove the result for collections of $n + 1$ sets. To this end, consider subsets $C_1, \dots, C_n, C_{n+1}$ of X with $\operatorname{diam}(C_i) \leqslant \beta - \varepsilon$ for every i. Observe that $\operatorname{diam}(C_i) = \operatorname{diam}(\overline{C_i}^{w}) \leqslant \beta - \varepsilon$ by w-lower semicontinuity of the norm of X, so that we may and do assume that C_i is weakly closed for every i.

Observe that by the case $n = 1$ we have $W \not\subset C_{n+1}$, which means that $W \setminus C_{n+1}$ is non-empty. Moreover, it is a weakly open subset of B_X since C_{n+1} is assumed to be weakly closed, and by induction hypothesis we conclude that $W \setminus C_{n+1} \not\subset \bigcup_{i=1}^{n} C_i$. In particular, $W \not\subset \bigcup_{i=1}^{n+1} C_i$ and the theorem is proved. ■

Next, let us establish the analogue of Theorem 6.3 for convex combinations of slices. To this end, observe that by the Bourgain lemma (see Lemma 2.2) every convex combination of non-empty relatively weakly open subsets of B_X contains a convex combination of slices of B_X. This assertion yields the following lemma which allows us to focus on convex combinations of weakly open subsets.

Lemma 6.5. *Let X be a Banach space and $r > 0$.*

(1) *The following are equivalent:*
   (a) *Every convex combination of slices of B_X has diameter at least r.*
   (b) *Every convex combination of non-empty relatively weakly open subsets of B_X has diameter at least r.*

(2) *The following are equivalent:*
   (a) *$\alpha(C) \geqslant r$ for every convex combination C of slices of B_X.*
   (b) *$\alpha(D) \geqslant r$ for every convex combination D of non-empty relatively weakly open subsets of B_X.*

Now we are able to give the following result.

Theorem 6.6. *Let X be a Banach space and let $\beta \in (0, 2]$. The following assertions are equivalent:*

(1) *Every convex combination of slices of B_X has diameter at least β.*
(2) *Every convex combination of slices of B_X has Kuratowski measure at least β.*

Proof. (2)⇒(1) is immediate, so let us prove (1)⇒(2). To this end, fix $\beta \in (0, 2]$ and assume that every convex combination of non-empty relatively weakly open subsets of B_X has diameter at least β. Then pick $\varepsilon > 0$, and let us prove by induction on n that for every convex combination D of non-empty relatively weakly open subsets of B_X and for every finite collection $C_1, \ldots, C_n$ of subsets of X with $\operatorname{diam}(C_i) \leqslant \beta - \varepsilon$ for every i, we have $D \not\subset \bigcup_{i=1}^n C_i$.

For $n = 1$ it is clear since by assumption $\operatorname{diam}(D) \geqslant \beta > \beta - \varepsilon$ for every convex combination of non-empty relatively weakly open subsets of D of B_X.

Assume by inductive step that the result stands for n.

