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Some stability properties for the Bishop–Phelps–Bollobás property for Lipschitz maps

by

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Abstract. We study the stability behavior of the Bishop–Phelps–Bollobás property for Lipschitz maps (Lip-BPB property). This property is a Lipschitz version of the classical Bishop–Phelps–Bollobás property and deals with the possibility of approximating a Lipschitz map that almost attains its (Lipschitz) norm at a pair of distinct points by a Lipschitz map attaining its norm at a pair of distinct points (relatively) very close to the previous one. We first study the stability of this property under the (metric) sum of the domain spaces. Next, we study when it is possible to pass the Lip-BPB property from scalar functions to some vector-valued maps, getting some positive results related to the notions of Γ-flat operators and ACK structure. We get sharper results for the case of Lipschitz compact maps. The behavior of the property with respect to absolute sums of the target space is also studied. We also get results similar to the above for the density of strongly norm attaining Lipschitz maps and of Lipschitz compact maps.

1. Introduction. A *pointed metric space* is just a metric space M in which we have distinguished an element, denoted by 0, that acts like the center of the metric space. All along this paper, metric spaces will be complete and Banach spaces will be over the real scalars. Given a Banach space X, the closed unit ball and the sphere of X will be denoted by B_X and S_X respectively. If Y is another Banach space, $\mathrm{L}(X,Y)$ will denote the space of bounded linear operators from X to Y. In the case $Y=\mathbb{R}$, we simply write X^*.

Given a pointed metric space M and a Banach space Y, $\mathrm{Lip}_0(M,Y)$ denotes the vector space of Lipschitz maps from M to Y that vanish at 0. It becomes a Banach space when endowed with the norm

$$\|F\|_L=\sup\left\{\frac{\|F(p)-F(q)\|}{d(p,q)}: p,q\in M,\, p\neq q\right\}\qquad \forall F\in \mathrm{Lip}_0(M,Y).$$

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We say that a Lipschitz map $F\colon M \to Y$ *strongly attains its norm* if the above supremum is a maximum, that is, there exist distinct points $p,\, q \in M$ such that

$$\frac{\|F(p) - F(q)\|}{d(p,q)} = \|F\|_L.$$

We write $\mathrm{LipSNA}(M,Y)$ for the subset of those Lipschitz maps from M to Y which strongly attain their norm. If M is a finite metric space, it is clear that every Lipschitz map strongly attains its norm. On the other hand, for any infinite metric space M it is easy to find a Lipschitz function $F\colon M \to \mathbb{R}$ which does not strongly attain its norm (see [19, Corollary 3.46]).

In view of this, a natural question arises: Is it always possible to approximate a Lipschitz map by strongly norm attaining Lipschitz maps? The negative answer was given in [17, Example 2.1], where it is shown that this is not the case, e.g., for $M = [0,1]$ and $Y = \mathbb{R}$. On the other hand, the first positive examples appeared in [13, §5], which include the case when M is a compact Hölder metric space and Y is finite-dimensional. The question is then reformulated in [13, Problem 6.7]: for which metric spaces M and Banach spaces Y is the subset $\mathrm{LipSNA}(M,Y)$ dense in $\mathrm{Lip}_0(M,Y)$? Henceforth we will say that we have *strong density* for M and Y if the answer is affirmative. Several papers studying this problem have appeared (see [10, Section 7], [3], [6]). Analyzing those results (which we will briefly comment on), it is possible to extract a common idea in them: a Lipschitz map is identified with a bounded linear operator between Banach spaces. This allows one to apply some results of the classical theory of norm attaining linear operators to obtain results in the Lipschitz context. In order to understand this identification, we need to introduce the Lipschitz-free space over a metric space.

Let M be a pointed metric space. We denote by δ the canonical isometric embedding of M into $\mathrm{Lip}_0(M,\mathbb{R})^*$, which is given by

$$\langle f, \delta(p)\rangle = f(p) \qquad \forall p \in M,\ \forall f \in \mathrm{Lip}_0(M,\mathbb{R}).$$

We denote by $\mathcal{F}(M)$ the norm-closed linear span of $\delta(M)$,

$$\mathcal{F}(M) = \overline{\mathrm{span}}\,\{\delta(p)\colon p \in M\} \subseteq \mathrm{Lip}_0(M,\mathbb{R})^*,$$

usually called the *Lipschitz-free space over* M. It is well known that $\mathcal{F}(M)$ is an isometric predual of $\mathrm{Lip}_0(M,\mathbb{R})$. Moreover, in [19] it was shown that it is the unique isometric predual when M is bounded or a geodesic space. For background we refer to the papers [13] and [14], and to the book [19] (where these spaces are called *Arens–Eells spaces*). Given a Lipschitz map $F\colon M \to Y$, we can consider the unique bounded linear operator $\widehat{F}\colon \mathcal{F}(M) \to Y$ satisfying

$$\widehat{F}(\delta(p)) = F(p) \quad \forall p \in M.$$

It turns out that the mapping $F \mapsto \widehat{F}$ is an isometric isomorphism between $\mathrm{Lip}_0(M,Y)$ and $\mathrm{L}(\mathcal{F}(M),Y)$. Therefore, we can identify every Lipschitz map from M to Y with a bounded linear operator from $\mathcal{F}(M)$ to Y. In order to reformulate our question in terms of this identification, we need to introduce the notion of *molecule* of $\mathcal{F}(M)$, which is just an element of the form

$$m_{p,q} = \frac{\delta(p)-\delta(q)}{d(p,q)}, \qquad \text{where } p,q \in M,\ p \neq q.$$

We write $\mathrm{Mol}(M)$ for the set of all molecules of $\mathcal{F}(M)$. As a consequence of the Hahn–Banach theorem, every molecule has norm 1 and we can recover the unit ball of $\mathcal{F}(M)$ as the closed convex hull of the molecules,

$$B_{\mathcal{F}(M)} = \overline{\mathrm{co}}(\mathrm{Mol}(M)).$$

Now, we can reformulate our question using the notation introduced. It is enough to notice that F strongly attains its norm at a pair of distinct points (p,q) of M if and only if $\widehat{F}$ attains its norm (in the classical sense) at the molecule $m_{p,q}$, that is, $\|\widehat{F}\| = \|\widehat{F}(m_{p,q})\|$. Thus, a Lipschitz map strongly attains its norm if and only if its associated bounded linear operator attains its norm at some molecule. Hence, we are studying for which metric spaces M and Banach spaces Y the set of those bounded linear operators from $\mathcal{F}(M)$ to Y which attain their norm at some molecule is dense in $\mathrm{L}(\mathcal{F}(M),Y)$. Furthermore, we are also interested in the following stronger version of density, which was introduced in [7].

Definition 1.1 ([7]). Let M be a pointed metric space and let Y be a Banach space. We say that the pair (M,Y) has the *Lipschitz Bishop–Phelps–Bollobás property* (*Lip-BPB property* for short) if given $\varepsilon > 0$ there exists $\eta(\varepsilon) > 0$ such that for every norm-one $F \in \mathrm{Lip}_0(M,Y)$ and any $p,q \in M$, $p \neq q$, such that $\|F(p)-F(q)\| > (1-\eta(\varepsilon))d(p,q)$, one may find $G \in \mathrm{Lip}_0(M,Y)$ and $r,s \in M$, $r \neq s$, such that

$$\frac{\|G(r)-G(s)\|}{d(r,s)} = \|G\|_L = 1, \qquad \|G-F\|_L < \varepsilon, \qquad \frac{d(p,r)+d(q,s)}{d(p,q)} < \varepsilon.$$

Equivalently (see [7, Remark 1.2.a]), (M,Y) has the Lip-BPB property if and only if given $\varepsilon > 0$ there is $\eta(\varepsilon) > 0$ such that for every norm-one $\widehat{F} \in \mathrm{L}(\mathcal{F}(M),Y)$ and every $m \in \mathrm{Mol}(M)$ such that $\|\widehat{F}(m)\| > 1-\eta(\varepsilon)$, there exist $\widehat{G} \in \mathrm{L}(\mathcal{F}(M),Y)$ and $u \in \mathrm{Mol}(M)$ such that

$$\|\widehat{G}(u)\| = \|G\|_L = 1, \qquad \|\widehat{F}-\widehat{G}\| < \varepsilon, \qquad \|m-u\| < \varepsilon.$$

The definition of the Lip-BPB property can be understood as a nonlinear generalization of the classical Bishop–Phelps–Bollobás property (BPBp for short). It is clear that if a pair (M,Y) has the Lip-BPB property, then $\mathrm{LipSNA}(M,Y)$ is norm-dense in $\mathrm{Lip}_0(M,Y)$. On the other hand, the converse is far from being true. In fact, if M is a finite pointed metric space, while it is clear that $\mathrm{LipSNA}(M, Y) = \mathrm{Lip}_0(M, Y)$ for every Banach space Y, Example 2.5 in [7] shows that one can find finite pointed metric spaces M and Banach spaces Y such that (M, Y) fails to have the Lip-BPB property. For this reason, throughout this paper each of these notions of approximation by strongly norm attaining Lipschitz maps will be discussed separately.

We will also study the following version of the Lip-BPB property: given a pointed metric space M and a Banach space Y, we say that the pair (M, Y) has the *Lip-BPB property for Lipschitz compact maps* if the requirements of Definition 1.1 are satisfied when the map F is Lipschitz compact and the relevant map G is Lipschitz compact too. Recall that $F\colon M \to Y$ is *Lipschitz compact* when its Lipschitz image, that is, the set

$$\left\{ \frac{F(p) - F(q)}{d(p, q)} : p, q \in M,\, p \neq q \right\} \subseteq Y,$$

is relatively compact. We denote by $\mathrm{Lip}_{0K}(M, Y)$ the space of Lipschitz compact maps from M to Y. Some results related to this notion appear in [15]. One can easily check that $F\colon M \to Y$ is Lipschitz compact if and only if its associated linear operator $\widehat{F}\colon \mathcal{F}(M) \to Y$ is compact, a fact that we will use without further mention. Therefore, if Y is finite-dimensional then every Lipschitz map from M to Y is Lipschitz compact. We denote by $\mathrm{LipSNA}_K(M, Y)$ the set of those Lipschitz compact maps from M to Y which strongly attain their norm, that is,

$$\mathrm{LipSNA}_K(M, Y) = \mathrm{LipSNA}(M, Y) \cap \mathrm{Lip}_{0K}(M, Y).$$

We are also interested in studying for which pointed metric spaces M and Banach spaces Y, either the set $\mathrm{LipSNA}_K(M, Y)$ is dense in $\mathrm{Lip}_{0K}(M, Y)$ or (M, Y) has the Lip-BPB property for Lipschitz compact maps.

1.1. Known results. As already mentioned, all these questions have been studied before in the papers [3, 6, 7, 10, 13], among others. Let us present some of the known results concerning these types of density.

First, it was shown in [10, Proposition 7.4], extending results from [13], that if M is a pointed metric space such that $\mathcal{F}(M)$ has the Radon–Nikodym property (RNP), then $\mathrm{LipSNA}(M, Y)$ is dense in $\mathrm{Lip}_0(M, Y)$ for every Banach space Y. Some cases of metric spaces M for which $\mathcal{F}(M)$ has the RNP are: uniformly discrete, countable and compact, and compact Hölder metric spaces (see [3, Example 1.2] to get references for each result).