Now pick a convex combination D of non-empty relatively weakly open subsets of B_X and a finite collection $C_1, \ldots, C_{n+1}$ of subsets of X with $\operatorname{diam}(C_i) \leqslant \beta - \varepsilon$ for every i. We can assume as in the proof of Theorem 6.4 that every C_i is weakly closed. Write $D = \sum_{i=1}^k \lambda_i W_i$. Observe that by the case $n = 1$ we have $D \not\subseteq C_{n+1}$, so there exists $z \in D \setminus C_{n+1}$. Since $z \in D$ we can write $z = \sum_{i=1}^k \lambda_i x_i$ where $x_i \in W_i$ holds for every $1 \leqslant i \leqslant k$. Moreover, since $z = \sum_{i=1}^k \lambda_i x_i \notin C_{n+1}$, this means that $z = \sum_{i=1}^k \lambda_i x_i \in X \setminus C_{n+1}$, and the latter is a weakly open set. By a weak-continuity argument for the sum we can find weakly open subsets V_i of B_X, with $x_i \in V_i$ for every $1 \leqslant i \leqslant k$, such that $z = \sum_{i=1}^k \lambda_i x_i \in \sum_{i=1}^n \lambda_i V_i \subseteq X \setminus C_{n+1}$. Up to taking smaller V_i, we can assume $V_i \subseteq W_i$ for every i. Now set $\tilde{D} := \sum_{i=1}^k \lambda_i V_i$, which is a convex combination of weakly open subsets of B_X. By the inductive step, $\tilde{D} \not\subseteq \bigcup_{j=1}^n C_j$, so there exists $y \in \tilde{D}$ with $y \notin C_j$ for $1 \leqslant j \leqslant n$. Observe that the condition $V_i \subseteq W_i$ implies $\tilde{D} \subseteq D$, so $y \in D$ indeed. Moreover, $\tilde{D} \subseteq X \setminus C_{n+1}$ implies in particular $y \notin C_{n+1}$. This implies that $y \in D \setminus \bigcup_{i=1}^{n+1} C_i$, which is precisely what we wanted to prove. ∎

6.2. Kuratowski measure and Δ-notions. We now prove an analogue to Theorem 6.1 for super Δ-points.

THEOREM 6.7. *Let X be a Banach space and let $x \in S_X$ be a super Δ-point. Then every non-empty relatively weakly open subset W of B_X containing x has $\alpha(W) = 2$.*

The proof will be an obvious consequence of the following result.

PROPOSITION 6.8. *Let X be a Banach space, $x \in S_X$ be a super Δ point, and W be a weakly open subset of B_X such that $x \in W$. Then, for every $\varepsilon > 0$, there exists a sequence $\{x_n\} \subseteq W$ such that $\|x_i - x_j\| > 2 - \varepsilon$ for every $i \neq j$.*

Proof. Set $\varepsilon > 0$. Let us construct by induction a sequence $\{x_n\}$ such that $\|x - x_i\| > 2 - \varepsilon/2$ and $\|x_i - x_j\| > 2 - \varepsilon$ for $i \neq j$.

Since x is a super Δ point, select, by the definition of super Δ, a point $x_1 \in W$ with $\|x - x_1\| > 2 - \varepsilon/2$.

Now, assume that $x_1, \ldots, x_n$ have been constructed and let us construct x_{n+1}. By the properties defining the sequence observe that, given $1 \leqslant i \leqslant n$, we have $\|x - x_i\| > 2 - \varepsilon/2$, so we can find $g_i \in S_{X^*}$ with $\operatorname{Re} g_i(x - x_i) > 2 - \varepsilon/2$, which implies $\operatorname{Re} g_i(x) > 1 - \varepsilon/2$ and $\operatorname{Re} g_i(x_i) < -1 + \varepsilon/2$. Consequently,

$$x \in V := W \cap \bigcap_{i=1}^{n} S(g_i, \varepsilon/2),$$

which is a weakly open set. Since x is super Δ we can find $x_{n+1} \in V$ such that $\|x - x_{n+1}\| > 2 - \varepsilon/2$. In order to finish the construction we only have to prove that $\|x_i - x_{n+1}\| > 2 - \varepsilon$ for every $1 \leqslant i \leqslant n$. But this is clear because, given $1 \leqslant i \leqslant n$, the condition $x_{n+1} \in V$ implies that $\operatorname{Re} g_i(x_{n+1}) > 1 - \varepsilon/2$, so

$$\|x_{n+1} - x_i\| \geqslant \operatorname{Re} g_i(x_{n+1} - x_i) > 1 - \varepsilon/2 + 1 - \varepsilon/2 = 2 - \varepsilon,$$

and the proof is finished. ∎

Note that a similar statement to Theorem 6.7 can be established for ccw Δ points.