Furthermore, properties of $\mathcal{F}(M)$, different from the RNP, that also imply strong density for every Banach space, were given in [3]. More precisely, it is shown there that if $\mathcal{F}(M)$ has either property α or property quasi-α, then $\mathrm{LipSNA}(M, Y)$ is dense in $\mathrm{Lip}_0(M, Y)$ for every Banach space Y. Also, we get the same result if we can find inside $\mathcal{F}(M)$ a uniformly strongly exposed 1-norming set of molecules, that is, a uniformly strongly exposed set generating the unit ball of $\mathcal{F}(M)$ by taking the closed absolutely convex hull. We refer to [3, Section 3], where these sufficient conditions and the relationship between them are discussed. Also, we refer to [6, Theorem 2.5], where it is shown that the Radon–Nikodym property and having a uniformly strongly exposed 1-norming set of molecules are not necessary for strong density, and they are distinct properties.

On the other hand, let us comment on some negative results. Recall that [17, Example 2.1] proves that for $M = [0, 1]$ with its usual metric and $Y = \mathbb{R}$, the subset $\mathrm{LipSNA}([0, 1], \mathbb{R})$ is not dense in $\mathrm{Lip}_0([0, 1], \mathbb{R})$. In fact, [17] proved the same result for geodesic spaces. Later, it was generalized to length spaces (see [3, Theorem 2.2]), where the lack of strongly exposed points of $B_{\mathcal{F}(M)}$ seems to be essential in the proofs. However, if we consider $\mathbb{T} = \{x \in \mathbb{R}^2 \colon \|x\|_2 = 1\}$ endowed with the Euclidean metric, then the unit ball of $\mathcal{F}(\mathbb{T})$ is generated by its strongly exposed molecules (in fact, every molecule is strongly exposed), but [6, Theorem 2.1] shows that $\mathrm{LipSNA}(\mathbb{T}, \mathbb{R})$ is not dense in $\mathrm{Lip}_0(\mathbb{T}, \mathbb{R})$.

Given a metric space M, the last cited result shows that the requirement that $B_{\mathcal{F}(M)}$ is generated by its extreme molecules, or even its strongly exposed molecules, does not guarantee strong density. Interestingly, [6, Theorem 3.3] proves that if M is a metric space for which $\mathrm{LipSNA}(M, \mathbb{R})$ is dense in $\mathrm{Lip}_0(M, \mathbb{R})$, then $B_{\mathcal{F}(M)}$ is generated by its extreme molecules, so the converse holds. Moreover, if we assume that M is compact, then [6, Theorem 3.15] shows that $B_{\mathcal{F}(M)}$ is indeed generated by its strongly exposed molecules.

As regards the Lip-BPB property, given a finite pointed metric space M and a Banach space Y, [7, Theorem 2.1] shows that if $(\mathcal{F}(M), Y)$ has the classical BPBp, then (M, Y) has the Lip-BPB property. This shows, for instance, that if M is finite and Y is finite-dimensional, then (M, Y) has the Lip-BPB property. Moreover, Examples 2.5 and 2.6 in [7] show that both finiteness assumptions are needed. It is also proved in that paper that if M is a pointed metric space such that $\mathrm{Mol}(M)$ is a uniformly strongly exposed set, then (M, Y) has the Lip-BPB property for any Banach space Y. This is the case e.g. when M is concave and $\mathcal{F}(M)$ has property α, M is ultrametric, or M is a Hölder metric space.

We may also find in [7] some results on the vector-valued case and on Lipschitz compact maps. Further, [7, Proposition 4.4] states that if M is a metric space such that $(M, \mathbb{R})$ has the Lip-BPB property and Y is a Banach space with property β, then (M, Y) also has the Lip-BPB property. It is also shown that this result holds when we replace Lip-BPB by strong density (in both the hypothesis and the conclusion) and property β by the weaker property quasi-β. Analogous results are given for Lipschitz compact maps.

Moreover, we can find some results only valid in this context. For instance, [7, Proposition 4.17] says that if M is a metric space such that $(M, \mathbb{R})$ has the Lip-BPB property and Y^* is isometrically isomorphic to an L_1-space, then (M, Y) has the Lip-BPB property for Lipschitz compact maps.

1.2. Outline of the present paper. The main aim of this work is to study the behavior under some natural operations on the domain or on the range spaces of the density of strongly norm attaining Lipschitz maps, of the Lip-BPB property, and of the corresponding versions for Lipschitz compact maps. The results complement the study initiated in [7].

In the second section we focus on the domain space, studying the behavior of the properties under sums of metric spaces (this sum can be understood as the ℓ_1-sum of the metric spaces after identifying their centers and can be found in [19, Definition 1.13]). First, Proposition 2.2 shows that if M is the sum of two metric spaces M_1 and M_2 and Y is a Banach space such that (M, Y) has the LipBPB property, then so does (M_i, Y) for $i = 1, 2$, but the converse is not true, as shown in Example 2.4. For the density of norm attaining Lipschitz maps, the situation is better: if M is the (metric) sum of a family $\{M_i \colon i \in I\}$ of metric spaces and Y is a Banach space, then $\operatorname{LipSNA}(M, Y)$ is dense in $\operatorname{Lip}_0(M, Y)$ if and only if $\operatorname{LipSNA}(M_i, Y)$ is dense in $\operatorname{Lip}_0(M_i, Y)$ for every $i \in I$ (Theorem 2.5). We also obtain analogous results for Lipschitz compact maps.

In the third section we study the problem of strong density or the Lip-BPB property for vector-valued Lipschitz maps for some Banach space Y assuming that we have that property for scalar-valued Lipschitz functions. Notice that [7, Proposition 4.1] shows that if M is a pointed metric space such that (M, Y) has the Lip-BPB property for some Banach space $Y \neq 0$, then $(M, \mathbb{R})$ has the Lip-BPB property. Moreover, [7, Proposition 4.2] gives an analogous result for strong density. Therefore, in order to get the vector-valued versions of the last properties, these assumptions are necessary. It is not known if the assumption that $\operatorname{LipSNA}(M, \mathbb{R})$ is dense in $\operatorname{Lip}_0(M, \mathbb{R})$ is also sufficient to guarantee that $\operatorname{LipSNA}(M, Y)$ is dense in $\operatorname{Lip}_0(M, Y)$ for every Banach space Y. However, [7, Example 2.5] shows that the Lip-BPB property for $(M, \mathbb{R})$ does not imply the Lip-BPB property for (M, Y) in general. In order to get positive results, we follow the recent paper [4], where the class of *Γ-flat operators* and the notion of *ACK structure* on Banach spaces are introduced. Our main results in this section are Theorems 3.5 and 3.10, from which we extract several implications about approximability of Γ-flat maps into spaces of continuous functions, spaces of sequences, and injective tensor products. Also, we state versions of these theorems for Lipschitz compact maps, for which we obtain more satisfactory results.

Finally, in the fourth section we focus on stability properties related to the range space. First, given a pointed metric space M, Proposition 4.3 shows that if Y_1 is an absolute summand of a Banach space Y and (M, Y) has the Lip-BPB property (respectively, $\mathrm{LipSNA}(M, Y)$ is dense), then so does (M, Y_1) (resp. $\mathrm{LipSNA}(M, Y_1)$ is dense). Conversely, Propositions 4.9 and 4.10 show that if Y is the c_0 or ℓ_∞ sum of a family $\{Y_i : i \in I\}$ of Banach spaces such that all the pairs (M, Y_i) have the Lip-BPB property with the same function $\varepsilon \mapsto \eta(\varepsilon)$ (respectively, $\mathrm{LipSNA}(M, Y_i)$ is dense in $\mathrm{Lip}_0(M, Y_i)$ for all $i \in I$), then (M, Y) has the Lip-BPB property (respectively, $\mathrm{LipSNA}(M, Y)$ is dense in $\mathrm{Lip}_0(M, Y)$). Furthermore, we give some stability results involving spaces of continuous functions. The corresponding results for Lipschitz compact maps are also established.

2. Results concerning the domain space: relationship with metric sums. In this section we will study the behavior of the Lip-BPB property with respect to "metric" sums on the domain of Lipschitz maps. We need the following definition.

Definition 2.1 ([19, Definition 1.13]). Given a family $\{(M_i, d_i)\}_{i\in I}$ of pointed metric spaces, the (metric) *sum* of the family, denoted by $\coprod_{i\in I} M_i$, is the disjoint union of all M_i's, with the base points identified, endowed with the metric $d(x, y) = d_i(x, y)$ if $x, y \in M_i$, and $d(x, y) = d_i(x, 0) + d_j(0, y)$ if $x \in M_i$, $y \in M_j$ and $i \neq j$.

It is known (see e.g. [18] or [19, Proposition 3.9]) that Lipschitz-free spaces behave well with respect to sums of metric spaces. Indeed, if $M = \coprod_{i\in I} M_i$, then

$$\mathcal{F}(M) \cong \Big[\bigoplus_{i\in I} \mathcal{F}(M_i)\Big]_{\ell_1}$$

isometrically.

Now, we give some results which show the good behavior of sums of metric spaces with respect to the Lip-BPB property and the density of $\mathrm{LipSNA}(M, Y)$.

Proposition 2.2. *Let $M = M_1 \amalg M_2$ be the sum of two pointed metric spaces and let Y be a Banach space. If (M, Y) has the Lip-BPB property, then so do (M_1, Y) and (M_2, Y).*

Proof. Fix $0 < \varepsilon < 1$ and let $\eta(\varepsilon)$ be the constant given by the Lip-BPB property of (M, Y), which we may suppose to satisfy $\eta(\varepsilon) < \varepsilon$. Let $\widehat{F_1} \in \mathrm{L}(\mathcal{F}(M_1), Y)$ with $\|F_1\|_L = 1$ and $m \in \mathrm{Mol}(M_1)$ such that $\|\widehat{F_1}(m)\| > 1 - \eta(\varepsilon)$. Now, define $\widehat{F} \in \mathrm{L}(\mathcal{F}(M), Y)$ by

$$F(p) = \begin{cases} F_1(p) & \text{if } p \in M_1, \\ 0 & \text{if } p \in M_2. \end{cases}$$

It is easy to see that $\|F\|_L = 1$ and $\|\widehat{F}(m)\| > 1 - \eta(\varepsilon)$, where we see m as a molecule of $\mathcal{F}(M)$. By hypothesis, there exist $\widehat{G} \in \mathrm{L}(\mathcal{F}(M), Y)$ and a molecule $u \in \mathrm{Mol}(M)$ such that

$$\|\widehat{G}(u)\| = \|G\|_L = 1, \qquad \|m - u\| < \varepsilon, \qquad \|F - G\|_L < \varepsilon.$$

Let $\widehat{G}_1 \in \mathrm{L}(\mathcal{F}(M_1), Y)$ be the restriction of $\widehat{G}$ to the subspace $\mathcal{F}(M_1)$. Then it is clear that

$$\|G_1\|_L \leq \|G\|_L = 1 \quad \text{and} \quad \|F_1 - G_1\|_L \leq \|F - G\|_L < \varepsilon.$$

Hence, it will be enough to show that $\widehat{G}_1$ attains its norm at a molecule close enough to m. Let

$$u = \frac{\delta_p - \delta_q}{d(p,q)},$$

where p, $q \in M$, $p \neq q$. We distinguish four cases:

(1) $p, q \in M_1$: In this case u can be seen as a molecule of M_1 and so $\widehat{G}_1$ attains its norm at u.