THEOREM 6.9. *Let X be a Banach space and let $x \in S_X$ be a ccw Δ-point. Then every non-empty convex combination D of relatively weakly open subsets of B_X containing x has $\alpha(D) = 2$.*

As in the previous case, the proof follows directly from the next result.

PROPOSITION 6.10. *Let X be a Banach space, $x \in S_X$ be a ccw Δ point, and D a ccw of B_X such that $x \in D$. Then, for every $\varepsilon > 0$, there exists a sequence $\{x_n\} \subseteq D$ such that $\|x_i - x_j\| > 2 - \varepsilon$ for every $i \neq j$.*

Proof. Set $\varepsilon > 0$. Write $D := \sum_{i=1}^{k} \lambda_i W_i$ with $\lambda_i \neq 0$ for every i. Set $\delta := \frac{\varepsilon}{2 \min_{1 \leqslant i \leqslant k} \lambda_i}$. Let us construct by induction a sequence $\{x_n\} \subseteq D$ such that $\|x - x_i\| > 2 - \delta$ and $\|x_i - x_j\| > 2 - \varepsilon$ for $i \neq j$. Since x is a ccw Δ point select, by the definition of ccw Δ, a point $x_1 \in D$ with $\|x - x_1\| > 2 - \delta$.

Now assume that $x_1, \ldots, x_n$ have been constructed and let us construct x_{n+1}. We can write $x = \sum_{j=1}^{k} \lambda_j x_j$ and $x_i := \sum_{j=1}^{k} \lambda_j x_j^i$ as elements of D.

By the properties defining the sequence, observe that given $1 \leqslant i \leqslant n$ we have $\|x - x_i\| > 2 - \delta$, so we can find $g_i \in S_{X^*}$ with

$$\operatorname{Re} g_i(x - x_i) = \sum_{j=1}^{k} \lambda_j \operatorname{Re} g_i(x_j - x_j^i) > 2 - \delta = 2 - \varepsilon/2 \min_{1 \leqslant j \leqslant n} \lambda_j.$$

A convexity argument implies that $\operatorname{Re} g_i(x_j - x_j^i) > 2 - \varepsilon/2$ for every $1 \leqslant j \leqslant k$, which implies that

$$\operatorname{Re} g_i(x_j) > 1 - \varepsilon/2 \quad \text{and} \quad \operatorname{Re} g_i(x_j^i) < -1 + \varepsilon/2.$$

Observe that

$$x_j \in V_i := W_i \cap \bigcap_{i=1}^{n} S(g_i, \varepsilon/2),$$

which is a weakly open set. Since x is a ccw Δ-point and $x \in \sum_{j=1}^{k} \lambda_j V_j$, we can find a point $x_{n+1} = \sum_{j=1}^{k} \lambda_j z_j \in \sum_{j=1}^{k} \lambda_j V_j \subseteq D$ such that $\|x - x_{n+1}\| > 2 - \delta$. In order to finish the construction we only have to prove that $\|x_i - x_{n+1}\| > 2 - \varepsilon$ holds for every $1 \leqslant i \leqslant n$. Given $1 \leqslant j \leqslant k$, the condition $z_j \in V_j$ implies $\operatorname{Re} g_i(z_j) > 1 - \varepsilon/2$. On the other hand, $\operatorname{Re} g_i(x_j^i) < -1 + \varepsilon/2$, so

$$\|x_{n+1} - x_i\| \geqslant \operatorname{Re} g_i(x - x_i) = \sum_{j=1}^{k} \lambda_j \operatorname{Re} g_i(z_j - x_j^i) > (2 - \varepsilon) \sum_{j=1}^{k} \lambda_j = 2 - \varepsilon,$$

and the proof is finished. ∎

7. Commented open questions

The only implication between properties which is still unclear is the following one (see Figure 2 on page 45).

QUESTION 7.1. Let X be a Banach space and let $x \in S_X$ be a ccs Δ-point. Is x a super Δ-point?

Let us give some comments on this question. On the one hand, it may seem that the answer is positive by Bourgain's lemma (Lemma 2.2), but this lemma *does not say* that, in general, given an element x of a relative weak open subset W of B_X, there is a convex combination of slices of B_X contained in W and *containing* x. The latter happens when $x \in \operatorname{co}(\text{pre-ext}(B_X))$ (see Remark 2.3), so the answer to Question 7.1 is positive in this case. On the other hand, a possible counterexample to this problem could be the molecule in Example 4.16, which is known to be a ccs Δ-point and it is an extreme point but not a preserved extreme point (hence it does not belong to the convex hull of the set of preserved extreme points). A way to show that this molecule is not super Δ-point would be to investigate whether RNP spaces may contain super Δ-points.

Moreover, we do not know whether the global properties related to super Δ-points and ccs Δ-points (i.e. DD2P and restricted DSD2P) are equivalent, or even whether any of the implications holds. Actually, it is not known whether restricted DSD2P implies D2P.

QUESTION 7.2. Does restricted DSD2P imply DD2P or even D2P? Does DD2P imply restricted DSD2P?

Theorem 3.19 states that *real* Banach spaces with a one-unconditional basis contain neither super Δ-points nor ccs Δ-points. It is likely that this result also holds true in the complex setting since we believe that the preliminary results from [6] are also valid for complex scalars, provided that one works with a suitable notion of one-unconditional basis (for which [6, Proposition 2.3] holds). Also, we also expect that the results there can be easily extended to one-uncondional FDDs. Yet, since a sharper version of this result was obtained in Proposition 3.20 for super Δ-points in a very general setting, it is natural to ask whether improved results could be simultaneously obtained in both directions for ccs Δ-points by proving an analogue to Proposition 3.20. So let us ask the following.

QUESTION 7.3. Let X be a Banach space, and assume that there exists a subset $\mathcal{A} \subseteq \mathcal{F}(X, X)$ such that $\sup\{\|\mathrm{Id} - T\| \colon T \in \mathcal{A}\} < 2$ and for every $\varepsilon > 0$ and every $x \in X$, there exists $T \in \mathcal{A}$ such that $\|x - Tx\| < \varepsilon$. Can X contain a ccs Δ-point?

A negative answer to this question would be interesting, since it would provide an example of a ccs Δ-point that is not a super Δ-point, hence a negative answer to Question 7.1.

Another interesting question could be if a point of continuity could be a ccs Δ-point. Let us formalize the questions.

QUESTION 7.4. Let X be a Banach space.

(1) Does X fail RNP (or even CPCP) if X contains a super Δ-point or a super Daugavet point?
(2) Is it possible for a point of continuity to be a ccs Δ-point?

The surprising examples given in Chapter 4 show that the mere existence of some diametral notions (but ccs Daugavet points) on a Banach space does not imply that the whole space has any diameter-two property or the Daugavet property. Our question here is how many diametral points a Banach space has to contain in order to have any diameter-two property or the Daugavet property or fail to have RNP or one-unconditional basis.

QUESTION 7.5. How big can be the set of Daugavet points, super Daugavet points, Δ-points, super Δ-points, or ccs Δ-points in a Banach space with the Radon–Nikodým property, or with CPCP, or being strongly regular, or having one-unconditional basis?

Concerning isometric consequences of the existence of diametral points, there are some recent results showing that a Banach space containing a Δ-point cannot be uniformly non-square [5] or even locally uniformly non-square [37], or asymptotic uniformly smooth [5, 52]. Also, a Banach space having an unconditional basis with suppression-unconditional constant less than 2 cannot contain super Δ-points and a Banach space containing a ccs Daugavet point has SD2P. Taking into account that it is not known if there exists a strictly convex Banach space with the Daugavet property (see [33, Section 5]), the following question makes sense. Recall that Section 4.3.2 gives an example of a strictly convex Banach space in which every norm-one element is a ccs Δ-point and a super Δ-point, but it does not contain any Daugavet point by the way it is constructed.