(2) $p, q \in M_2$: Then

$$\widehat{F}(u) = \frac{F(p) - F(q)}{d(p,q)} = 0,$$

from which we deduce that $\|\widehat{G}(u)\| < \varepsilon$, a contradiction.

(3) $p \in M_1, q \in M_2$: Let us write u as the following convex combination:

$$\begin{aligned} u = \frac{\delta_p - \delta_q}{d(p,q)} &= \frac{\delta_p - \delta_0}{d(p,0)} \frac{d(p,0)}{d(p,q)} + \frac{\delta_0 - \delta_q}{d(0,q)} \frac{d(0,q)}{d(p,q)} \\ &= m_{p,0} \frac{d(p,0)}{d(p,q)} + m_{0,q} \frac{d(0,q)}{d(p,q)}. \end{aligned}$$

Since $\widehat{G}$ attains its norm at u, it also attains its norm at $m_{p,0} \in \mathrm{Mol}(M_1)$. Hence, $\widehat{G}_1$ attains its norm at $m_{p,0}$. Also, note that

$$\|\widehat{F}(u)\| = \frac{d(p,0)}{d(p,q)} \|\widehat{F}(m_{p,0})\| \leq \frac{d(p,0)}{d(p,q)}.$$

On the other hand, $\|\widehat{F}(m)\| > 1 - \eta(\varepsilon)$ and $\|m - u\| < \varepsilon$. Therefore, we must have $\|\widehat{F}(u)\| > 1 - \eta(\varepsilon) - \varepsilon$, which yields $\frac{d(p,0)}{d(p,q)} > 1 - \eta(\varepsilon) - \varepsilon$. Consequently, $\frac{d(0,q)}{d(p,q)} < \eta(\varepsilon) + \varepsilon$. Now, note that

$$\begin{aligned} \|m - m_{p,0}\| &= \left\| (m - u) + \left(\frac{d(p,0)}{d(p,q)} - 1 \right) m_{p,0} + \frac{d(0,q)}{d(p,q)} m_{0,q} \right\| \\ &\leq \|m - u\| + 2 \frac{d(0,q)}{d(p,q)} \leq \|m - u\| + 2\eta(\varepsilon) + 2\varepsilon \\ &< 2\eta(\varepsilon) + 3\varepsilon < 5\varepsilon. \end{aligned}$$

(4) $p \in M_2$, $q \in M_1$: We just have to repeat the previous argument.

Consequently, (M_1, Y) has the Lip-BPB property. By symmetry, so does (M_2, Y). ∎

Note that from this result we obtain the next corollary by just observing that for every $j \in I$, we have $\coprod_{i\in I} M_i \equiv M_j \amalg Z$ for some pointed metric space Z.

COROLLARY 2.3. *Let $M = \coprod_{i\in I} M_i$ be the sum of a family $\{M_i\}_{i\in I}$ of pointed metric spaces and let Y be a Banach space. If (M, Y) has the Lip-BPB property, then so does (M_i, Y) for every $i \in I$.*

The converse of Proposition 2.2 is false, as the next example shows.

EXAMPLE 2.4. Let $M_1 = \{0, 1\}$ and $M_2 = \{1, 2\}$ viewed as subsets of $\mathbb{R}$ with the usual metric and consider 1 as the base point for both spaces. First, observe that $M = M_1 \amalg M_2$ is isometric to the subset $\{0, 1, 2\}$ of $\mathbb{R}$ with the usual metric. Now, (M_i, Y) has the Lip-BPB property for $i = 1, 2$ and every Banach space Y (this is obvious since the spaces $\mathcal{F}(M_1)$ and $\mathcal{F}(M_2)$ are one-dimensional), but for every strictly convex Banach space Y which is not uniformly convex, (M, Y) fails the Lip-BPB property [7, Example 2.5].

For the density of $\mathrm{LipSNA}(M, Y)$, we actually get a characterization, as the following result shows.

THEOREM 2.5. *Let $\{M_i\}_{i\in I}$ be a family of pointed metric spaces, let $M = \coprod_{i\in I} M_i$ and let Y be a Banach space. Then the following are equivalent:*

(1) $\mathrm{LipSNA}(M_i, Y)$ *is dense in* $\mathrm{Lip}_0(M_i, Y)$ *for every* $i \in I$.
(2) $\mathrm{LipSNA}(M, Y)$ *is dense in* $\mathrm{Lip}_0(M, Y)$.

Proof. (1)⇒(2) Consider the natural embeddings $E_i : \mathcal{F}(M_i) \to \mathcal{F}(M)$ and the natural projections $P_i : \mathcal{F}(M) \to \mathcal{F}(M_i)$ for every $i \in I$. Fix $\varepsilon > 0$ and take $\widehat{F} \in L(\mathcal{F}(M), Y) \cong \mathrm{Lip}_0(M, Y)$. Without loss of generality, we may assume that $\|F\|_L = 1$. Using $\|F\|_L = \sup\{\|\widehat{F}E_i\| : i \in I\}$ we can find $h \in I$ such that $\|\widehat{F}E_h\| > \|F\|_L - \varepsilon$. By hypothesis, we can find $G_h \in \mathrm{LipSNA}(M_h, Y)$ satisfying $\|G_h\|_L = \|\widehat{F}E_h\|$ and $\|\widehat{G}_h - \widehat{F}E_h\| \le \varepsilon$. Define $\widehat{G} \in \mathrm{L}(\mathcal{F}(M), Y)$ by

$$\widehat{G}E_i = (1-\varepsilon)\widehat{F}E_i \quad \text{for } i \in I,\ i \neq h, \quad \text{and} \quad \widehat{G}E_h = \widehat{G}_h.$$

Then $\|G\|_L = \sup\{\|\widehat{G}E_i\| : i \in I\} = \|G_h\|_L$ and

$$\|G - F\|_L = \sup\{\|(\widehat{G} - \widehat{F})E_i\| : i \in I\} \le \varepsilon.$$

Moreover, if we take a molecule $m_{p_h,q_h} \in \mathcal{F}(M_h)$ such that $\|\widehat{G}_h(m_{p_h,q_h})\| = \|G_h\|_L$, and we consider the molecule $E_h(m_{p_h,q_h}) \in \mathcal{F}(M)$, we have

$$\|\widehat{G}(E_h(m_{p_h,q_h}))\| = \|\widehat{G}_h(m_{p_h,q_h})\| = \|G_h\|_L = \|G\|_L.$$

Hence, $G \in \mathrm{LipSNA}(M, Y)$.

(2)⇒(1) Pick a summand M_i of M and let $Z = \coprod_{j\in I\setminus\{i\}} M_j$. Then $M = M_i \amalg Z$. Hence, we can simply repeat the arguments in the proof of Proposition 2.2, but forgetting about the distance between the molecules. ∎

Let us show that the previous results also work for Lipschitz compact maps.

Proposition 2.6. *Let $M = M_1 \amalg M_2$ be the sum of two pointed metric spaces and let Y be a Banach space. If (M, Y) has the Lip-BPB property for Lipschitz compact maps, then so do (M_1, Y) and (M_2, Y).*

Proof. We repeat the proof of Proposition 2.2 for a Lipschitz compact map F_1 observing that, in that case, the strongly norm attaining Lipschitz map which approximates F_1 is Lipschitz compact too. ∎

From this result we obtain the following corollary.

Corollary 2.7. *Let $M = \coprod_{i\in I} M_i$ be the sum of pointed metric spaces and let Y be a Banach space. If (M, Y) has the Lip-BPB property for Lipschitz compact maps, then so does (M_i, Y) for every $i \in I$.*

In the same way as in the general case, the converse of Proposition 2.6 is not true, as the same Example 2.4 shows. And again, the analogous result for the density of $\mathrm{LipSNA}_K(M, Y)$ is more satisfactory.

Proposition 2.8. *Let $\{M_i\}_{i\in I}$ be a family of pointed metric spaces, let Y be a Banach space, and let $M = \coprod_{i\in I} M_i$. Then the following are equivalent:*

(1) $\mathrm{LipSNA}_K(M_i, Y)$ *is dense in* $\mathrm{Lip}_{0K}(M_i, Y)$ *for every* $i \in I$.
(2) $\mathrm{LipSNA}_K(M, Y)$ *is dense in* $\mathrm{Lip}_{0K}(M, Y)$.

Proof. It is enough to note that the operators $\widehat{G}$ and $\widehat{G}_h$ defined in the proof of Theorem 2.5 are compact whenever $\widehat{F}$ and $\widehat{F}_h$ are. ∎

3. From scalar functions to vector-valued maps. Our aim in this section is to study the problem of passing from the Lip-BPB property for scalar-valued functions to some vector-valued maps, and the problem of passing from the density of $\mathrm{LipSNA}(M, \mathbb{R})$ to the density of $\mathrm{LipSNA}(M, Y)$ for some Y's. In [7] one can find examples of metric spaces M for which there are Banach spaces Y such that $(M, \mathbb{R})$ has the Lip-BPB property, but (M, Y) does not. Actually, our Example 2.4 contains one of such metric spaces: $M = \{0, 1, 2\}$ with the usual metric. We will present sufficient conditions on a Banach space Y ensuring that (M, Y) has the Lip-BPB property whenever $(M, \mathbb{R})$ does, formally extending [7, Proposition 4.4] in which the result is proved for Y having Lindenstrauss' property β. As regards the density of strongly norm attaining Lipschitz maps, we do not know of any metric space M such that $\mathrm{LipSNA}(M, \mathbb{R})$ is dense in $\mathrm{Lip}_0(M, \mathbb{R})$ but there is Y such that $\mathrm{LipSNA}(M, Y)$ is not dense in $\mathrm{Lip}_0(M, Y)$. Nevertheless, we will also present sufficient conditions on Y ensuring that the density of $\mathrm{LipSNA}(M,\mathbb{R})$ implies the density of $\mathrm{LipSNA}(M,Y)$.

Our work is based on the recent paper [4]. First of all, we need to recall the necessary notions.

DEFINITION 3.1. Let A be a topological space and (M,d) be a metric space. A function $f\colon A\to M$ is said to be *openly fragmented* if for every nonempty open subset $U\subset A$ and every $\varepsilon>0$ there exists a nonempty open subset $V\subset U$ with $\mathrm{diam}(f(V))<\varepsilon$.

It is clear that every continuous function $f\colon A\to M$ is openly fragmented. In particular, if A is a discrete topological space then every function $f\colon A\to M$ is openly fragmented.

DEFINITION 3.2. Let X, Y be Banach spaces and $\Gamma\subset Y^*$. An operator $T\in \mathrm{L}(X,Y)$ is said to be *Γ-flat* if $T^*|_\Gamma\colon(\Gamma,w^*)\to(X^*,\|\cdot\|_{X^*})$ is openly fragmented. In other words, if for every w^*-open subset $U\subseteq Y^*$ with $U\cap\Gamma\neq\emptyset$ and every $\varepsilon>0$ there exists a w^*-open subset $V\subset U$ with $V\cap\Gamma\neq\emptyset$ such that $\mathrm{diam}(T^*(V\cap\Gamma))<\varepsilon$. The set of all Γ-flat operators in $\mathrm{L}(X,Y)$ will be denoted by $\mathrm{Fl}_\Gamma(X,Y)$.

In [4] it is shown that every Asplund operator $T\in\mathrm{L}(X,Y)$ is Γ-flat for every $\Gamma\subseteq B_{Y^*}$. Consequently, every compact operator is Γ-flat for every $\Gamma\subseteq B_{Y^*}$. In addition, it is shown that if (Γ,w^*) is discrete then every bounded operator $T\in\mathrm{L}(X,Y)$ is Γ-flat. Let us also mention that the recently introduced notion of *dentable map* [11] implies Γ-flatness.

Finally, [4] introduces the notion of ACK_ρ structure, which has the structural properties of $C(K)$ and its uniform subalgebras that are essential for the BPB property to hold. Recall that a subset Γ of the unit ball of the dual of a Banach space Y is 1-*norming* if the weak-star closed absolutely convex hull of Γ equals the whole of B_{Y^*} or, equivalently, if $\|y\|=\sup\{|f(y)|\colon f\in\Gamma\}$ for every $y\in Y$.

DEFINITION 3.3. We say that a Banach space Y has *ACK structure* with parameter $\rho\in[0,1)$ ($Y\in ACK_\rho$ for short) whenever there exists a 1-norming set $\Gamma\subset B_{Y^*}$ such that for every $\varepsilon>0$ and every nonempty relatively w^*-open subset $U\subset\Gamma$ there exist a nonempty subset $V\subset U$, vectors $y_1^*\in V$, $e\in S_X$, and an operator $F\in\mathrm{L}(Y,Y)$ such that

(1) $\|Fe\|=\|F\|=1$;
(2) $y_1^*(Fe)=1$;
(3) $F^*y_1^*=y_1^*$;
(4) denoting $V_1=\{y^*\in\Gamma\colon\|F^*y^*\|+(1-\varepsilon)\|(\mathrm{Id}_{Y^*}-F^*)(y^*)\|\le 1\}$, we have $|v^*(Fe)|\le\rho$ for every $v^*\in\Gamma\setminus V_1$;
(5) $d(F^*y^*,\mathrm{aco}\{0,V\})<\varepsilon$ for every $y^*\in\Gamma$; and
(6) $|v^*(e)-1|\le\varepsilon$ for every $v^*\in V$.

The Banach space Y has *simple ACK structure* ($X \in ACK$) if $V_1 = \mathit{\Gamma}$ (and so ρ is redundant).

The following statement is a compilation of results that can be found in [4]. We introduce some notation. Given a Banach space Y, we write $c_0(Y,w)$ for the Banach space of all weakly null sequences in Y; if K is a compact Hausdorff topological space, $C_w(K,Y)$ is the Banach space of all Y-valued weakly continuous functions from K to Y.

Proposition 3.4 ([4]).

1. (1) *$C(K)$ has simple ACK structure for every compact Hausdorff topological space K.*
2. (2) *Finite injective tensor products of Banach spaces which have ACK_ρ structure also have ACK_ρ structure.*
3. (3) *Given a compact Hausdorff topological space K, if $Y \in ACK_\rho$ then $C(K,Y) \in ACK_\rho$.*
4. (4) *Let Y be a Banach space having ACK_ρ structure. Then $c_0(Y)$, $\ell_\infty(Y)$, and $c_0(Y,w)$ have ACK_ρ structure.*
5. (5) *Given a compact Hausdorff topological space K, if $Y \in ACK_\rho$, then $C_w(K,Y)$ has ACK_ρ structure.*

The main result of this section is the following.

Theorem 3.5. *Let M be a pointed metric space such that $(M,\mathbb{R})$ has the Lip-BPB property, let Y be a Banach space in ACK_ρ with associated 1-norming set $\mathit{\Gamma} \subseteq B_{Y^*}$ of Definition 3.3, and let $\varepsilon > 0$. Then there exists $\eta(\varepsilon,\rho) > 0$ such that if $\widehat{T} \in \mathrm{L}(\mathcal{F}(M),Y)$ is a $\mathit{\Gamma}$-flat operator with $\|T\|_L = 1$ and $m \in \mathrm{Mol}(M)$ satisfying $\|\widehat{T}(m)\| > 1-\eta(\varepsilon,\rho)$, then there exist an operator $\widehat{S} \in \mathrm{L}(\mathcal{F}(M),Y)$ and a molecule $u \in \mathrm{Mol}(M)$ such that*

$$\|\widehat{S}(u)\| = \|S\|_L = 1, \qquad \|m-u\| < \varepsilon, \qquad \|T-S\|_L < \varepsilon.$$

By abuse of language, we will say that *$\mathit{\Gamma}$-flat operators on (M,Y) have the Lip-BPB property* when the conclusion of the theorem above is satisfied. Observe that it is not the same kind of version of the Lip-BPB property as the one we gave in the introduction for Lipschitz compact maps since we do not require the approximating operator to be $\mathit{\Gamma}$-flat.

Before proving the result, we present the main consequences of Theorem 3.5.

Corollary 3.6. *Let M be a pointed metric space such that $(M,\mathbb{R})$ has the Lip-BPB property. The following statements hold.*

1. (1) *For every compact Hausdorff topological space K, $\mathit{\Gamma}$-flat operators on $(M,C(K))$ have the Lip-BPB property.*

(2) *Let Z be a finite injective tensor product of Banach spaces which have ACK_ρ structure. Then Γ-flat operators on (M, Z) have the Lip-BPB property.*
(3) *Let K be a compact Hausdorff topological space. If $Y \in ACK_\rho$, then Γ-flat operators on $(M, C(K, Y))$ and on $(M, C_w(K, Y))$ have the Lip-BPB property.*
(4) *Let $Y \in ACK_\rho$. Then Γ-flat operators on $(M, c_0(Y))$, on $(M, \ell_\infty(Y))$, and on $(M, c_0(Y, w))$, have the Lip-BPB property.*

In each case, Γ is any 1*-norming set for which the corresponding space satisfies Definition* 3.3.

This follows immediately from Theorem 3.5 and Proposition 3.4.

Identifying the set Γ of Definition 3.3 in concrete examples is not an easy task in general. One exception is the case of $C(K)$ spaces which we discuss in the next remark.

Remark 3.7. Let us comment on assertion (1) of Corollary 3.6. First, the set Γ of Definition 3.3 for $Y = C(K)$ is just $\Gamma = \{\delta_t : t \in K\} \subset S_{C(K)^*}$ (this follows from the results in [4]), so given $T \in \mathrm{L}(X, C(K))$, $T^*|_\Gamma$ is just the usual representation function of the operator T, that is, $\mu_T : K \to X^*$ given by $\mu_T(t) = T^*(\delta_t)$ for all $t \in K$. This procedure actually gives an identification between $\mathrm{L}(X, C(K))$ and the space of weak-star continuous functions $\mu : K \to X^*$. Norm continuous functions correspond to compact operators (which are Γ-flat). We do not know which functions are openly fragmented or, equivalently, which functions correspond to Γ-flat operators, but there is an intermediate condition which has been studied widely in the literature: quasi-continuity. A function $\mu : K \to X^*$ is *quasi-continuous* if for every nonempty open subset $U \subset K$, every $s \in U$, and every neighborhood V of $\mu(s)$, there exists a nonempty open subset $W \subset U$ such that $\mu(W) \subset V$. This is a classical notion which is still investigated; see [2] and references therein for a sample. Quasi-continuous functions are openly fragmented and they form a class more general than the one of continuous functions.

There is one more consequence of Theorem 3.5 that was already stated in [7, Proposition 4.4] with a different proof. Indeed, if a Banach space Y has Lindenstrauss' property β (see [7, Definition 4.3] for instance), then $Y \in ACK_\rho$ for a discrete 1-norming set Γ, so every operator mapping to Y is Γ-flat. Therefore, Theorem 3.5 yields another proof of the following fact given in [7, Proposition 4.4]: if $(M, \mathbb{R})$ has the Lip-BPB property and Y has property β, then (M, Y) has the Lip-BPB property.

Let us now prepare for the proof of Theorem 3.5 by presenting some preliminary results. The next easy lemma shows that if the Lip-BPB property is satisfied for $(M,\mathbb{R})$, then we can also approximate Lipschitz functions whose norm is less than 1 as long as it is close enough to 1.

LEMMA 3.8. *Let M be a pointed metric space and let $\varepsilon > 0$. Suppose that $(M,\mathbb{R})$ has the Lip-BPB property witnessed by a function $\varepsilon \mapsto \eta(\varepsilon) > 0$. Then, given $f \in \mathrm{Lip}_0(M,\mathbb{R})$ with $\|f\|_L \le 1$ and $m \in \mathrm{Mol}(M)$ such that $|\widehat{f}(m)| > 1-\eta(\varepsilon)$, there exist $g \in \mathrm{Lip}_0(M,\mathbb{R})$ with $\|g\|_L = 1$ and $u \in \mathrm{Mol}(M)$ satisfying*

$$|\widehat{g}(u)| = 1, \qquad \|f-g\|_L < \varepsilon + \eta(\varepsilon), \qquad \|m-u\| < \varepsilon.$$

Proof. If $\|f\|_L = 1$ then it is enough to apply the Lip-BPB property. If $\|f\|_L < 1$, then by the Lip-BPB property there exist $g \in S_{\mathrm{Lip}_0(M,\mathbb{R})}$ and $u \in \mathrm{Mol}(M)$ satisfying

$$\left\| g - \frac{f}{\|f\|} \right\|_L < \varepsilon, \qquad \|u-m\| < \varepsilon.$$

Then

$$\|g-f\|_L \le \left\| g - \frac{f}{\|f\|} \right\|_L + \left\| \frac{f}{\|f\|} - f \right\|_L < \varepsilon + |1-\|f\|_L| \le \varepsilon + \eta(\varepsilon). \ \blacksquare$$

LEMMA 3.9. *Let M be a pointed metric space such that $(M,\mathbb{R})$ has the Lip-BPB property, let Y be a Banach space, and let $\Gamma \subseteq B_{Y^*}$ be a 1-norming set. Fix $\varepsilon > 0$ and let $\eta(\varepsilon)$ be the constant given by the Lip-BPB property of $(M,\mathbb{R})$. Let $\widehat{T} \in \mathrm{Fl}_\Gamma(\mathcal{F}(M),Y)$ be a Γ-flat operator with $\|T\|_L = 1$ and $m \in \mathrm{Mol}(M)$ such that*

$$\|\widehat{T}(m)\| > 1-\eta(\varepsilon).$$

Then for every $r > 0$ there exist

(1) *a w^*-open subset $U_r \subset Y^*$ with $U_r \cap \Gamma \neq \emptyset$,*
(2) *$\widehat{f}_r \in S_{\mathcal{F}(M)^*}$ and $u \in \mathrm{Mol}(M)$ satisfying*

$$\widehat{f}_r(u) = 1, \quad \|m-u\| \le \varepsilon, \quad \|\widehat{T}^* z^* - \widehat{f}_r\| \le r+\varepsilon+\eta(\varepsilon) \quad \forall z^* \in U_r \cap \Gamma. \tag{3.1}$$

Proof. Since Γ is a 1-norming set, we can pick $y_0^* \in \Gamma$ such that

$$|\widehat{T}^*(y_0^*(m))| = |y_0^*(\widehat{T}(m))| > 1-\varepsilon.$$

Set $U = \{y^* \in Y^* \colon |(\widehat{T}^* y^*)(m)| > 1-\varepsilon\}$. Then, according to Definition 3.2, for every $r > 0$ there exists a w^*-open subset $U_r \subseteq U$ with $U_r \cap \Gamma \neq \emptyset$ such that $\mathrm{diam}(\widehat{T}^*(U_r \cap \Gamma)) < r$. Fix $y_1^* \in U_r \cap \Gamma$ and set $\widehat{f}_1 = \widehat{T}^*(y_1^*)$. Then

$$1 \ge \|f_1\|_L \ge |\widehat{f}_1(m)| > 1-\varepsilon.$$

Now we obtain the function $\widehat{f}_r$ and the molecule u by applying Lemma 3.8. It is easily checked that they satisfy the required properties (for details see [4, proof of Lemma 2.9]). $\blacksquare$