QUESTION 7.6. Is there a strictly convex Banach space containing a Daugavet point?

In view of Proposition 3.13 and of Theorem 4.14, the following question makes sense.

QUESTION 7.7. Let X be a Banach space. Suppose that $x \in \operatorname{ext}(B_X)$ is a Δ-point; does this imply that x is a ccs Δ-point or a super Δ-point?

By now, the only *isomorphic* restriction which is known for a Banach space to contain Δ-points or even Daugavet points is that it cannot be finite-dimensional. It would be interesting to find some more.

QUESTION 7.8. Find isomorphic restrictions for a Banach space to contain Δ-points or any of the other diametral notions. In particular, is it possible for a reflexive or even super-reflexive Banach space to contain Δ-, super Δ-, ccs Δ-, Daugavet or super Daugavet points?

The results about absolute sums in Section 3.2 are not complete in the case of super Daugavet points and they are even less clear in the case of ccs notions. Here are two possible questions.

QUESTION 7.9. Let X, Y be Banach spaces and let N be an absolute sum.

(1) If N is A-octahedral, $x \in S_X$ and $y \in S_Y$ are super Daugavet points, is (ax, by) a super Daugavet point in $X \oplus_N Y$ when a, b satisfy the conditions in the definition of A-octahedrality?
(2) If N is the ℓ_∞-sum, $x \in S_X$ and $y \in S_Y$ are ccs Δ-points, are the elements of the form (ax, by) ccs Δ-points in $X \oplus_\infty Y$ for $a, b \in [0, 1]$ with $\max\{a, b\} = 1$?

It would also be desirable to study the reversed results to those in Section 3.2 as it is done in [45] for Δ-points and Daugavet points (see [45, tables on pp. 86–87]).

QUESTION 7.10. Let X, Y be Banach spaces, let N be an absolute sum, $x \in S_X$, $y \in S_Y$, and $a, b \geqslant 0$ such that $N(a, b) = 1$. Discuss what happens if (ax, by) satisfies any of the six diametral notions.

It may be the case that some of the arguments in Sections 4.1 and 4.2 can be adapted to other classes of Banach spaces. We propose some possibilities.

QUESTION 7.11. Characterize the six diametral notions in uniform algebras, in Lorentz spaces and their isometric preduals, and in some vector-valued function spaces, like $C(K, X)$, $L_1(\mu, X)$ or $L_\infty(\mu, X)$-spaces.

The relations between the weak* versions of the diametral points (see Remark 2.6) are not yet clear. For instance, the following questions arise.

QUESTION 7.12. Let X be a Banach space and $x \in S_X$.

(1) Is $J_X(x)$ a ccs Δ-point in X^{**} if x is a ccs Δ-point?
(2) Is there any relationship between DD2P in X and weak* super Δ-points in S_{X^*}?

As commented in Remark 4.18, there is a Banach space X containing a sequence (y_n) of super Δ-points such that the distance of y_n to the set of strongly exposed points of B_X tends to zero. But the following question remains open.

QUESTION 7.13. Can a super Δ-point (or even a Δ-point) belong to the closure of the set of denting points?

The answer to the next question on the behaviour of Δ- and super Δ-points in rays is still unknown, as we commented in Chapter 5.

QUESTION 7.14. Let X be a Banach space and let $x \in S_X$.

(1) If rx is a Δ-point for some $0 < r < 1$, does this imply that x is a Δ-point?
(2) If rx is a super Δ-point for some $0 < r < 1$, does this imply that x is a auper Δ-point?
(3) If x is a super Δ-point, does this imply that rx is a super Δ-point for all $0 < r < 1$?

As we proved in Chapter 6, every relatively weakly open subset which contains a super Δ-point (respectively, a ccw Δ-point) has Kuratowski measure 2. Our proofs do not seem to work for convex combination of slices, so let us ask the following.

QUESTION 7.15. If a *ccs* of the unit ball contains a ccs Δ-point, does it necessarily have maximal Kuratowski measure?

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