Proof of Theorem 3.5. Given $\varepsilon > 0$, let $\widehat{\eta}(\varepsilon) > 0$ be the constant associated to the Lip-BPB property of $(M, \mathbb{R})$. Fix $0 < \varepsilon_0 < \varepsilon$ and take $\varepsilon_1 > 0$ such that

$$\max\left\{\varepsilon_1, 2\left((\varepsilon_1 + \eta(\varepsilon_1)) + \frac{2(\varepsilon_1 + \eta(\varepsilon_1))}{1 - \rho + (\varepsilon_1 + \eta(\varepsilon_1))}\right)\right\} \leq \varepsilon_0.$$

Take $r > 0$ and $0 < \varepsilon_2 < 2/3$. Let $\widehat{T} \in \mathrm{L}(\mathcal{F}(M), Y)$ be a Γ-flat operator with $\|T\|_L = 1$ and a molecule $m \in \mathrm{Mol}(M)$ such that $\|\widehat{T}(m)\| > 1 - \widehat{\eta}(\varepsilon)$. Then, applying Lemma 3.9 with Y, Γ, r and ε_1, we obtain a w^*-open subset $U_r \subseteq Y^*$ with $U_r \cap \Gamma \neq \emptyset$, and $\widehat{f}_r \in S_{\mathcal{F}(M)^*}$, $u_r \in \mathrm{Mol}(M)$ satisfying

$$\widehat{f}_r(u_r) = 1, \quad \|m - u_r\| \leq \varepsilon_1, \quad \|\widehat{T}^* z^* - \widehat{f}_r\| \leq r + \varepsilon_1 + \eta(\varepsilon_1) \quad \forall z^* \in U_r \cap \Gamma.$$

On the other hand, since $U_r \cap \Gamma \neq \emptyset$, by applying the definition of ACK_ρ structure to $U = U_r \cap \Gamma$ and ε_2, we obtain a nonempty subset $V \subseteq U$, points $y_1^* \in V$ and $e \in S_Y$, an operator $F \in \mathrm{L}(Y, Y)$, and a subset $V_1 \subseteq \Gamma$ satisfying the properties of Definition 3.3.

Define a linear operator $\widehat{S} \colon \mathcal{F}(M) \to Y$ by

$$\widehat{S}(x) = \widehat{f}_r(x) Fe + (1 - \delta)(\mathrm{Id}_Y - F)\widehat{T}(x),$$

where $\delta \in [\varepsilon_2, 1)$. We will show that one can choose δ so that $\|\widehat{S}\| \leq 1$. Recall that since Γ is a 1-norming set, we have

$$\|\widehat{S}\| = \|\widehat{S}^*\| = \sup\{\|\widehat{S}^* y^*\| \colon y^* \in \Gamma\}.$$

Therefore, we take $y^* \in \Gamma$ and estimate

$$\|\widehat{S}^* y^*\| = \|y^*(Fe)\widehat{f}_r + (1 - \delta)\widehat{T}^*(\mathrm{Id}_{Y^*} - F^*)(y^*)\|.$$

If $y^* \in V_1$, then $\|\widehat{S}^* y^*\| \leq 1$ follows from the definition of V_1 (see (4) in Definition 3.3) and from $\delta \geq \varepsilon_2$. Therefore, we only have to consider the case when $y^* \in \Gamma \setminus V_1$. By Definition 3.3(5), for every $y^* \in \Gamma$ there exists a point $v^* = \sum_{k=1}^n \lambda_k v_k^*$ satisfying

$$\{v_1^*, \ldots, v_n^*\} \subseteq V, \qquad \sum_{k=1}^n |\lambda_k| \leq 1, \qquad \|F^* y^* - v^*\| < \varepsilon_2. \tag{3.2}$$

Now, by Definition 3.3(6) and by (3.1), we obtain

$$\begin{aligned}
\|v^*(e)\widehat{f}_r - \widehat{T}^* v^*\| &\leq \sum_{k=1}^n |\lambda_k| \, \|v_k^*(e)\widehat{f}_r - \widehat{T}^* v_k^*\| \\
&\leq \sum_{k=1}^n |\lambda_k| (\|v_k^*(e)\widehat{f}_r - \widehat{f}_r\| + \|\widehat{f}_r - \widehat{T}^* v_k^*\|) \\
&\leq \varepsilon_2 + \sum_{k=1}^n |\lambda_k| \, \|\widehat{f}_r - \widehat{T}^* v_k^*\| \leq \varepsilon_2 + r + \varepsilon_1 + \eta(\varepsilon_1).
\end{aligned}$$

Using Definition 3.3(4) and (3.2), for every $y^* \in \Gamma \setminus V_1$ we have

$$\begin{aligned}\|\widehat{S}^* y^*\| &\leq \delta |y^*(Fe)| + (1-\delta)\|y^*(Fe)\widehat{f_r} + \widehat{T}^* y^* - \widehat{T}^* F^* y^*\| \\ &\leq \delta\rho + (1-\delta)\|\widehat{T}^* y^*\| + (1-\delta)\|(F^*y^*)(e)\widehat{f_r} - \widehat{T}^* F^* y^*\| \\ &\leq \delta\rho + (1-\delta) + 2\varepsilon_2(1-\delta) + (1-\delta)\|v^*(e)\widehat{f_r} - \widehat{T}^* v^*\| \\ &\leq \delta\rho + (1-\delta) + 2\varepsilon_2(1-\delta) + (1-\delta)(\varepsilon_2 + r + \varepsilon_1 + \eta(\varepsilon_1)) \\ &\leq \delta\rho + (1-\delta)(1 + 3\varepsilon_2 + r + \varepsilon_1 + \eta(\varepsilon_1)).\end{aligned}$$

Therefore, if we choose

$$\delta = \frac{3\varepsilon_2 + r + \varepsilon_1 + \eta(\varepsilon_1)}{1 - \rho + 3\varepsilon_2 + r + \varepsilon_1 + \eta(\varepsilon_1)},$$

then $\|\widehat{S}\| \leq 1$. Moreover, if ε_2 is small enough, we will have $\delta \geq \varepsilon_2$ which was needed before. In this case,

$$1 = |\widehat{f_r}(u_r)| = |y_1^*(\widehat{f_r}(u_r)Fe)| = |y_1^*(\widehat{S}(u_r))| \leq \|\widehat{S}(u_r)\| \leq 1,$$

from which we deduce that $\|\widehat{S}\| = 1$ and $\widehat{S}$ attains its norm at the molecule u_r, which we already know to satisfy $\|m - u_r\| \leq \varepsilon_1 \leq \varepsilon_0 < \varepsilon$.

Finally, let us estimate $\|\widehat{S} - \widehat{T}\|$. First,

$$\begin{aligned}\|\widehat{S} - \widehat{T}\| = \|\widehat{S}^* - \widehat{T}^*\| &= \sup\{|\widehat{S}^* y^* - \widehat{T}^* y^*| \colon y^* \in \Gamma\} \\ &\leq 2\delta + \sup\{\|y^*(Fe)\widehat{f_r} - \widehat{T}^* F^* y^*\| \colon y^* \in \Gamma\}.\end{aligned}$$

Second,

$$\|(F^*y^*)(e)\widehat{f_r} - \widehat{T}^* F^* y^*\| \leq 2\varepsilon_2 + \|v^*(e)\widehat{f_r} - \widehat{T}^* v^*\| \leq 3\varepsilon_2 + r + \varepsilon_1 + \eta(\varepsilon_1).$$

Therefore,

$$\|\widehat{S} - \widehat{T}\| \leq 2\delta + 3\varepsilon_2 + r + \varepsilon_1 + \eta(\varepsilon_1).$$

Since ε_2 and r were arbitrary, by taking these constants with $3\varepsilon_2 + r \leq \varepsilon_1 + \eta(\varepsilon_1)$, we will have

$$\begin{aligned}\|\widehat{S} - \widehat{T}\| &\leq 2(\varepsilon_1 + \eta(\varepsilon_1) + \delta) \\ &\leq 2\left((\varepsilon_1 + \eta(\varepsilon_1)) + \frac{2(\varepsilon_1 + \eta(\varepsilon_1))}{1 - \rho + \varepsilon_1 + \eta(\varepsilon_1)}\right) \leq \varepsilon_0 < \varepsilon. \ \blacksquare\end{aligned}$$

Given a pointed metric space M and a Banach space Y, it is possible to give a result analogous to Theorem 3.5 but for the density of $\mathrm{LipSNA}(M, Y)$. We just have to repeat the previous proof using the fact that $\mathrm{LipSNA}(M, \mathbb{R})$ is dense in $\mathrm{Lip}_0(M, \mathbb{R})$ instead of the Lip-BPB property of $(M, \mathbb{R})$, forgetting the estimation on the distance between molecules.

THEOREM 3.10. *Let M be a pointed metric space such that* $\mathrm{LipSNA}(M, \mathbb{R})$ *is dense in* $\mathrm{Lip}_0(M, \mathbb{R})$*, let Y be a Banach space in ACK_ρ, and let $\Gamma \subseteq B_{Y^*}$*

be the 1*-norming set given by Definition* 3.3*. Then*

$$\mathrm{Fl}_\Gamma(\mathcal{F}(M), Y) \subseteq \overline{\mathrm{LipSNA}(M, Y)}.$$

As before, we obtain a series of consequences.

Corollary 3.11. *Assume that M is a pointed metric space such that $\mathrm{LipSNA}(M,\mathbb{R})$ is dense in $\mathrm{Lip}_0(M,\mathbb{R})$.*

(1) *Let K be a compact Hausdorff topological space and $\Gamma = \{\delta_t : t \in K\}$ (see Remark* 3.7*). Then $\mathrm{Fl}_\Gamma(\mathcal{F}(M), C(K)) \subseteq \overline{\mathrm{LipSNA}(M, C(K))}$.*
(2) *Let Z be a finite injective tensor product of Banach spaces which have ACK_ρ structure. If Γ is the* 1*-norming set given by Definition* 3.3*, then $\mathrm{Fl}_\Gamma(\mathcal{F}(M), Z) \subseteq \overline{\mathrm{LipSNA}(M, Z)}$.*
(3) *Let K be a compact Hausdorff topological space. If $Y \in ACK_\rho$ and Γ is the* 1*-norming set given by Definition* 3.3*, then*
$$\mathrm{Fl}_\Gamma(\mathcal{F}(M), C(K,Y)) \subseteq \overline{\mathrm{LipSNA}(M, C(K,Y))}.$$
(4) *Let $Y \in ACK_\rho$. If Γ is the* 1*-norming set given by Definition* 3.3*, then*
$$\mathrm{Fl}_\Gamma(\mathcal{F}(M), c_0(Y)) \subseteq \overline{\mathrm{LipSNA}(M, c_0(Y))},$$
$$\mathrm{Fl}_\Gamma(\mathcal{F}(M), \ell_\infty(Y)) \subseteq \overline{\mathrm{LipSNA}(M, \ell_\infty(Y))},$$
$$\mathrm{Fl}_\Gamma(\mathcal{F}(M), c_0(Y,w)) \subseteq \overline{\mathrm{LipSNA}(M, c_0(Y,w))}.$$
(5) *Let K be a compact Hausdorff topological space. If $Y \in ACK_\rho$ and Γ is the* 1*-norming set given by Definition* 3.3*, then*
$$\mathrm{Fl}_\Gamma(\mathcal{F}(M), C_w(K,Y)) \subseteq \overline{\mathrm{LipSNA}(M, C_w(K,Y))}.$$

As in Corollary 3.6, the following consequence also follows: if the set $\mathrm{LipSNA}(M,\mathbb{R})$ is dense in $\mathrm{Lip}_0(M,\mathbb{R})$ and a Banach space Y has property β, then $\mathrm{LipSNA}(M,Y)$ is dense in $\mathrm{Lip}_0(M,Y)$. This result also appeared in [7]. Actually, a more general result dealing with a property weaker than property β called property quasi-β holds [7, Proposition 4.7].

We next deal with Lipschitz compact maps. Observe that one of the disadvantages of Theorems 3.5 and 3.10 and their consequences in Corollaries 3.6 and 3.11 is the need to deal with Γ-flat operators. But now this requirement disappears: given a Banach space Y, every compact operator with codomain Y is Γ-flat for every $\Gamma \subseteq B_{Y^*}$ [4, Example A]. Moreover, if we take a compact operator $\widehat{T}$ in the proof of Theorem 3.5, then the operator $\widehat{S}$ that approximates $\widehat{T}$ will also be compact. Consequently, we obtain the following result.

Proposition 3.12. *Let M be a pointed metric space such that $(M,\mathbb{R})$ has the Lip-BPB property and let Y be an ACK_ρ Banach space. Then (M,Y) has the Lip-BPB property for Lipschitz compact maps.*

Again, in view of Proposition 3.4, we obtain a series of implications.

Corollary 3.13. *Let M be a pointed metric space such that $(M,\mathbb{R})$ has the Lip-BPB property.*

(1) *Let K be a compact Hausdorff topological space. Then $(M,C(K))$ has the Lip-BPB property for Lipschitz compact maps.*
(2) *Let Z be a finite injective tensor product of Banach spaces which have ACK_ρ structure. Then (M,Z) has the Lip-BPB property for Lipschitz compact maps.*
(3) *Let K be a compact Hausdorff topological space. If $Y\in ACK_\rho$, then $(M,C(K,Y))$ and $(M,C_w(K,Y))$ have the Lip-BPB property for Lipschitz compact maps.*
(4) *Let $Y\in ACK_\rho$. Then $(M,c_0(Y))$, $(M,\ell_\infty(Y))$, and $(M,c_0(Y,w))$ have the Lip-BPB property for Lipschitz compact maps.*

The case when Y has property β in the above proposition already appeared in [7, Proposition 4.13]. Also, item (1) above is covered by [7, Proposition 4.17].

Just as with Proposition 3.12, the proof of Theorem 3.10 can be easily adapted to the density of Lipschitz compact maps.

Proposition 3.14. *Let M be a pointed metric space such that* $\operatorname{LipSNA}(M,\mathbb{R})$ *is dense in* $\operatorname{Lip}_0(M,\mathbb{R})$ *and $Y\in ACK_\rho$ be a Banach space. Then the set* $\operatorname{LipSNA}_K(M,Y)$ *is dense in* $\operatorname{Lip}_{0K}(M,Y)$.

As before, this result has many consequences.

Corollary 3.15. *Assume that M is a pointed metric space such that* $\operatorname{LipSNA}(M,\mathbb{R})$ *is dense in* $\operatorname{Lip}_0(M,\mathbb{R})$.

(1) $\operatorname{LipSNA}_K(M,C(K))$ *is dense in* $\operatorname{Lip}_{0K}(M,C(K))$ *for every compact Hausdorff topological space K.*
(2) *Let Z be a finite injective tensor product of Banach spaces which have ACK_ρ structure. Then* $\operatorname{LipSNA}_K(M,Z)$ *is dense in* $\operatorname{Lip}_{0K}(M,Z)$.
(3) *Let K be a compact Hausdorff topological space. If $Y\in ACK_\rho$, then* $\operatorname{LipSNA}_K(M,C(K,Y))$ *and* $\operatorname{LipSNA}_K(M,C_w(K,Y))$ *are dense in* $\operatorname{Lip}_{0K}(M,C(K,Y))$ *and* $\operatorname{Lip}_{0K}(M,C_w(K,Y))$, *respectively.*
(4) *Let $Y\in ACK_\rho$. Then* $\operatorname{LipSNA}_K(M,c_0(Y))$, $\operatorname{LipSNA}_K(M,\ell_\infty(Y))$, *and* $\operatorname{LipSNA}_K(M,c_0(Y,w))$ *are dense in the spaces* $\operatorname{Lip}_{0K}(M,c_0(Y))$, $\operatorname{Lip}_{0K}(M,\ell_\infty(Y))$, *and* $\operatorname{Lip}_{0K}(M,c_0(Y,w))$, *respectively.*

Let us mention another result which follows from Proposition 3.14: property β of Y is enough to pass from the density of $\operatorname{LipSNA}(M,\mathbb{R})$ to the density of $\operatorname{LipSNA}_K(M,Y)$. Actually, [7, Proposition 4.15] gives a stronger result: let M be a pointed metric space such that $\operatorname{LipSNA}(M,\mathbb{R})$ is dense in $\operatorname{Lip}_0(M,\mathbb{R})$ and let Y be a Banach space with property quasi-β (weaker than property β); then $\overline{\mathrm{LipSNA}_K(M,Y)} = \mathrm{Lip}_{0K}(M,Y)$. Moreover, item (1) above is covered by [7, Corollary 4.19].

We finish this section by presenting some more results on the Lip-BPB property for Lipschitz compact maps and on the density of strongly norm attaining Lipschitz compact maps. They easily follow from the results of [7]. We start with a result on the Lip-BPB property for Lipschitz compact maps.

PROPOSITION 3.16 ([7, Proposition 4.16]). *Let M be a pointed metric space and let Y be a Banach space. Suppose that there exists a net $\{Q_\lambda\}_{\lambda\in\Lambda} \subset \mathrm{L}(Y,Y)$ of norm-one projections such that $\{Q_\lambda(y)\} \to y$ in norm for every $y \in Y$. If there is a function $\eta\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that for every $\lambda \in \Lambda$, the pair $(M, Q_\lambda(Y))$ has the Lip-BPB property for Lipschitz compact maps witnessed by η, then (M,Y) has the Lip-BPB property for Lipschitz compact maps.*

The following result collects several consequences of the proposition above. None of them was included in [7]. Observe that item (1) below formally extends items (1) and (3) of our Corollary 3.13.

COROLLARY 3.17. *Let M be a pointed metric space and let Y be a Banach space such that (M,Y) has the Lip-BPB property for Lipschitz compact maps.*

(1) *For every compact Hausdorff topological space K, $(M, C(K,Y))$ has the Lip-BPB property for Lipschitz compact maps.*
(2) *For $1 \le p < \infty$, if $(M, \ell_p(Y))$ has the Lip-BPB property for Lipschitz compact maps, then so does $(M, L_p(\mu,Y))$ for every positive measure μ.*
(3) *For every σ-finite positive measure μ, the pair $(M, L_\infty(\mu,Y))$ has the Lip-BPB property for Lipschitz compact maps.*

Proof. This proof is based on [9, proof of Theorem 3.15]. To prove (1), following [16, proof of Theorem 4], by using peaked partitions of unity in $C(K)$ we can find a net $\{Q_\lambda\}_{\lambda\in\Lambda}$ of norm-one projections on $C(K,Y)$ such that $\{Q_\lambda(f)\} \to f$ in norm for every $f \in C(K,Y)$ and $Q_\lambda(C(K,Y))$ is isometrically isomorphic to $\ell_\infty^m(Y)$. Consequently, (1) follows from Propositions 4.11 and 3.16.

To prove (2), fix $1 \le p < \infty$. If $L_1(\mu)$ is finite-dimensional, the result is a consequence of Proposition 4.3. Otherwise, by using [9, Lemma 3.12] we may find a net $\{Q_\lambda\}_{\lambda\in\Lambda}$ of norm-one projections on $L_p(\mu,Y)$ such that $\{Q_\lambda\} \to f$ in norm for every $f \in L_p(\mu,Y)$ and $Q_\lambda(L_p(\mu,Y))$ is isometrically isomorphic to $\ell_p(Y)$. Therefore, it is enough to apply Proposition 3.16.

As before, if $L_\infty(\mu)$ is finite-dimensional, the result is a consequence of Proposition 4.11. If $L_\infty(\mu)$ is infinite-dimensional, we may suppose that the measure is finite by using [5, Proposition 1.6.1]. Then [9, Lemma 3.12] provides a net $\{Q_\lambda\}_{\lambda\in\Lambda}$ of norm-one projections on $L_\infty(\mu,Y)$ such that $\{Q_\lambda\} \to f$ in norm for every $f \in L_\infty(\mu,Y)$ and $Q_\lambda(L_p(\mu,Y))$ is isometrically isomorphic to $\ell_\infty(Y)$. Consequently, the result follows from Propositions 4.11 and 3.16. ■

An analogous result to Proposition 3.16 also appeared in [7] for the density of strongly norm attaining Lipschitz compact maps.

PROPOSITION 3.18 ([7, Proposition 4.18]). *Let M be a pointed metric space and Y be a Banach space. Suppose that there exists a net $\{Q_\lambda\}_{\lambda\in\Lambda} \subset \mathrm{L}(Y,Y)$ of norm-one projections such that $\{Q_\lambda(y)\} \to y$ in norm for every $y \in Y$. If $\mathrm{LipSNA}_K(M, Q_\lambda(Y))$ is dense in $\mathrm{Lip}_{0K}(M, Q_\lambda(Y))$ for every $\lambda \in \Lambda$, then*

$$\overline{\mathrm{LipSNA}_K(M,Y)} = \mathrm{Lip}_{0K}(M,Y).$$

Now, by using this proposition instead of Proposition 3.16 and replacing the necessary results by their versions for the density of $\mathrm{LipSNA}_K(M,Y)$, the proof of Corollary 3.17 can be easily adapted to get the following results about the density of strongly norm attaining Lipschitz compact maps. None of them appeared in [7].

COROLLARY 3.19. *Let M be a pointed metric space and let Y be a Banach space such that $\mathrm{LipSNA}_K(M,Y)$ is dense in $\mathrm{Lip}_{0K}(M,Y)$.*

(1) $\mathrm{LipSNA}_K(M, C(K,Y))$ *is dense in* $\mathrm{Lip}_{0K}(M, C(K,Y))$ *for every compact Hausdorff topological space K.*
(2) *For $1 \le p < \infty$, if* $\mathrm{LipSNA}_K(M, \ell_p(Y))$ *is dense in* $\mathrm{Lip}_{0K}(M, \ell_p(Y))$, *then* $\mathrm{LipSNA}_K(M, L_p(\mu,Y))$ *is dense in* $\mathrm{Lip}_{0K}(M, L_p(\mu,Y))$ *for every positive measure μ.*
(3) $\mathrm{LipSNA}_K(M, L_\infty(\mu,Y))$ *is dense in* $\mathrm{Lip}_{0K}(M, L_\infty(\mu,Y))$ *for every σ-finite positive measure μ.*

Item (1) above extends items (1) and (3) of Corollary 3.15 and it is indeed stronger. Actually, for Y having property quasi-β but not property β, item (1) above implies that $\mathrm{LipSNA}_K(M, C(K,Y))$ is dense in $\mathrm{Lip}_{0K}(M, C(K,Y))$ for every compact Hausdorff topological space K, and this result does not follow from Corollary 3.15.

Proof of Corollary 3.19. We proceed as in the proof of Corollary 3.17. To prove (1), [16, Theorem 4] shows that we can find a net $\{Q_\lambda\}_{\lambda\in\Lambda}$ of norm-one projections on $C(K,Y)$ such that $\{Q_\lambda(f)\} \to f$ in norm for every $f \in C(K,Y)$ and $Q_\lambda(C(K,Y))$ is isometrically isomorphic to $\ell_p(Y)$. Consequently, we can apply Propositions 4.12 and 3.18.

For (2), fix $1 \le p < \infty$. If $\mathrm{L}_1(\mu)$ is finite-dimensional, the result is a consequence of Proposition 4.5. Otherwise, using [9, Lemma 3.12] we find a net $\{Q_\lambda\}_{\lambda\in\Lambda}$ of norm-one projections on $L_p(\mu,Y)$ such that $\{Q_\lambda\} \to f$ in norm for every $f \in L_p(\mu,Y)$ and $Q_\lambda(L_p(\mu,Y))$ is isometrically isomorphic to $\ell_p(Y)$. Consequently, we can apply Proposition 3.18.

Finally, if $L_\infty(\mu)$ is finite-dimensional, (3) follows from Proposition 4.12 below. If $L_\infty(\mu)$ is infinite-dimensional, we may suppose that the measure is finite by using [5, Proposition 1.6.1]. Then [9, Lemma 3.12] provides a net $\{Q_\lambda\}_{\lambda\in\Lambda}$ of norm-one projections on $L_\infty(\mu, Y)$ such that $\{Q_\lambda\} \to f$ in norm for every $f \in L_\infty(\mu, Y)$ and $Q_\lambda(L_p(\mu, Y))$ is isometrically isomorphic to $\ell_\infty(Y)$. Consequently, we can apply Propositions 4.12 and 3.18. ■

4. Absolute sums of codomains. In this last section we study the behavior of the Lip-BPB property and the density of $\mathrm{LipSNA}(M, Y)$ with respect to absolute sums of the codomain. We need some definitions.

Definition 4.1. An *absolute norm* is a norm $|\cdot|_a$ in $\mathbb{R}^2$ such that

$$|(1,0)|_a = |(0,1)|_a = 1 \quad \text{and} \quad |(s,t)|_a = |(|s|,|t|)|_a \text{ for every } s,t \in \mathbb{R}.$$

Given two Banach spaces W and Z and an absolute norm $|\cdot|_a$, the *absolute sum* of W and Z with respect to $|\cdot|_a$, denoted by $W \oplus_a Z$, is the Banach space $W \times Z$ endowed with the norm

$$\|(w,z)\|_a = |(\|w\|, \|z\|)|_a \quad \forall w \in W, \forall z \in Z.$$

A closed subspace Y_1 of a Banach space Y is said to be an *absolute summand* of Y whenever there exists a closed subspace Z of Y and an absolute norm $|\cdot|_a$ in $\mathbb{R}^2$ such that $Y \equiv Y_1 \oplus_a Z$.

We will need an easy lemma (for a proof, see [12, Lemma 2.2]).

Lemma 4.2. *Let W and Z be Banach spaces and $|\cdot|_a$ be any absolute norm in $\mathbb{R}^2$. If $(w,z) \in S_{W\oplus_a Z}$ and $(w^*,z^*) \in S_{W^*\oplus_{a^*} Z^*}$ are such that $\langle (w,z),(w^*,z^*)\rangle = 1$, then*

$$w^*(w) = \|w^*\|\,\|w\| \quad \textit{and} \quad z^*(z) = \|z^*\|\,\|z\|.$$

Our first result is the following lifting of the Lip-BPB property from a space to its absolute summands.

Proposition 4.3. *Let M be a pointed metric space, Y be a Banach space and Y_1 be an absolute summand of Y. If (M,Y) has the Lip-BPB property with a function $\varepsilon \mapsto \eta(\varepsilon)$, then so does (M,Y_1) with $\varepsilon \mapsto \eta(\varepsilon/3)$.*

The case of $\oplus_a = \oplus_\infty$ appeared in [7, Lemma 4.8].

Proof of Proposition 4.3. Since Y_1 is an absolute summand of Y, we can write $Y = Y_1 \oplus_a Y_2$ for some Banach space Y_2. Fix $\varepsilon > 0$ and consider $\widehat{F}_1 \in \mathrm{L}(\mathcal{F}(M), Y_1)$ with $\|F_1\|_L = 1$ and $m \in \mathrm{Mol}(M) \subset S_{\mathcal{F}(M)}$ satisfying

$$\|\widehat{F}_1(x_0)\| > 1 - \eta(\varepsilon/3).$$

Define $\widehat{T} \in \mathrm{L}(\mathcal{F}(M), Y)$ by $\widehat{T}(x) = (\widehat{F}_1(x), 0)$ for all $x \in \mathcal{F}(M)$, and note that $\|\widehat{T}\| = 1$ and

$$\|\widehat{T}(m)\| = \|(\widehat{F}_1(m), 0)\|_a = \|\widehat{F}_1(m)\| > 1 - \eta(\varepsilon/3).$$

Since (M,Y) has the Lip-BPB property, there are $\widehat{H}\in \mathrm{L}(\mathcal{F}(M),Y)$ and $m'\in \mathrm{Mol}(M)$ such that

$$\|\widehat{H}(m')\|=\|H\|_L=1,\qquad \|T-H\|_L<\varepsilon/3,\qquad \|m-m'\|<\varepsilon/3.$$

Write $\widehat{H}=(\widehat{H}_1,\widehat{H}_2)$, where $H_i\in \mathrm{L}(\mathcal{F}(M),Y_i)$ for $i=1,2$. For all $x\in B_{\mathcal{F}(M)}$ we have

$$\|(\widehat{H}_1(x)-\widehat{F}_1(x),\widehat{H}_2(x))\|_\infty\leq \|(\widehat{H}_1(x)-\widehat{F}_1(x),\widehat{H}_2(x))\|_a\leq \|\widehat{H}-\widehat{T}\|.$$

Then $\|F_1-H_1\|_L<\varepsilon/3$ and $\|H_2\|_L<\varepsilon/3$. Now consider $y^*=(y_1^*,y_2^*)\in Y_1^*\oplus_{a^*}Y_2^*$ with $\|y^*\|_{a^*}=1$ satisfying

$$1=\|\widehat{H}(m')\|=y_1^*(\widehat{H}_1(m'))+y_2^*(\widehat{H}_2(m')).$$

Define $\widehat{G}_1\in \mathrm{L}(\mathcal{F}(M),Y_1)$ by

$$\widehat{G}_1(x)=\|y_1^*\|\widehat{H}_1(x)+y_2^*(\widehat{H}_2(x))\frac{\widehat{H}_1(x)}{\|\widehat{H}_1(x)\|}\qquad \forall x\in \mathcal{F}(M).$$

Now, following [8, proof of Theorem 2.1] one can verify that

$$\|\widehat{G}_1(m')\|=\|G_1\|_L=1,\qquad \|F_1-G_1\|_L<\varepsilon,\qquad \|m-m'\|<\varepsilon/3.$$

Hence, (M,Y_1) has the Lip-BPB property with the function $\varepsilon\mapsto\eta(\varepsilon/3)$. ∎

Note that, as proved in the above proposition, essentially the same function η from the Lip-BPB property of (M,Y) works for the Lip-BPB property of (M,Y_1). This is the key fact to obtain the following consequence.

Corollary 4.4. *Let M be a pointed metric space such that (M,Y) has the Lip-BPB property for all Banach spaces Y. Then there exists a function $\eta_M(\varepsilon)$, which only depends on M, such that (M,Y) has the Lip-BPB property witnessed by $\eta_M(\varepsilon)$ for every Banach space Y.*

Proof. Suppose this is not the case. Then there is a sequence Y_n of Banach spaces such that whenever each pair (M,Y_n) has the Lip-BPB property witnessed by $\eta_n(\varepsilon)>0$, one has $\inf_n\eta_n(\varepsilon)=0$ for some $0<\varepsilon<1$. Let $Y=[\bigoplus_{n\in\mathbb{N}}Y_n]_{c_0}$ and observe that, by hypothesis, (M,Y) has the Lip-BPB property witnessed by a function $\varepsilon\mapsto\eta(\varepsilon)>0$. As each Y_n is clearly an absolute summand of Y, Proposition 4.3 shows that for every $n\in\mathbb{N}$, each pair (M,Y_n) has the Lip-BPB property witnessed by $\varepsilon\mapsto\eta(\varepsilon/3)>0$, contrary to assumption. ∎

For the density of $\mathrm{LipSNA}(M,Y)$, we can give the following result whose proof is a modification of that of [8, Proposition 2.5], in the same way as the proof of Proposition 4.3 follows from [8, Theorem 2.1].

Proposition 4.5. *Let M be a pointed metric space, let Y be a Banach space, and let Y_1 be an absolute summand of Y. If* $\mathrm{LipSNA}(M,Y)$ *is dense in* $\mathrm{Lip}_0(M,Y)$*, then* $\mathrm{LipSNA}(M,Y_1)$ *is dense in* $\mathrm{Lip}_0(M,Y_1)$.

Another result in the same direction is the following modification of [1, Proposition 2.8].

PROPOSITION 4.6. *Let M be a pointed metric space, Y be a Banach space, and K be a compact Hausdorff topological space. If $(M, C(K,Y))$ has the Lip-BPB property witnessed by a function η, then (M,Y) has the Lip-BPB property witnessed by the same function.*

Proof. Fix $\varepsilon > 0$. Consider $\widehat{F}_1 \in \mathrm{L}(\mathcal{F}(M), Y)$ with $\|\widehat{F}_1\| = 1$ and $m \in \mathrm{Mol}(M)$ satisfying

$$\|\widehat{F}_1(m)\| > 1 - \eta(\varepsilon).$$

Define $\widehat{F} \colon \mathcal{F}(M) \to C(K,Y)$ by

$$[\widehat{F}(x)](t) = \widehat{F}_1(x) \quad \text{for every } x \in \mathcal{F}(M),\ t \in K.$$

It is clear that $\|\widehat{F}\| = \|\widehat{F}_1\| = 1$. Furthermore, $\|\widehat{F}(m)\| > 1 - \eta(\varepsilon)$. By the assumption, there exist $\widehat{G} \in \mathrm{L}(\mathcal{F}(M), C(K,Y))$ and $u \in \mathrm{Mol}(M)$ such that

$$\|\widehat{G}(u)\| = \|\widehat{G}\| = 1, \quad \|\widehat{F} - \widehat{G}\| < \varepsilon, \quad \|m - u\| < \varepsilon.$$

Moreover, since K is compact, there is $t_1 \in K$ such that

$$1 = \|\widehat{G}(u)\| = \|[\widehat{G}(u)](t_1)\|.$$

Now, define $\widehat{G}_1 \colon \mathcal{F}(M) \to Y$ by $\widehat{G}_1(x) = [\widehat{G}(x)](t_1)$ for every $x \in \mathcal{F}(M)$. Note that

$$\|\widehat{G}_1\| = \sup_{x \in B_{\mathcal{F}(M)}} \|[\widehat{G}(x)](t_1)\| = \|[\widehat{G}(u)](t_1)\| = \|\widehat{G}_1(u)\| = 1.$$

In addition, we have

$$\begin{aligned} \|G_1 - F_1\|_L &= \sup_{x \in B_{\mathcal{F}(M)}} \|[\widehat{G}(x)](t_1) - [\widehat{F}(x)](t_1)\| \\ &\le \sup_{x \in B_{\mathcal{F}(M)}} \|\widehat{G}(x) - \widehat{F}(x)\| = \|\widehat{G} - \widehat{F}\| < \varepsilon. \end{aligned}$$

As we already know that $\|m - u\| < \varepsilon$, we conclude that (M,Y) has the Lip-BPB property witnessed by η. ∎

The previous proposition also has an analogous formulation for the density of strongly norm-attaining Lipschitz maps.

PROPOSITION 4.7. *Let M be a pointed metric space, let Y be a Banach space, and let K be a compact Hausdorff topological space. Assume that* $\mathrm{LipSNA}(M, C(K,Y))$ *is dense in* $\mathrm{Lip}_0(M, C(K,Y))$. *Then* $\mathrm{LipSNA}(M,Y)$ *is dense in* $\mathrm{Lip}_0(M,Y)$.

Proof. Given $\varepsilon > 0$, consider $\widehat{F}_1 \in \mathrm{L}(\mathcal{F}(M), Y)$ with $\|\widehat{F}_1\|_L = 1$. Define $\widehat{F}$ as in the proof of Proposition 4.6. By hypothesis, there exist $\widehat{G} \in$

$\mathrm{L}(\mathcal{F}(M), C(K,Y))$ and $u \in \mathrm{Mol}(M)$ such that

$$\|\widehat{G}(u)\| = \|\widehat{G}\| = 1 \quad \text{and} \quad \|\widehat{G} - \widehat{F}\| < \varepsilon.$$

Since K is compact, there is $t_1 \in K$ such that $1 = \|\widehat{G}(u)\| = \|[\widehat{G}(u)](t_1)\|$. Now, define a bounded linear operator $\widehat{G}_1 \colon \mathcal{F}(M) \to Y$ by $\widehat{G}_1(x) = [\widehat{G}(x)](t_1)$ for every $x \in \mathcal{F}(M)$. By repeating the argument in Proposition 4.6, we find that $\widehat{G}_1$ attains its norm at $u \in \mathrm{Mol}(M)$ and $\|\widehat{G}_1 - \widehat{F}_1\| \leq \|\widehat{G} - \widehat{F}\| < \varepsilon$. Consequently, $\mathrm{LipSNA}(M,Y)$ is dense in $\mathrm{Lip}_0(M,Y)$. ∎

A careful inspection of the proofs in this section shows that if one starts with a Lipschitz compact map, then one also gets a Lipschitz compact map in each case. Thus, every result has a version for the Lip-BPB property for Lipschitz compact maps and for the density of strongly norm attaining Lipschitz compact maps. We summarize all the results in the following proposition, whose proof we skip.

Proposition 4.8. *Let M be a pointed metric space and let Y be a Banach space.*

(a) *Let Y_1 be an absolute summand of Y. If (M,Y) has the Lip-BPB property for Lipschitz compact maps with a function η, then so does (M,Y_1).*
(b) *If for all Banach spaces Z the pair (M,Z) has the Lip-BPB property for Lipschitz compact maps, then there exists a function η, which only depends on M, such that for every Banach space Z the pair (M,Z) has the Lip-BPB property for Lipschitz compact maps witnessed by η.*
(c) *Let Y_1 be an absolute summand of Y. If* $\mathrm{LipSNA}_K(M,Y)$ *is dense in* $\mathrm{Lip}_{0K}(M,Y)$, *then* $\mathrm{LipSNA}_K(M,Y_1)$ *is dense in* $\mathrm{Lip}_{0K}(M,Y_1)$.
(d) *If for some compact Hausdorff space K the pair $(M,C(K,Y))$ has the Lip-BPB property for Lipschitz compact maps witnessed by a function η, then (M,Y) has the Lip-BPB property for Lipschitz compact maps witnessed by the same function.*
(e) *If* $\mathrm{LipSNA}_K(M,C(K,Y))$ *is dense in* $\mathrm{Lip}_{0K}(M,C(K,Y))$ *for some compact Hausdorff space K, then* $\mathrm{LipSNA}_K(M,Y)$ *is dense in* $\mathrm{Lip}_{0K}(M,Y)$.

Let M be a pointed metric space, let $\{Y_i\}_{i\in I}$ be a family of Banach spaces and let $Y = [\bigoplus_{i\in I} Y_i]_{c_0}$ or $Y = [\bigoplus_{i\in I} Y_i]_{\ell_\infty}$. By Proposition 4.3, if (M,Y) has the Lip-BPB property, then all (M,Y_i) have the Lip-BPB property witnessed by the same function. By Proposition 4.5, if $\mathrm{LipSNA}(M,Y)$ is dense in $\mathrm{Lip}_0(M,Y)$, then $\mathrm{LipSNA}(M,Y_i)$ is dense in $\mathrm{Lip}_0(M,Y_i)$ for all $i \in I$. Our next aim is to show that the converse results also hold. We start with the Lip-BPB property.

Proposition 4.9. *Let M be a pointed metric space, let $\{Y_i\}_{i\in I}$ be a family of Banach spaces, and let Y be $[\bigoplus_{i\in I} Y_i]_{c_0}$ or $[\bigoplus_{i\in I} Y_i]_{\ell_\infty}$. Assume that (M,Y_i) has the Lip-BPB property witnessed by a function η_i for every*

$i \in I$. *If* $\inf \{\eta_i(\varepsilon) : i \in I\} > 0$ *for every* $\varepsilon > 0$, *then* (M, Y) *has the Lip-BPB property.*

Proof. Fix $\varepsilon > 0$, take $\eta(\varepsilon) := \inf \{\eta_i(\varepsilon) : i \in I\} > 0$ and note that we have $\eta_i(\varepsilon) \geq \eta(\varepsilon)$ for every $i \in I$. Let $Q_i : Y \to Y_i$ be the natural projection and $E_i : Y_i \to Y$ the natural embedding for every $i \in I$. Take $\widehat{F} \in \mathrm{L}(\mathcal{F}(M), Y)$ with $\|F\|_L = 1$ and $m \in \mathrm{Mol}(M)$ such that

$$\|\widehat{F}(m)\| > 1 - \eta(\varepsilon).$$

Then there exists $k \in I$ such that $\|Q_k \widehat{F}(m)\| > 1 - \eta(\varepsilon)$. By hypothesis, there exist $\widehat{G}_k \in \mathrm{L}(\mathcal{F}(M), Y_k)$ and $u \in \mathrm{Mol}(M)$ satisfying

$$\|\widehat{G}_k(u)\| = \|G_k\|_L = 1, \quad \|Q_k \widehat{F} - \widehat{G}_k\| < \varepsilon, \quad \|m - u\| < \varepsilon.$$

Now, define $\widehat{G} : \mathcal{F}(M) \to Y$ by

$$\widehat{G}(x) = \sum_{i \neq k} E_i(Q_i(\widehat{F}))(x) + E_k \widehat{G}_k(x) \quad \forall x \in \mathcal{F}(M).$$

Then $\|G\|_L \leq 1$ and $\|\widehat{G}(u)\| \geq \|\widehat{G}_k(u)\| = 1$. Therefore, $\widehat{G}$ attains its norm at $u \in \mathrm{Mol}(M)$. Finally,

$$\|F - G\|_L = \sup \{\|Q_i(\widehat{F} - \widehat{G})\| : i \in I\} = \|Q_k(\widehat{F} - \widehat{G})\| < \varepsilon,$$

that is, (M, Y) has the Lip-BPB property. ∎

It is possible to give a result analogous to Proposition 4.9 for the density of $\mathrm{LipSNA}(M, Y)$.

PROPOSITION 4.10. *Let M be a pointed metric space, let $\{Y_i\}_{i \in I}$ be a family of Banach spaces, and let Y be either $[\bigoplus_{i \in I} Y_i]_{c_0}$ or $[\bigoplus_{i \in I} Y_i]_{\ell_\infty}$. If $\overline{\mathrm{LipSNA}(M, Y_i)} = \mathrm{Lip}_0(M, Y_i)$ for every $i \in I$, then*

$$\overline{\mathrm{LipSNA}(M, Y)} = \mathrm{Lip}_0(M, Y).$$

Proof. For each $i \in I$, let $Q_i : Y \to Y_i$ be the natural projection and $E_i : Y_i \to Y$ the natural embedding. Fix $\varepsilon > 0$ and $\widehat{F} \in \mathrm{L}(\mathcal{F}(M), Y)$ with $\|F\|_L = 1$. There exists $k \in I$ such that $\|Q_k \widehat{F}\| > 1 - \varepsilon/2$. Since $\overline{\mathrm{LipSNA}(M, Y_k)} = \mathrm{Lip}_0(M, Y_K)$ we may find $G_k \in \mathrm{Lip}_0(M, Y_k)$ and $u \in \mathrm{Mol}(M)$ such that

$$\|\widehat{G}_k(u)\| = \|G_k\|_L = 1, \quad \|\widehat{G}_k - Q_k \widehat{F}\| < \varepsilon.$$

Now, define $\widehat{G} : \mathcal{F}(M) \to Y$ by

$$\widehat{G}(x) = \sum_{i \neq k} E_i(Q_i(\widehat{F}))(x) + E_k \widehat{G}_k(x) \quad \forall x \in \mathcal{F}(M).$$

Then $\|G\|_L \leq 1$ and $\|\widehat{G}(u)\| \geq \|\widehat{G}_k(u)\| = 1$. Therefore, $\widehat{G}$ attains its norm at u. Finally,

$$\|F - G\|_L = \sup \{\|Q_i(\widehat{F} - \widehat{G})\| : i \in I\} = \|Q_k(\widehat{F} - \widehat{G})\| < \varepsilon. \ \blacksquare$$

A look at the above two proofs shows that the analogous results for Lipschitz compact maps are also valid.

Proposition 4.11. *Let M be a pointed metric space, let $\{Y_i\}_{i\in I}$ be a family of Banach spaces, and let Y be $[\bigoplus_{i\in I} Y_i]_{c_0}$ or $[\bigoplus_{i\in I} Y_i]_{\ell_\infty}$. Assume that for each $i \in I$ the pair (M, Y_i) has the Lip-BPB property for Lipschitz compact maps witnessed by a function $\eta_i(\varepsilon)$. If $\inf\{\eta_i(\varepsilon)\colon i \in I\} > 0$ for every $\varepsilon > 0$, then (M, Y) has the Lip-BPB property for Lipschitz compact maps.*

Proposition 4.12. *Let M be a pointed metric space, let $\{Y_i\}_{i\in I}$ be a family of Banach spaces, and let Y be either $[\bigoplus_{i\in I} Y_i]_{c_0}$ or $[\bigoplus_{i\in I} Y_i]_{\ell_\infty}$. If $\mathrm{LipSNA}_K(M, Y_i)$ is dense in $\mathrm{Lip}_{0K}(M, Y_i)$ for every $i \in I$, then the set $\mathrm{LipSNA}_K(M, Y)$ is dense in $\mathrm{Lip}_{0K}(M, Y)$.*

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